The Gradient Discretisation Method

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University Paris 6, October 2016

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1. **Presentation of the gradient discretisation method: linear stationary diffusion**
   - From FEM to GDM
   - Measures of gradient scheme accuracy, error estimate

2. **Gradient schemes for non-linear models**
   - Semi-linear equation
   - Quasi-linear equations (and time-stepping)

3. **Do gradient discretisations exist?**
   - A few examples
   - Proving coercivity, limit-conformity, compactness: polytopal toolboxes
   - Proving consistency: local linearly exact GD

4. **About time-dependent problems**

5. **New results obtained through gradient schemes**
Plan

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Model problem

\[
\begin{aligned}
\begin{cases}
-\text{div}(\Lambda \nabla u) &= f \quad \text{in } \Omega, \\
\bar{u} &= 0 \quad \text{on } \partial \Omega.
\end{cases}
\end{aligned}
\]

- \(\Omega\) open bounded in \(\mathbb{R}^d\),
- \(\Lambda : \Omega \rightarrow M_d(\mathbb{R})\) bounded uniformly coercive,
- \(f \in L^2(\Omega)\).
Weak formulation

Find \( \bar{u} \in H^1_0(\Omega) \) s.t., for all \( \bar{v} \in H^1_0(\Omega) \), \( \int_{\Omega} \nabla \bar{u} \cdot \nabla \bar{v} = \int_{\Omega} f \bar{v} \).

▶ Existence and uniqueness of the weak solution (Riesz representation theorem, or Lax-Milgram).
Conforming Finite Element Method (e.g., $P_1$)

Find $\bar{u} \in H^1_0(\Omega)$ s.t., for all $\bar{v} \in H^1_0(\Omega)$, $\int_\Omega \Lambda \nabla \bar{u} \cdot \nabla \bar{v} = \int_\Omega f \bar{v}$.

**Computational issue:** $H^1_0(\Omega)$ is an infinite-dimensional space.

$\leadsto$ Cannot be understood/manipulated by computer.
Conforming Finite Element Method (e.g., $P_1$)

Find $\bar{u} \in H^1_0(\Omega)$ s.t., for all $\bar{v} \in H^1_0(\Omega)$,
$$\int_{\Omega} \Lambda \nabla \bar{u} \cdot \nabla \bar{v} = \int_{\Omega} f \bar{v}.$$  

**Computational issue:** $H^1_0(\Omega)$ is an infinite-dimensional space.

$\Rightarrow$ Cannot be understood/manipulated by computer.

**Easy solution:** replace $H^1_0(\Omega)$ be a finite-dimensional subspace $E = V_h$.

Find $u_h \in V_h$ s.t., for all $v_h \in V_h$,
$$\int_{\Omega} \Lambda \nabla u_h \cdot \nabla v_h = \int_{\Omega} f v_h.$$
Conforming Finite Element Method (e.g., $P_1$)

Find $\bar{u} \in H_0^1(\Omega)$ s.t., for all $\bar{v} \in H_0^1(\Omega)$,
\[ \int_{\Omega} \Lambda \nabla \bar{u} \cdot \nabla \bar{v} = \int_{\Omega} f \bar{v}. \]

**Computational issue:** $H_0^1(\Omega)$ is an infinite-dimensional space.
\[ \Rightarrow \text{Cannot be understood/manipulated by computer.} \]

**Easy solution:** replace $H_0^1(\Omega)$ be a finite-dimensional subspace $E = V_h$.

Find $u_h \in V_h$ s.t., for all $v_h \in V_h$,
\[ \int_{\Omega} \Lambda \nabla u_h \cdot \nabla v_h = \int_{\Omega} f v_h. \]

If $u_h = \sum_i U_i \phi_i$ on a basis $(\phi_i)_i$ of $V_h$, this leads to the square system
\[ AU = B \]
where $A_{ij} = \int_{\Omega} \Lambda \nabla \phi_i \cdot \phi_j$ and $B_i = \int_{\Omega} f \phi_i$. 
Non-conforming FEM (e.g., non-conforming $\mathcal{P}_1$/Crouzet-Raviart)

Non-conforming approximation: $V_h$ is a space of functions, but not a subspace of $H_0^1(\Omega)$.

$\rightsquigarrow$ Need to define $\nabla v_h$ if $v_h \in V_h$?

