Mathematical study of wind-driven oceanic motions

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Modèles mathématiques en mécanique des fluides
Plan

Introduction

The almost-periodic, resonant case

The random stationary, non-resonant case
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**Introduction**
- Presentation of the model
- General strategy

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Main assumptions in the interior

- **Starting point**: Ocean = homogeneous, incompressible fluid in a rotating frame.
  → 3D Navier-Stokes equations with Coriolis force $\Omega \wedge u$.

- **Coriolis acceleration**:
  → $f$-plane approximation: $f = 2|\Omega| \sin(\theta)$ homogeneous ("small" geographical zone, midlatitudes),
  → effect of horizontal component of $\Omega$ is neglected.

- **Frictional forces $F$**: notion of "turbulent viscosity":

$$F = A_v \partial_z^2 u + A_h \Delta_h u, \quad A_h, A_v > 0, \ A_h \neq A_v.$$  

- **Conclusion**: the velocity $u$ of currents inside the ocean is described by

$$\partial_t u + (u \cdot \nabla)u + \Omega \wedge u - A_v \partial^2_z u - A_h \Delta_h u + \nabla p = 0,$$

$$\nabla \cdot u = 0. \quad (1)$$
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Boundary conditions

- **Bottom of the ocean**: flat \((h_B \equiv 0)\).
  Homogeneous Dirichlet boundary condition (no-slip):
  \[
  u|_{z=0} = 0.
  \]

- **Surface of the ocean**: rigid lid approximation: \(h \equiv D\).
  Description of wind-stress:
  \[
  \partial_z u_h|_{z=D} = \frac{1}{\rho A_v} \sigma_h,
  \]
  \[
  u_3|_{z=D} = 0.
  \]

- **Horizontal boundaries**: box \(\rightarrow\) horizontal domain:
  \(\omega_h = [0, La_1) \times [0, La_2)\) with periodic boundary conditions.
Scaling assumptions

- High rotation limit: Rossby number \( \varepsilon := \frac{U}{2\Omega|L|} \ll 1 \).

- Horizontal and vertical viscosities:
  \[ \frac{A_h}{\rho UL} \approx 1, \quad \nu := \frac{LA_v}{\rho UD^2} \ll 1. \]

- Amplitude of wind stress: \( \alpha := \frac{\sigma_0 D}{\rho A_v} \gg 1 \).

\( \Omega \) Earth rotation vector
\( L \) Horizontal length scale
\( U \) Horizontal velocity scale
\( D \) Vertical length scale
\( A_h \) Turbulent horizontal viscosity
\( A_v \) Turbulent vertical viscosity
\( \rho \) Density
\( \sigma_0 \) Amplitude of wind velocity
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- **Conclusion**: the system in rescaled variables becomes
  \[
  \partial_t u + u \cdot \nabla u + \frac{1}{\varepsilon} e_3 \wedge u + \nabla p - \Delta_h u - \nu \partial_z^2 u = 0, \\
  \text{div} u = 0, \\
  u|_{z=0} = 0, \\
  \partial_z u|_{h,z=a} = \alpha \sigma, \\
  u_3|_{z=a} = 0.
  \]
Modelization of the wind stress

- Full atmosphere/ocean coupled model is out of reach...
  → Effect of a **given wind stress** on ocean dynamics.

- **Time dependance** of wind stress:
  Coriolis op. $\sim$ fast oscillations in time (freq. $\sim 1/\varepsilon$).
  → Interesting scaling : $\sigma = \sigma(t, t/\varepsilon, x_h)$.

- **First choice**: $\sigma$ almost-periodic : [Masmoudi, 2000]
  
  $$\sigma(t, \tau, x_h) = \sum_{\mu \in M} \sum_{k_h \in \mathbb{Z}^2} \hat{\sigma}(t, \mu, k_h) e^{i k_h \cdot x_h} e^{i \mu \tau}$$

- **Second choice**: $\sigma$ stationary :
  
  $$\sigma(t, \tau, x_h; \omega) = S(t, x_h, \theta_\tau \omega),$$

where

- $\omega \in E$, and $(E, \mathcal{A}, \mu)$ is a probability space,
- $(\theta_\tau)_{\tau \in \mathbb{R}}$ is a measure preserving transformation group actig on $E$.

Interest : introduce some **randomness** in the equation.
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Introduction

Brief review of results on rotating fluids

Ref: Chemin, Desjardins, Gallagher, Grenier.