Example: non-conforming $\mathcal{P}_1$:

$$V_h = \{ v_h : \Omega \to \mathbb{R} : v_h \text{ piecewise linear, continuous at edge midpoints, zero at boundary edge midpoints} \}$$
Non-conforming FEM (e.g., non-conforming $P_1$/Crouzet-Raviart)

Non-conforming approximation: $V_h$ is a space of functions, but not a subspace of $H^1_0(\Omega)$.

~~> Need to define $\nabla v_h$ if $v_h \in V_h$?

Example: non-conforming $P_1$:

\[ \nabla u_h \text{ replaced with broken gradient } \nabla_h u_h, \text{ computed in each cell.} \]

Find $u_h \in V_h$ s.t., for all $v_h \in V_h$,

\[ \int_{\Omega} \nabla_h u_h \cdot \nabla_h v_h = \int_{\Omega} f v_h. \]
Mass-lumping

Consider diffusion-reaction (from, e.g., Euler time-stepping):

\[-\text{div}(\Lambda \nabla \bar{u}) + \bar{u} = f.\]

**Issues:**

- Requires computation of $M_{ij} = \int_{\Omega} \phi_i \phi_j$: non-diagonal mass-matrix (costly for explicit time-stepping).
- Also problematic for non-linear models $-\text{div}(\Lambda \nabla \bar{u}) + \beta(\bar{u}) = f$ (e.g. Richards’ equation).
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**Standard mass-lumping:** replace $M$ with diagonal matrix by summing over columns. Why is that justified?
Consider diffusion-reaction (from, e.g., Euler time-stepping):

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**Standard mass-lumping:** replace $M$ with diagonal matrix by summing over columns. Why is that justified?

**Better vision:** reconstruct piecewise constant functions $\Pi_h u_h$ and $\Pi_h v_h$ from $u_h$ and $v_h$.

Find $u_h \in V_h$ s.t., for all $v_h \in V_h$,

\[
\int_{\Omega} \Lambda \nabla_h u_h \cdot \nabla_h v_h + \int_{\Omega} \Pi_h u_h \Pi_h v_h = \int_{\Omega} f \Pi_h v_h.
\]
The gradient discretisation method (GDM) in a nutshell

In the weak formulation of the PDE, replace continuous space and operators (gradient, function) by discrete space and reconstructed operators.

- Set of discrete space and reconstructed operators: *gradient discretisation* (GD).
- A large number of possible gradient discretisations.
A gradient discretisation is $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ with

- $X_{\mathcal{D},0}$ finite dimensional space (of degrees of freedom), taking into account the Dirichlet BC,
- $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \to L^2(\Omega)$ reconstruction of function,
- $\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \to L^2(\Omega)^d$ reconstructed gradient,

such that $v \to \|\nabla_{\mathcal{D}} v\|_{L^2(\Omega)^d}$ is a norm on $X_{\mathcal{D},0}$. 
Definition (GD)
A gradient discretisation is $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ with

- $X_{\mathcal{D},0}$ finite dimensional space (of degrees of freedom), taking into account the Dirichlet BC,
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such that $v \rightarrow \|\nabla_{\mathcal{D}} v\|_{L^2(\Omega)^d}$ is a norm on $X_{\mathcal{D},0}$.

Definition (GS)
If $\mathcal{D}$ is a GD, the corresponding gradient scheme for $-\text{div}(\Lambda \nabla \bar{u}) = f$ with homogeneous Dirichlet BC is

Find $u_{\mathcal{D}} \in X_{\mathcal{D},0}$ s.t., for all $v_{\mathcal{D}} \in X_{\mathcal{D},0}$,

$$
\int_{\Omega} \Lambda \nabla_{\mathcal{D}} u_{\mathcal{D}} \cdot \nabla_{\mathcal{D}} v_{\mathcal{D}} = \int_{\Omega} f \Pi_{\mathcal{D}} v_{\mathcal{D}}.
$$
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3 measures of accuracy

**Measure of coercivity** (discrete Poincaré’s constant)

\[ C_D = \max_{v_D \in X_D, 0 \{0\}} \frac{\| \Pi_D v_D \|_{L^2}}{\| \nabla_D v_D \|_{L^2}}. \]

**Measure of consistency** ("interpolation error" in FEM vocabulary)

\[ S_D(\varphi) = \min_{v_D \in X_D, 0} (\| \Pi_D v_D - \varphi \|_{L^2} + \| \nabla_D v_D - \nabla \varphi \|_{L^2}) . \]