- Dominant process: Coriolis operator:

\[ L = P(e_3 \wedge \cdot); \]

Spectrum \( \{ \lambda_k := \frac{k_3}{|k|}, k \in \mathbb{Z}^3 \setminus \{0\} \} \).

→ Creation of waves propagating at speed \( \varepsilon^{-1} \).

- Filtering method [Grenier; Schochet]:
  Equation for \( u_L = \exp \left( \frac{t}{\varepsilon} L \right) u. \)

→ Passage to the limit as \( \varepsilon, \nu \to 0 \): envelope equations;

→ Problem: \( u_L \) does not match the boundary conditions.

- Construction of boundary layers [Colin-Fabrie; Desjardins-Grenier; Grenier-Masmoudi; Masmoudi ...]

→ Creation of source terms (Ekman pumping) in envelope equation.
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Coupling between interior and boundary layer terms

Consider the following **Ansatz**

\[
    u(t, x, y, z) \approx u_{\text{int}} \left( t, \frac{t}{\varepsilon}, x, y, z \right) + u_{\text{BL}} \left( t, \frac{t}{\varepsilon}, x, y, z \right),
\]

where

- \( u_{\text{int}}(t, \tau) = \exp(-\tau L) w(t) + v_{\text{int}}(t, \tau), \quad v_{\text{int}} = O(\varepsilon) \);
  - **Role**: \( u_{\text{int}}(t, t/\varepsilon) \) satisfies the evolution equation (up to \( O(\varepsilon) \)) ;

- \( u_{\text{BL}}(\cdot, z) = u_T(\cdot, (a - z)/\eta) + u_B(\cdot, z/\eta), \quad \eta \ll 1 \);
  - **Role**: \( u_{\text{BL}} \) matches the horizontal boundary conditions.

**Remarks** :

- The horizontal BC for \( u_{\text{BL}} \) depend on \( u_{\text{int}} \);
- The vertical BC for \( v_{\text{int}} \) depends on \( u_{\text{BL}} \), and creates a source term (Ekman pumping) in equation for \( w \).

\( \rightarrow \) Coupling between \( u_{\text{int}} \) and \( u_{\text{BL}} \).
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Method of resolution

Idea : define a **boundary layer operator** \( B \):
- **Input** : arbitrary horizontal boundary conditions.
- **Output** : divergence-free boundary layer term, matching the horizontal boundary conditions and equation at leading order.

and an **interior operator** \( U \):
- **Input** : arbitrary initial data and vertical boundary conditions.
- **Output** : interior term matching the vertical boundary conditions and equation at leading order.

→ “Loop” construction :
- at each step, adapt inputs of \( U \) and \( B \) such that BC and eq. are satisfied (at leading order).
- iterate this step until all error terms are sufficiently small.
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Convergence result

**Theorem**: [D., Saint-Raymond, 2007] Let \( u = u^{\varepsilon, \nu} \) be the solution of

\[
\begin{align*}
\partial_t u + \frac{1}{\varepsilon} Lu - \nu \partial_z^2 u - \Delta_h u + \nabla p &= 0, \\
\text{div} u &= 0, \\
u|_{z=0} &= 0, \\
u_3|_{z=a} &= 0, \\
\partial_z u_h|_{z=a}(t) &= \frac{1}{(\varepsilon \nu)^{\frac{1}{4}}} \sum_{\mu, k_h} \hat{\sigma}(\mu, k_h) e^{i\mu \frac{t}{\varepsilon}} e^{i k_h \cdot x_h}.
\end{align*}
\]

Let \( w \) be the solution of the envelope equation. There exists a function \( u^{\text{sing}} \), of order \( (\varepsilon \nu)^{-\frac{1}{4}} \) in \( L^\infty \), such that as \( \varepsilon, \nu \to 0 \),

\[
u^{\varepsilon, \nu} - \left( \exp \left( \frac{t}{\varepsilon} L \right) w(t) + u^{\text{sing}} \right) \to 0,
\]

in \( L^\infty(0, \infty; L^2) \cap L^2(0, \infty; H^1_h) \).
Remarks on the convergence result

- No *a priori* bounds for $u^{\varepsilon, \nu}$;
- In general, $u^{\varepsilon, \nu}$ does not remain bounded: destabilization of the whole fluid inside the domain.
- The **singular profile** $u^{\text{sing}}$ is explicit. Linear response to forcing on the mode

$$k_h = 0, \mu = \pm 1.$$ 

In particular, $u^{\text{sing}}$ does not depend on $x_h$ and $u_3^{\text{sing}} \equiv 0$.
→ No singular Ekman transpiration velocity.

In the sequel:
- Construction of operators $B$ (boundary layer), $U$ (interior).
- Focus on uncommon behaviour: apparition of atypical boundary layers, singular profile.
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The random stationary, non-resonant case
The almost-periodic, resonant case

**General setting**

**Ansatz:**

\[ u_{BL} = u_B \left( t, \frac{t}{\varepsilon}, x_h, \frac{z}{\sqrt{\varepsilon \nu}} \right) + u_T \left( t, \frac{t}{\varepsilon}, x_h, \frac{a - z}{\sqrt{\varepsilon \nu}} \right), \]

and

\[ u_T / u_B = \sum_{k_h, \mu} \hat{u}_T / \hat{u}_B(t, k_h, \mu) e^{i\mu \tau} e^{ik_h \cdot x_h} \exp(-\lambda z). \]

**Linearity:** work with fixed \( k_h \) and \( \mu \) (\( \lambda = \lambda(k_h, \mu) \)).

**Equation in rescaled variables:**

\[ i\mu \hat{u}_1 - \lambda^2 \hat{u}_1 - \hat{u}_2 + \varepsilon k_h^2 \hat{u}_1 + \varepsilon \nu \frac{k_1 k_2 \hat{u}_1 - k_2^2 \hat{u}_2}{\lambda^2 - \varepsilon \nu k_h^2} = 0, \]

\[ i\mu \hat{u}_2 - \lambda^2 \hat{u}_2 + \hat{u}_1 + \varepsilon k_h^2 \hat{u}_2 + \varepsilon \nu \frac{-k_1 k_2 \hat{u}_2 + k_2^2 \hat{u}_1}{\lambda^2 - \varepsilon \nu k_h^2} = 0, \]  

\[ \sqrt{\varepsilon \nu} (ik_1 \hat{u}_1 + ik_2 \hat{u}_2) \pm \lambda \hat{u}_3 = 0. \]
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General setting - 2

**Question**: find $\lambda \in \mathbb{C}$ such that $\det A_\lambda = 0$, where

$$A_\lambda = \begin{pmatrix} i\mu - \lambda^2 + \varepsilon k_h^2 + \frac{\varepsilon \nu k_1 k_2}{\lambda^2 - \varepsilon \nu k_h^2} & -1 - \frac{\varepsilon \nu k_1^2}{\lambda^2 - \varepsilon \nu k_h^2} \\ 1 + \frac{\varepsilon \nu k_2^2}{\lambda^2 - \varepsilon \nu k_h^2} & i\mu - \lambda^2 + \varepsilon k_h^2 - \frac{\varepsilon \nu k_1 k_2}{\lambda^2 - \varepsilon \nu k_h^2} \end{pmatrix}$$

**Different cases**:

- $\mu \neq \pm 1$: eigenvalues of $\begin{pmatrix} i\mu & -1 \\ 1 & i\mu \end{pmatrix}$ are non-zero.
  → Stability by small linear perturbations.
  Conclusion: $\lambda = O(1)$ (bounded away from $0$).

- $\mu = \pm 1$: one of the eigenvalues of $\begin{pmatrix} i\mu & -1 \\ 1 & i\mu \end{pmatrix}$ is zero.
  → Two sub-cases:
    - $k_h \neq 0$: atypical boundary layer ($\lambda = O(\sqrt{\varepsilon} + (\varepsilon \nu)^{1/4})$)
    - $k_h = 0$: $\lambda = 0$ is a solution !– singular profile (bifurcation).
**General setting - 2**

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\end{pmatrix}$$

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- $\mu \neq \pm 1$: eigenvalues of $\begin{pmatrix} i\mu & -1 \\ 1 & i\mu \end{pmatrix}$ are non zero.