**Measure of limit-conformity**

\[ W_D(\psi) = \max_{v_D \in X_D, 0 \{0\}} \frac{1}{\| \nabla_D v_D \|_{L^2}} \left| \int_\Omega \nabla_D v_D \cdot \psi + \Pi_D v_D \text{div} \psi \right| . \]
Error estimate

\[\|\Pi_D u_D - \bar{u}\|_{L^2} + \|\nabla_D u_D - \nabla \bar{u}\|_{L^2} \leq C(1 + C_D) [S_D(\bar{u}) + W_D(\Lambda \nabla \bar{u})].\]
\[ \| \Pi_D u_D - \bar{u} \|_{L^2} + \| \nabla_D u_D - \nabla \bar{u} \|_{L^2} \leq C(1 + C_D) \left[ S_D(\bar{u}) + W_D(\Lambda \nabla \bar{u}) \right]. \]

**Convergence:** if a sequence \((D_m)_{m \in \mathbb{N}}\) of gradient discretisations is

\begin{align*}
(P1) \text{ Coercive: } & (C_{D_m})_{m \in \mathbb{N}} \text{ bounded,} \\
(P2) \text{ Consistent: } & S_{D_m}(\varphi) \to 0 \text{ for all } \varphi \in H^1_0(\Omega), \\
(P3) \text{ Limit-conforming: } & W_{D_m}(\psi) \to 0 \text{ for all } \psi \in H_{\text{div}}(\Omega),
\end{align*}

then \(\Pi_{D_m} u_m \to \bar{u}\) and \(\nabla_{D_m} u_m \to \nabla \bar{u}\) in \(L^2\).
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Semi-linear model:

\[-\text{div}(\Lambda(\bar{u})\nabla \bar{u}) = f\]  with homogeneous Dirichlet BC.
Semi-linear model: Weak form:

Find $\bar{u} \in H^1_0(\Omega)$ such that, for all $\bar{v} \in H^1_0(\Omega)$,

$$\int_{\Omega} \Lambda(\bar{u}) \nabla \bar{u} \cdot \nabla \bar{v} = \int_{\Omega} f \bar{v}.$$
Semi-linear model: Weak form:

Find $\bar{u} \in H^1_0(\Omega)$ such that, for all $\bar{v} \in H^1_0(\Omega)$,

$$
\int_{\Omega} \Lambda(\bar{u}) \nabla \bar{u} \cdot \nabla \bar{v} = \int_{\Omega} f \bar{v}.
$$

Gradient scheme: if $D$ is a GD,

Find $u_D \in X_{D,0}$ s.t., for all $v_D \in X_{D,0}$,

$$
\int_{\Omega} \Lambda(\Pi_D u_D) \nabla_D u_D \cdot \nabla_D v_D = \int_{\Omega} f \Pi_D v_D.
$$
To deal with low-order non-linearities, \((\mathcal{D}_m)_{m \in \mathbb{N}}\) must be \(\text{(P4) Compact: (discrete Rellich theorem)}\)

for all \(v_m \in X_{\mathcal{D}_m,0}\) such that \((\|\nabla_{\mathcal{D}_m} v_m\|_{L^2})_{m \in \mathbb{N}}\) is bounded, the sequence \((\Pi_{\mathcal{D}_m} v_m)_{m \in \mathbb{N}}\) is relatively compact in \(L^2\).
To deal with low-order non-linearities, \((\mathcal{D}_m)_{m \in \mathbb{N}}\) must be (P4) \textbf{Compact}: (discrete Rellich theorem) for all \(v_m \in X_{\mathcal{D}_m,0}\) such that \((\|\nabla_{\mathcal{D}_m} v_m\|_{L^2})_{m \in \mathbb{N}}\) is bounded, the sequence \((\Pi_{\mathcal{D}_m} v_m)_{m \in \mathbb{N}}\) is relatively compact in \(L^2\).

**Theorem (Convergence of the GDM for semi-linear equations)**

If \((\mathcal{D}_m)_{m \in \mathbb{N}}\) is coercive, consistent, limit-conforming and compact, then, up to a subsequence,

\[
\Pi_{\mathcal{D}_m} u_m \rightarrow \bar{u} \quad \text{and} \quad \nabla_{\mathcal{D}_m} u_m \rightarrow \nabla \bar{u} \quad \text{strongly in} \ L^2(\Omega).
\]
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With $\beta(s)s \geq 0$,

$$-\text{div}(\Lambda \nabla \bar{u}) + \beta(\bar{u}) = f.$$  

Weak form:

$$\bar{u} \in H^1_0(\Omega) \text{ s.t. } \forall \nu \in H^1_0(\Omega), \quad \int_\Omega \Lambda \nabla \bar{u} \cdot \nabla \nu + \int_\Omega \beta(\bar{u})\nu = \int_\Omega f \nu.$$
\[ \overline{u} \in H_0^1(\Omega) \text{ s.t. } \forall v \in H_0^1(\Omega), \int_{\Omega} \Lambda \nabla \overline{u} \cdot \nabla v + \int_{\Omega} \beta(\overline{u}) v = \int_{\Omega} f v. \]