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$$A_\lambda = \begin{pmatrix}
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    \frac{\varepsilon k_2^2}{\lambda^2 - \varepsilon \nu k_h^2} & i\mu - \lambda^2 + \varepsilon k_h^2 - \frac{\varepsilon \nu k_1 k_2}{\lambda^2 - \varepsilon \nu k_h^2}
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The almost-periodic, resonant case

General setting - 2

**Question**: find $\lambda \in \mathbb{C}$ such that $\det A_\lambda = 0$, where

$$A_\lambda = \begin{pmatrix}
i\mu - \lambda^2 + \varepsilon k_h^2 + \frac{\varepsilon \nu k_1 k_2}{\lambda^2 - \varepsilon \nu k_h^2} & -1 - \frac{\varepsilon k_1^2}{\lambda^2 - \varepsilon \nu k_h^2} \\
1 + \frac{\varepsilon \nu k_2^2}{\lambda^2 - \varepsilon \nu k_h^2} & i\mu - \lambda^2 + \varepsilon k_h^2 - \frac{\varepsilon \nu k_1 k_2}{\lambda^2 - \varepsilon \nu k_h^2}
\end{pmatrix}$$

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Question: find $\lambda \in \mathbb{C}$ such that $\det A_{\lambda} = 0$, where

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i \mu - \lambda^2 + \varepsilon k^2_h & \frac{\varepsilon \nu k_1 k_2}{\lambda^2 - \varepsilon \nu k^2_h} & -1 - \frac{\varepsilon k^2_1}{\lambda^2 - \varepsilon \nu k^2_h} \\
1 + \frac{\varepsilon \nu k^2_2}{\lambda^2 - \varepsilon \nu k^2_h} & i \mu - \lambda^2 + \varepsilon k^2_h - \frac{\varepsilon \nu k_1 k_2}{\lambda^2 - \varepsilon \nu k^2_h} & \end{pmatrix}$$

Different cases:

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**The almost-periodic, resonant case**

**General setting - 2**

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$$A_\lambda = \begin{pmatrix}
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Classical Ekman layers: $\mu \neq \pm 1$

At first order,

$$A_\lambda = \begin{pmatrix} i\mu - \lambda^2 & -1 \\ 1 & i\mu - \lambda^2 \end{pmatrix}.$$  

**Eigenvalues:** $\lambda^2_{\pm} = i(\mu \pm 1) + o(1)$;

**Eigenvectors:** $w_{\pm} = (1, \pm i) + o(1)$.

**Conclusion:** basis of eigenvectors in $\mathbb{C}^2$.

**Method:** decompose the boundary condition $\delta_h$ (input of $B$) onto basis $\{w_+, w_-\}$:

$$\hat{\delta}_h(k_h, \mu) = \alpha_+ w_+ + \alpha_- w_-.$$  

Horizontal part of the boundary layer term is given by

$$u_{B,h} = \left( \alpha_+ w_+ e^{-\lambda_+ z} + \alpha_- w_- e^{-\lambda_- z} \right) e^{i\mu \tau} e^{ik_h \cdot x_h}$$

$$u_{T,h} = \left( \varepsilon \nu \right)^{1/4} \left( \frac{\alpha_+}{\lambda_+} w_+ e^{-\lambda_+ z} + \frac{\alpha_-}{\lambda_-} w_- e^{-\lambda_- z} \right) e^{i\mu \tau} e^{ik_h \cdot x_h}.$$
Classical Ekman layers: $\mu \neq \pm 1$

At first order,

\[ A_\lambda = \begin{pmatrix} i\mu - \lambda^2 & -1 \\ 1 & i\mu - \lambda^2 \end{pmatrix}. \]

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The almost-periodic, resonant case

Atypical boundary layers: \( \mu = \pm 1, k_h \neq 0 \)

\[
\det A_\lambda = 0 \Rightarrow \left\{ \begin{array}{l}
\lambda^2 = \pm 2i + o(1) \\
\text{or } \lambda^2 = \mathcal{O}(\varepsilon + \sqrt{\varepsilon \nu}).
\end{array} \right.
\]

"Eigenvectors": \( w_{\pm} = (1, \pm i) + o(1). \)
→ Basis of \( \mathbb{C}^2 \) for \( \varepsilon, \nu \) small enough.

Method: decompose the boundary condition (input of \( B \)) onto basis \( \{ w_+, w_- \} \).
Same formulas as before.
→ Uniform bounds in \( L^\infty, L^2 \).

Novelty: keep exact (\( \neq \) approximated) values for \( w_+, w_- \).
→ No error term of order \( \frac{1}{\lambda^2} \).
Atypical boundary layers: $\mu = \pm 1, k_h \neq 0$

$$\text{det } A_\lambda = 0 \Rightarrow \begin{cases} \lambda^2 = \pm 2i + o(1) \\
or \lambda^2 = O((\varepsilon + \sqrt{\varepsilon \nu})) \end{cases}.$$ 

"Eigenvectors": $w_{\pm} = (1, \pm i) + o(1)$. 
→ Basis of $\mathbb{C}^2$ for $\varepsilon, \nu$ small enough.

**Method**: decompose the boundary condition (input of $\mathcal{B}$) onto basis $\{w_+, w_-\}$.
Same formulas as before.
→ Uniform bounds in $L^\infty$, $L^2$.

**Novelty**: keep exact ($\neq$ approximated) values for $w_+, w_-$. 
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The almost-periodic, resonant case

**Apparition of a singular profile**: $\mu = \pm 1, k_h = 0$

Choosing for example $\mu = 1$, we derive

$$A_\lambda = \begin{pmatrix} i - \lambda^2 & -1 \\ 1 & i - \lambda^2 \end{pmatrix}.$$  

**Eigenvalues**: $\lambda_-^2 = 2i$, $\lambda_+^2 = 0$;  
**Eigenvectors**: $w_{\pm} = (1, \pm i)$.

**Remark**: define $\bar{u}^{\text{sing}} := \frac{Z}{(\varepsilon \nu)^{1/4}} e^{rac{it}{\varepsilon}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$. Then

$$\bar{u}^{\text{sing}}_{|z=0} = 0, \quad \partial_z \bar{u}^{\text{sing}}_{h|z=a} = \frac{1}{(\varepsilon \nu)^{1/4}} e^{rac{it}{\varepsilon}} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$  

**Conclusion**: decompose the BC onto basis $\{w_+, w_-\}$.  
**Singular part** of the “boundary layer” term is given by

$$u_{BL,h} = \left( \alpha B, + \frac{\alpha T, + Z}{(\varepsilon \nu)^{1/4}} \right) w_+ e^{rac{it}{\varepsilon}}.$$
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Choosing for example \( \mu = 1 \), we derive

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i - \lambda^2 & -1 \\
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\]

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**Eigenvectors**: \( w_\pm = (1, \pm i) \).

**Remark**: define \( \bar{u}^{\text{sing}} := \frac{Z}{(\varepsilon \nu)^{\frac{1}{4}}} e^{it\frac{1}{\varepsilon}} \begin{pmatrix} 1 \\ i \end{pmatrix} \).

Then \( \bar{u}^{\text{sing}}|_{z=0} = 0, \partial_z \bar{u}^{\text{sing}}|_{z=a} = \frac{1}{(\varepsilon \nu)^{\frac{1}{4}}} e^{it\frac{1}{\varepsilon}} \begin{pmatrix} 1 \\ i \end{pmatrix} \).

**Conclusion**: decompose the BC onto basis \( \{w_+, w_-\} \).

Singular part of the “boundary layer” term is given by

\[
U_{BL,h} = \left( \alpha_{B,+} + \frac{\alpha_{T,+}Z}{(\varepsilon \nu)^{\frac{1}{4}}} \right) w_+ e^{it\frac{1}{\varepsilon}}.
\]
The almost-periodic, resonant case

Apparition of a singular profile: $\mu = \pm 1$, $k_h = 0$

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**Remark:** define $\bar{u}^{\text{sing}} := \frac{Z}{(\varepsilon \nu)^{\frac{1}{4}}} e^{i t_{\varepsilon}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$. Then

$$\bar{u}^{\text{sing}}|_{z=0} = 0, \quad \partial_z \bar{u}^{\text{sing}}|_{z=a} = \frac{1}{(\varepsilon \nu)^{\frac{1}{4}}} e^{i t_{\varepsilon}} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$ 

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**Singular part** of the “boundary layer” term is given by

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Plan

Introduction

The almost-periodic, resonant case
  Main result in the linear case
  The boundary layer operator
  The interior operator

The random stationary, non-resonant case
Decomposition of $u_{\text{int}}$ for $k_h \neq 0$

Explicit construction:

$$u_{\text{int}} = \mathcal{U}[v_B, v_T, u_0]$$

such that $u_{\text{int}}$ is a solution of the evolution equation and satisfies

$$u_{\text{int}}(t = 0) = u_0 + o(1), \; u_{\text{int},3}|_{z=0} = v_B, \; u_{\text{int},3}|_{z=a} = v_T.$$

Decomposition: $u_{\text{int}} = \mathcal{L}\left(\frac{t}{\varepsilon}\right)w(t) + v_{\text{int}} + u_{\text{int}}^{\text{osc}}$ where