**Application of GDM:**

- **Option 1:**
  \[ u \in X_{D,0} \text{ s.t. } \forall v \in X_{D,0}, \int_{\Omega} \Lambda \nabla_D u \cdot \nabla_D v + \int_{\Omega} \beta(\Pi_D u) \Pi_D v = \int_{\Omega} f \Pi_D v. \]
\( \bar{u} \in H^1_0(\Omega) \) s.t. \( \forall v \in H^1_0(\Omega) \), 
\[ \int_{\Omega} \Lambda \nabla \bar{u} \cdot \nabla v + \int_{\Omega} \beta(\bar{u}) v = \int_{\Omega} f v. \]

**Application of GDM:**

▶ Option 1:

\( u \in X_{D,0} \) s.t. \( \forall v \in X_{D,0} \),

\[ \int_{\Omega} \Lambda \nabla D u \cdot \nabla D v + \int_{\Omega} \beta(\Pi_D u) \Pi_D v = \int_{\Omega} f \Pi_D v. \]

▶ Option 2:

\( u \in X_{D,0} \) s.t. \( \forall v \in X_{D,0} \),

\[ \int_{\Omega} \Lambda \nabla D u \cdot \nabla D v + \int_{\Omega} \Pi_D \beta(u) \Pi_D v = \int_{\Omega} f \Pi_D v \]

with \( \beta(u_D) \in X_{D,0} \) constructed DOF by DOF.
Stability vs. Computability

**Stability:** with Option 1, since $\beta(s)s \geq 0$, with $v_D = u_D$,

$$\int_{\Omega} \beta(\Pi_D u_D) \Pi_D u_D \geq 0.$$ 

▶ Problem: $\beta(\Pi_D u)$ non-linear function of $\Pi_D u$, no exact quadrature.

**Computability:** with Option 2

$\int_{\Omega} \Pi_D \beta(u_D) \Pi_D u_D$ is fully computable since $\Pi_D$ usually reconstructs piecewise polynomial functions.
**Stability vs. Computability**

**Stability**: with Option 1, since $\beta(s)s \geq 0$, with $\nu_D = u_D$,

$$\int_{\Omega} \beta(\Pi_D u_D) \Pi_D u_D \geq 0.$$

- Problem: $\beta(\Pi_D u)$ non-linear function of $\Pi_D u$, no exact quadrature.

**Computability**: with Option 2

$$\int_{\Omega} \Pi_D \beta(u_D) \Pi_D \nu_D$$

is fully computable since $\Pi_D$ usually reconstructs piecewise polynomial functions.

- Problem: $\Pi_D \beta(u_D) \Pi_D u_D \geq 0$? Stability?
Fifth and last property

To have $\Pi_D \beta(v_D) = \beta(\Pi_D v_D)$:

(P5) **Piecewise constant reconstruction**: there is a basis $(e_i)_{i \in I}$ of $X_{D,0}$ and a partition $(\omega_i)_i$ of $\Omega$ such that, if $v = \sum_i v_i e_i \in X_{D,0}$,

$$\forall i \in I, \quad (\Pi_D v_D)|_{\omega_i} = v_i.$$
5 properties to rule them all, and in the light of conciseness bind them

- (P1) Coercivity
- (P2) Consistency
- (P3) Limit-conformity
- (P4) Compactness
- (P5) Piecewise constant reconstruction
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Conforming Galerkin (incl. $P_k$ FE)

$V$ finite dimensional subspace of $H^1_0(\Omega)$, $(\phi_i)_{i \in I}$ basis of $V$.

- $X_{D,0} = \{v_D = (v_i)_{i \in I}\}$,
- $\Pi_D v_D = \sum_{i \in I} v_i \phi_i$,
- $\nabla_D v_D = \nabla (\Pi_D v_D) = \sum_{i \in I} v_i \nabla \phi_i$.