- $w(t)$: preponderant term; matches initial data $u_0$;
- $v_{\text{int}}$: known explicitely;
- $u_{\text{int}}^{\text{osc}}$: oscillating term, takes into account rest of equation.
Decomposition of $u_{int}$ for $k_h \neq 0$

Explicit construction:

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$$u_{int}(t = 0) = u_0 + o(1), \quad u_{int,3}|_{z=0} = v_B, \quad u_{int,3}|_{z=a} = v_T.$$

Decomposition:

$$u_{int} = L\left(\frac{t}{\epsilon}\right) w(t) + v_{int} + u_{int}^{osc}$$

where

- $w(t)$: preponderant term; matches initial data $u_0$;
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Decomposition of $u_{\text{int}}$ for $k_h \neq 0$

Explicit construction:

$$u_{\text{int}} = \mathcal{U}[\nu_B, \nu_T, u_0]$$

such that $u_{\text{int}}$ is a solution of the evolution equation and satisfies

$$u_{\text{int}}(t = 0) = u_0 + o(1), \quad u_{\text{int},3|z=0} = \nu_B, \quad u_{\text{int},3|z=a} = \nu_T.$$ 

Decomposition:

$$u_{\text{int}} = \mathcal{L} \left( \frac{t}{\varepsilon} \right) w(t) + \nu_{\text{int}} + u_{\text{osc}}$$

where

- $w(t)$: preponderant term; matches initial data $u_0$;
- $\nu_{\text{int}}$: known explicitly;
- $u^{\text{osc}}_{\text{int}}$: oscillating term, takes into account rest of equation.

$$\hat{\nu}_{\text{int}}(t, k_h, \mu) := \begin{pmatrix} ik_h \frac{\hat{\nu}_T(t, k_h, \mu) - \hat{\nu}_B(t, k_h, \mu)}{|k_h|^2} \\ \hat{\nu}_T(t, k_h, \mu) z + \hat{\nu}_B(t, k_h, \mu)(1 - z) \end{pmatrix}.$$
The almost-periodic, resonant case

**Decomposition of** $u_{\text{int}}$ **for** $k_h \neq 0$

**Explicit construction:**

$$u_{\text{int}} = U[v_B, v_T, u_0]$$

such that $u_{\text{int}}$ is a solution of the evolution equation and satisfies

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- $w(t)$: preponderant term; matches initial data $u_0$;
- $\nu_{\text{int}}$: known explicitly;
- $u_{\text{int}}^{\text{osc}}$: oscillating term, takes into account rest of equation.

**Envelope equation:**

$$\partial_t w - \Delta_h w = -\frac{2}{\varepsilon} \sum_{l \in \mathbb{Z}^3} \frac{n_3(l)}{|l|} \left[ \hat{\nu}_B(t, l_h, \lambda_l) - (-1)^l \hat{\nu}_T(t, l_h, \lambda_l) \right].$$
The almost-periodic, resonant case

**Singular profile for** $k_h = 0$

**Problem**: recall singular profile

$$\bar{u}^{\text{sing}} = \sum_{\pm} \left( \alpha_{B,\pm} + \frac{\alpha_T,\pm Z}{(\varepsilon \nu)^{1/4}} \right) w_{\pm} e^{\pm i^t_{\varepsilon}}.$$

Does not match initial condition!

**Idea**: build $u^{\text{sing}} := \bar{u}^{\text{sing}} + u^{\text{sing}}_{\text{osc}}$, where

$$\partial_t u^{\text{sing}}_{\text{osc}} + \frac{1}{\varepsilon} Lu^{\text{sing}}_{\text{osc}} - \nu \partial^2_z u^{\text{sing}}_{\text{osc}} = 0$$

$$u^{\text{sing}}_{\text{osc}}(t = 0) = -\bar{u}^{\text{sing}}(t = 0),$$

$$u^{\text{sing}}_{\text{osc},h|z=0} = 0, \quad \partial_z u^{\text{sing}}_{\text{osc},h|z=a} = 0 \ (t > 0),$$

$$u^{\text{sing}}_{\text{osc},3} \equiv 0.$$

**Remark**: no stabilization.
Singular profile for \( k_h = 0 \)

**Problem:** recall singular profile

\[
\bar{u}^{\text{sing}} = \sum_{\pm} \left( \alpha_{B,\pm} + \alpha_{T,\pm} \frac{z}{(\varepsilon \nu)^{\frac{1}{4}}} \right) w_\pm e^{\pm \frac{t}{\varepsilon}}.
\]

Does not match initial condition!