Remark: $C_D \leq C_P$ (Poincaré constant in $H^1_0$) and $W_D \equiv 0$ (conforming method).
Conforming Galerkin (incl. $P_k \text{ FE}$)

$V$ finite dimensional subspace of $H^1_0(\Omega)$, $(\phi_i)_{i \in I}$ basis of $V$.

- $X_{D,0} = \{ v_D = (v_i)_{i \in I} \}$,
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▶ Remark: $C_D \leq C_P$ (Poincaré constant in $H^1_0$) and $W_D \equiv 0$ (conforming method).

▶ Mass-lumped method: change $\Pi_D$. 
Non-conforming $\mathbb{P}_1$

$\mathcal{T}_h$ triangulation, $(\phi_\sigma)_{\sigma \in \mathcal{E}_{\text{int}}}$ non-conforming $\mathbb{P}_1$ basis (one element per interior edge).

- $X_{D,0} = \{ v_D = (v_\sigma)_{\sigma \in \mathcal{E}_{\text{int}}} \}$,
- $\Pi_D v_D = \sum_{\sigma \in \mathcal{E}_{\text{int}}} v_\sigma \phi_\sigma$,
- $\nabla_D v_D = \nabla_h (\Pi_D v_D)$ broken gradient.

▶ Mass-lumped non-conforming $\mathbb{P}_1$: change $\Pi_D$. 
Finite volume/finite difference scheme ($\Lambda = \text{Id}$)

A rectangular mesh.

- $X_{D,0} = \{ v = ((v_K)_{K \in M}, (v_\sigma)_{\sigma \in E_{\text{int}}}) \}$, cell and interior edge DOFs.
- $\Pi_D v = v_K$ in $K \in M$ (piecewise constant),
- On $V_{K,s}$ as in the figure,

$$(\nabla_D v)|_{V_{K,s}} = \frac{v_\sigma - v_K}{d(\bar{x}_\sigma, \bar{x}_K)} n_{K,\sigma} + \frac{v_{\sigma'} - v_K}{d(\bar{x}_{\sigma'}, \bar{x}_K)} n_{K,\sigma'}.$$
Also...

- Mixed $\mathbb{RT}_k$ finite elements,
- Multi-point flux approximation-O (MPFA-O) finite volumes on cartesian or simplicial meshes,
- Discrete duality finite volumes,
- Hybrid mimetic mixed (HMM) schemes, including mixed-hybrid mimetic finite difference (MFD) schemes,
- Nodal mimetic finite difference methods,
- Vertex approximate gradient (VAG) schemes...
Take-home message: what can GDM do for you?

- (S1) Develop/take your favourite method, say $\mathcal{FM}$, for linear diffusion stationary equation (E),
- (S2) Identify a gradient discretisation $\mathcal{D}_{\mathcal{FM}}$ such that the correspond gradient scheme for (E) is precisely $\mathcal{FM}$,
- (S3) Prove that $\mathcal{D}_{\mathcal{FM}}$ satisfies the coercivity, consistency, limit-conformity, compactness and has, perhaps, piecewise constant reconstruction,

$\Rightarrow$ through $\mathcal{D}_{\mathcal{FM}}$, $\mathcal{FM}$ can be applied to any model studied in the framework of gradient scheme, and yields a converging scheme (without additional work).

*Exemple of models: stationary and transient Leray–Lions ($p$-Laplace), doubly degenerate parabolic, Stokes, Navier-Stokes, variational inequalities, diphasic flows in fractured networks...*
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5. **New results obtained through gradient schemes**
Set of scheme-independent results, for polytopal meshes, to prove (P1), (P3), (P4) (coercivity, limit-conformity, compactness).
Polytopal toolbox: discrete objects

- $\mathcal{T} = (\mathcal{M}, \mathcal{E}, \mathcal{V}, \mathcal{P})$ polytopal mesh: cells, faces, vertices, cell "centers".
- Face and cell DOF gathered in $X_{\mathcal{T},0}$:
  \[ X_{\mathcal{T},0} = \{ v = ((v_K)_{K \in \mathcal{M}}, (v_\sigma)_{\sigma \in \mathcal{E}}) \text{ with } v_\sigma = 0 \text{ if } \sigma \subset \partial \Omega \}. \]
Polytopal toolbox: discrete objects

- $\mathcal{I} = (\mathcal{M}, \mathcal{E}, \mathcal{V}, \mathcal{P})$ polytopal mesh: cells, faces, vertices, cell “centers”.

- Face and cell DOF gathered in $X_{\mathcal{I},0}$:
  
  $X_{\mathcal{I},0} = \{ \nu = ((\nu_K)_{K \in \mathcal{M}}, (\nu_\sigma)_{\sigma \in \mathcal{E}}) \text{ with } \nu_\sigma = 0 \text{ if } \sigma \subset \partial \Omega \}$.

- $\Pi_{\mathcal{I}} : X_{\mathcal{I},0} \to L^\infty(\Omega)$ defined by
  
  $(\Pi_{\mathcal{I}} \nu)|_K = \nu_K$ for all $K \in \mathcal{M}$,

- $\nabla_{\mathcal{I}} : X_{\mathcal{I},0} \to (L^\infty(\Omega))^d$ defined by
  
  $(\nabla_{\mathcal{I}} \nu)|_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| \nu_\sigma \mathbf{n}_{K,\sigma}$ for all $K \in \mathcal{M}$. 

Polytopal toolbox: discrete objects

- $\mathcal{X} = (\mathcal{M}, \mathcal{E}, V, P)$ polytopal mesh: cells, faces, vertices, cell “centers”.
- Face and cell DOF gathered in $X_{\mathcal{X},0}$:
  \[ X_{\mathcal{X},0} = \{ \nu = ((v_K)_{K \in \mathcal{M}}, (v_\sigma)_{\sigma \in \mathcal{E}}) \mid v_\sigma = 0 \text{ if } \sigma \subset \partial \Omega \} \]
- $\Pi_{\mathcal{X}} : X_{\mathcal{X},0} \to L^\infty(\Omega)$ defined by
  \[ (\Pi_{\mathcal{X}} \nu)|_K = v_K \text{ for all } K \in \mathcal{M}, \]
- $\nabla_{\mathcal{X}} : X_{\mathcal{X},0} \to (L^\infty(\Omega))^d$ defined by
  \[ (\nabla_{\mathcal{X}} \nu)|_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} |\sigma| v_\sigma n_{K,\sigma} \text{ for all } K \in \mathcal{M}. \]
- “$H^1_0$-norm” on $X_{\mathcal{X},0}$:
  \[ \| \nu \|_{\mathcal{X}} = \left( \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} |\sigma| d_{K,\sigma} \left| \frac{v_\sigma - v_K}{d_{K,\sigma}} \right|^2 \right)^{1/2}. \]
Under standard regularity assumptions on $\mathcal{T}$:

- Poincaré inequality:

$$\|\Pi_{\mathcal{T}} v\|_{L^2(\Omega)} \leq C \|v\|_{\mathcal{T}}.$$
Under standard regularity assumptions on $\mathcal{T}$:

- Poincaré inequality:
  \[ \| \Pi_{\mathcal{T}} v \|_{L^2(\Omega)} \leq C \| v \|_{\mathcal{T}}. \]

- Discrete (approximate) Stokes formula: for all $\varphi \in H_{\text{div}}(\Omega)^d$,
  \[ \left| \int_{\Omega} (\nabla_{\mathcal{T}} v \cdot \varphi + \Pi_{\mathcal{T}} v \text{div}(\varphi)) \right| \leq C \| \nabla \varphi \|_{L^2(\Omega)^d} \| v \|_{\mathcal{T}} h_M. \]
Under standard regularity assumptions on $\mathcal{X}$:

- **Poincaré inequality:**

  \[ \| \Pi_\mathcal{X} v \|_{L^2(\Omega)} \leq C \| v \|_{\mathcal{X}}. \]

- **Discrete (approximate) Stokes formula:** for all $\varphi \in H_{\text{div}}(\Omega)^d$,

  \[ \left| \int_{\Omega} \left( \nabla_\mathcal{X} v \cdot \varphi + \Pi_\mathcal{X} v \text{div}(\varphi) \right) \right| \leq C \| \nabla \varphi \|_{L^2(\Omega)^d} \| v \|_{\mathcal{X} h_M}. \]

- **Discrete Rellich theorem:** if $\left( \| v_m \|_{\mathcal{X}_m} \right)_{m \in \mathbb{N}}$ is bounded then $\left( \Pi_\mathcal{X}_m v_m \right)_{m \in \mathbb{N}}$ is relatively compact in $L^2(\Omega)$. 
**Definition (Control of a GD by a polytopal toolbox)**

A control of $\mathcal{D}$ by $\mathcal{T}$ is a linear mapping $\Phi : X_{\mathcal{D},0} \rightarrow X_{\mathcal{T},0}$. 
Definition (Control of a GD by a polytopal toolbox)

A control of $\mathcal{D}$ by $\mathcal{I}$ is a linear mapping $\Phi : \mathcal{X}_{\mathcal{D},0} \to \mathcal{X}_{\mathcal{I},0}$.