**Idea:** build \( u^{\text{sing}} := \bar{u}^{\text{sing}} + u^{\text{sing}}_{\text{osc}} \), where

\[
\partial_t u^{\text{sing}}_{\text{osc}} + \frac{1}{\varepsilon} Lu^{\text{sing}}_{\text{osc}} - \nu \partial_z^2 u^{\text{sing}}_{\text{osc}} = 0
\]

\[
u_{\text{osc}} (t = 0) = -\bar{u}^{\text{sing}}(t = 0),
\]

\[
u^{\text{sing}}_{\text{osc}, h|z=0} = 0, \quad \partial_z u^{\text{sing}}_{\text{osc}, h|z=a} = 0 (t > 0),
\]

\[
u^{\text{sing}}_{\text{osc}, 3} \equiv 0.
\]

**Remark:** no stabilization.
Singular profile for $k_h = 0$

**Problem**: recall singular profile

$$\bar{u}^{\text{sing}} = \sum_{\pm} \left( \alpha_{B,\pm} + \frac{\alpha_T,\pm Z}{(\varepsilon \nu)^{\frac{1}{4}}} \right) w_{\pm} e^{\pm i \frac{t}{\varepsilon}}.$$

Does not match initial condition!

**Idea**: build $u^{\text{sing}} := \bar{u}^{\text{sing}} + u^{\text{sing}}_\text{osc}$, where

$$\partial_t u^{\text{sing}}_\text{osc} + \frac{1}{\varepsilon} L u^{\text{sing}}_\text{osc} - \nu \partial_z^2 u^{\text{sing}}_\text{osc} = 0$$

$$u^{\text{sing}}_\text{osc} (t = 0) = -\bar{u}^{\text{sing}}(t = 0),$$

$$u^{\text{sing}}_{\text{osc},h|z=0} = 0, \quad \partial_z u^{\text{sing}}_{\text{osc},h|z=a} = 0 \ (t > 0),$$

$$u^{\text{sing}}_{\text{osc},3} \equiv 0.$$

**Remark**: no stabilization.
The almost-periodic, resonant case

Conclusion of the almost-periodic case

Linear problem:
- Apparition of atypical boundary layers due to resonant forcing ($\mu = \pm 1$) on the non-homogeneous modes ($k_h \neq 0$).
- Singular profile ($\mu = \pm 1, k_h = 0$) which destabilizes the whole fluid for arbitrary initial data.
- Linearity of the equation enables explicit calculations.

Nonlinear problem:
Recent result: stability of singular profile in $H^s$ norm and when the amplitude of the wind-stress is not too large.
Proof based on analysis of resonant modes: $\lambda_k - \lambda_l = \pm 1$. 
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The almost-periodic, resonant case

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  Convergence result
  The limit equation
The random stationary, non-resonant case

The stationary setting

Recall that

\[ \sigma = S \left( t, x_h, \theta \frac{t}{\varepsilon} \omega \right). \]

**Assumption of non-resonance:** (avoid singular profile)

Define approximate Fourier transform: for \( \gamma > 0 \),

\[ \hat{\sigma}_\gamma(\lambda, \omega) := \frac{1}{2\pi} \int_{\mathbb{R}} \exp(-\gamma |\tau|) e^{-i\lambda \tau} \sigma(\tau, \omega) \, d\tau. \]

Assume that

**(H1)** \( \forall \gamma > 0, \hat{\sigma}_\gamma \in L^\infty(E, L^1(\mathbb{R})) \), and

\[ \sup_{\gamma > 0} \| \hat{\sigma}_\gamma \|_{L^\infty(E, L^1(\mathbb{R}))} < +\infty. \]

**(H2)** \( \exists \) neighbourhoods \( V_\pm \) of \( \pm 1 \), independent of \( \gamma > 0 \), such that

\[ \lim_{\gamma \to 0} \sup_{\lambda \in V_+ \cup V_-} |\hat{\sigma}_\gamma(\lambda)| = 0. \]
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Convergence result in the nonlinear stationary case