Regularity factors:

\[
\|\Phi\|_{\mathcal{D},\mathcal{I}} = \max_{v \in \mathcal{X}_{\mathcal{D},0} \setminus \{0\}} \frac{\|\Phi(v)\|_{\mathcal{I}}}{\|\nabla_{\mathcal{D}} v\|_{L^2}},
\]

\[
\omega^\Pi(\mathcal{D}, \mathcal{I}, \Phi) = \max_{v \in \mathcal{X}_{\mathcal{D},0} \setminus \{0\}} \frac{\|\Pi_{\mathcal{D}} v - \Pi_{\mathcal{I}} \Phi(v)\|_{L^2(\Omega)}}{\|\nabla_{\mathcal{D}} v\|_{L^2}},
\]

\[
\omega^\nabla(\mathcal{D}, \mathcal{I}, \Phi) = \max_{v \in \mathcal{X}_{\mathcal{D},0} \setminus \{0\}} \left\{ \left( \sum_{K \in \mathcal{M}} |K|^{-1} \left\| \int_K \left[ \nabla_{\mathcal{D}} v - \nabla_{\mathcal{I}} \Phi(v) \right]^2 \right\| \right)^{1/2} \right\} \frac{1}{\|\nabla_{\mathcal{D}} v\|_{L^2}}.
\]
Polytopal toolbox: control of a GD

Estimates through control:

\[ C_D \leq \omega^\Pi(D, \mathcal{I}, \Phi) + C\|\Phi\|_{D,\mathcal{I}}. \]

\[ W_D(\varphi) \leq \|\varphi\|_{H^1(\Omega)^d} \left[ Ch_M (1 + \|\Phi\|_{D,\mathcal{I}}) + \omega^\Pi(D, \mathcal{I}, \Phi) \right. \]

\[ \left. + \omega^{\nabla}(D, \mathcal{I}, \Phi) \right]. \]
Polytopal toolbox: control of a GD

Estimates through control:

\[ C_D \leq \omega \Pi(D, \mathcal{I}, \Phi) + C \| \Phi \|_{D, \mathcal{I}}. \]

\[ W_D(\varphi) \leq \| \varphi \|_{H^1(\Omega)^d} \left[ Ch_M(1 + \| \Phi \|_{D, \mathcal{I}}) + \omega \Pi(D, \mathcal{I}, \Phi) \right. \]
\[ \left. + \omega \nabla(D, \mathcal{I}, \Phi) \right]. \]

Consequence for a sequence \((D_m)_m\) of GD: if \(\Phi_m\) control of \(D_m\) by \(\mathcal{I}_m\) such that

\[ h_M \rightarrow 0, \quad \sup_{m \in \mathbb{N}} \| \Phi_m \|_{D_m, \mathcal{I}_m} < +\infty, \]
\[ \lim_{m \rightarrow \infty} \omega \Pi(D_m, \mathcal{I}_m, \Phi_m) = 0, \quad \lim_{m \rightarrow \infty} \omega \nabla(D_m, \mathcal{I}_m, \Phi_m) = 0, \]

then \((D_m)_m\) is coercive, limit-conforming and compact.
Plan

1. Presentation of the gradient discretisation method: linear stationary diffusion
   - From FEM to GDM
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   - Semi-linear equation
   - Quasi-linear equations (and time-stepping)

3. Do gradient discretisations exist?
   - A few examples
   - Proving coercivity, limit-conformity, compactness: polytopal toolboxes
     - Proving consistency: local linearly exact GD

4. About time-dependent problems

5. New results obtained through gradient schemes
Local linearly exact (LLE) gradient discretisations

**Definition (LLE GD)**

Let $\mathcal{I}$ be a mesh. A GD $\mathcal{D}$ is local linearly exact if $X_{\mathcal{D},0} = \mathbb{R}^l$ and, for each cell $K$, there is $l_K \subset l$ s.t.

$$\forall x \in K, \quad \Pi_\mathcal{D} v(x) = \sum_{i \in l_K} v_i \alpha^i_K(x) \quad \text{and} \quad \nabla_\mathcal{D} v(x) = \sum_{i \in l_K} v_i G^i_K(x)$$

where, for some $(x_i)_{i \in l_K}$ close to $K$,

$$\forall x \in K, \quad \sum_{i \in l_K} \alpha^i_K(x) = 1 \quad \text{and} \quad \forall q_1 \in \mathbb{P}_1, \quad \sum_{i \in l_K} q_1(x_i) G^i_K(x) = \nabla q_1.$$

▶ A parameter $\text{reg}_{\text{LLE}}(\mathcal{D})$ measures, for all $K \in \mathcal{M}$,

(i) how far $(x_i)_{i \in l_K}$ are from $K$,

(ii) scaled $L^2(K)$ norms of $\alpha^i_K$ and $G^i_K$. 