**Theorem:** [D., 2007] Let \( u = u^{\varepsilon, \nu} \) be the solution of

\[
\begin{cases}
    \partial_t u + \frac{1}{\varepsilon} Lu + u \cdot \nabla u - \nu \partial_z^2 u - \Delta_h u + \nabla p = 0, \\
    \text{div} u = 0, \\
    u|_{z=0} = 0, \\
    u_3|_{z=a} = 0, \\
    \partial_z u_h|_{z=a}(t) = \frac{1}{(\varepsilon \nu)^{1/2}} \sigma \left( t, \frac{t}{\varepsilon}, x_h, \omega \right).
\end{cases}
\]

Let \( w \in L^{\infty}(0, T^*; H^s) \) (\( s > 5/2 \)) be the solution of the envelope equation, and assume that (H1)-(H2) are satisfied. Then as \( \varepsilon, \nu \to 0 \)

\[
u^{\varepsilon, \nu} - \left( \exp \left( \frac{t}{\varepsilon} L \right) w(t) \right) \to 0,
\]

in \( L^{\infty}(0, T; L^2) \cap L^2(0, T; H^1_h) \) for all \( T < T^* \).

**Remark:** \( w \) is random!
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The random stationary, non-resonant case

Elements of the proof

Same strategy as in almost-periodic case. Main features:

- No atypical boundary layer terms (non-resonance);
- Boundary layer terms are random stationary in time;
- Average behaviour of oscillating functions: ergodic

Theorem:

Lemma

Let $\phi \in L^1(E, \mu)$, and let $\lambda \in \mathbb{R}$. Then $\exists \bar{\phi}^\lambda \in L^1(E)$,

$$\frac{1}{T} \int_0^T \phi(\theta \tau \omega) e^{-i \lambda \tau} d\tau \rightarrow \bar{\phi}^\lambda$$

a.s. and in $L^1$. 
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The random stationary, non-resonant case
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Elements of the proof

Same strategy as in almost-periodic case. Main features:

▶ No atypical boundary layer terms (non-resonance);
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▶ Average behaviour of oscillating functions: \textit{ergodic Theorem}:

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Convergence result

The limit equation

The almost-periodic, resonant case
The function $w$ is a solution of

$$
\begin{cases}
\partial_t w + \bar{Q}(w, w) - \Delta_h w + \bar{S}_B(w) + \bar{S}_T(\omega) = 0, \\
\text{Ekman pumping}
\end{cases}
$$

$$
\left\{
\begin{aligned}
w(t = 0) &= w_0 \in H^s, \quad \text{div}w_0 = 0, \\
\text{div}w &= 0, \\
w_3|_{z=0} &= 0, \quad w_3|_{z=a} = 0,
\end{aligned}
\right.
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In general, $w$ is random... However, $\bar{w} = 1/a \int_0^a w$ is not !

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\partial_t \bar{w} + P(\bar{w} \cdot \nabla \bar{w}) - \Delta_h \bar{w} + \bar{S}_B(\bar{w}) + E[\bar{S}_T] = 0, \\
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$$

**Question :** equation on $E[w] - \bar{w}$? (vertical modes)
The random stationary, non-resonant case

The envelope equation

The function $w$ is a solution of

\[
\begin{align*}
\partial_t w + \tilde{Q}(w, w) - \Delta_h w + \tilde{S}_B(w) + \tilde{S}_T(\omega) &= 0, \\
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\]

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**Question**: equation on $\mathcal{E}[w] - \bar{w}$? (vertical modes)
If the torus is non-resonant, then
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\bar{Q}(\bar{w}, w) = \bar{Q}(\bar{w}, \bar{w}) + \bar{Q}(\bar{w}, w - \bar{w}) + \bar{Q}(w - \bar{w}, \bar{w}) =: q(\bar{w}, w - \bar{w}).
\]

→ The limit equation decouples: \( w = \bar{w} + \tilde{w}_1 + \tilde{w}_2 \), where

- \( \tilde{w} \): nonlinear deterministic equation;
- \( \tilde{w}_1 \): linear deterministic equation:
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  \partial_t \tilde{w}_1 + q(\bar{w}, \tilde{w}_1) - \Delta_h \tilde{w}_1 + \tilde{S}_B(\tilde{w}_1) = 0, \\
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The random stationary, non-resonant case

**Limit system in the case of non-resonant torus**

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Perspectives

- Include treatment of singular profile in the random case (avoid non-resonance assumptions);
- Use $\beta$-plane instead of $f$-plane model (variations of Coriolis parameter);
- Consider more general boundaries (different types of boundary layers are expected);
- Work with density-dependent models.