Theorem (LLE GD are consistent (i.e. satisfy (P2)))

If \((D_m)_m\) are LLE GD associated with meshes \((\mathcal{T}_m)_m\) such that \(h_{M_m} \to 0\) and \(\text{reg}_{\text{LLE}}(D_m)_m\) is bounded, then \((D_m)_m\) is consistent.
Entire proof of (P1)–(P4) for non-conforming $\mathbb{P}_1$ gradient discretisations. We drop the index $m$ from time to time for sake of legibility, and all constants below do not depend on $m$ or the considered cells/edges. Let us define a control of $\mathcal{D}$ by $\mathcal{T}$ in the sense of Definition 2.29, where $\mathcal{T}$ is the simplicial mesh associated to $\mathcal{D}$, with $x_K = \overline{x}_K = \frac{1}{d+1} \sum_{\sigma \in E_K} \overline{x}_\sigma$ the centres of gravity of the cells $K$. We define the linear (injective) mappings $\Phi : X_{\mathcal{D}, m, 0} \rightarrow X_{\mathcal{T}, m, 0}$ by $\Phi(u)_K = \frac{1}{d+1} \sum_{\sigma \in E_K} u_\sigma = \Pi_{\mathcal{D}} u(x_K)$ and $\Phi(u)_\sigma = u_\sigma = \Pi_{\mathcal{T}} u(\overline{x}_\sigma)$.

Since $\Phi(u)_K = \Pi_{\mathcal{D}} u(x_K)$ and $G_K u = \nabla (\Pi_{\mathcal{D}} u)$ in $K$, we get

$$\Phi(u)_\sigma - \Phi(u)_K = G_K u \cdot (\overline{x}_\sigma - x_K).$$

Therefore, since $|\overline{x}_\sigma - x_K| \leq \frac{h_K}{d_K, \sigma} \leq \theta_T$,

$$\sum_{\sigma \in E_K} |\sigma| d_{K, \sigma} \left| \frac{\Phi(u)_\sigma - \Phi(u)_K}{d_{K, \sigma}} \right|^p \leq \theta_T^p d |K| |G_K u|^p.$$

This implies (2.34). We now observe that the affine function $\alpha_\sigma$ reaches its extremal values at the vertices of $K$. It is easy to see that $\alpha_\sigma(v_\sigma) = 1 - d$, where $v_\sigma$ is the vertex opposite to the face $\sigma$, and that $\alpha_{\sigma'}(v_{\sigma'}) = 1$ for all $\sigma' \neq \sigma$. Therefore, for $x \in K$,

$$|\Pi_{\mathcal{D}} u(x) - \Phi(u)_K| = \left| \sum_{\sigma \in E_K} (\Phi(u)_\sigma - \Phi(u)_K) \alpha_\sigma(x) \right| \leq (d + 1) \max(1, d - 1) \max_{\sigma \in E_K} |G_K u \cdot (\overline{x}_\sigma - x_K)|.$$  

This inequality implies $\omega^{\Pi}(\mathcal{D}, \mathcal{T}, \Phi) \leq (d + 1) \max(1, d - 1) h_M$ and therefore (2.35) holds. Finally, recalling that $\Pi_{\mathcal{D}} u$ is affine in each simplex $K$ and that $\nabla_T$ is exact on interpolants of affine functions (cf. Lemma 2.28), we see that $\nabla_{\mathcal{D}} u = \nabla_{\mathcal{T}} \Phi(u)$ in $\Omega$. Hence $\omega^\nabla(\mathcal{D}, \mathcal{T}, \Phi) = 0$ and (2.36) holds. Proposition 2.31 therefore shows that $(\mathcal{D}_m)_{m \in \mathbb{N}}$ is coercive in the sense of Definition 2.2, limit-conforming in the sense of Definition 2.4, and compact in the sense of Definition 2.5.

Since non-conforming $\mathbb{P}_1$ gradient discretisations are LLE gradient discretisations, the consistency of $(\mathcal{D}_m)_{m \in \mathbb{N}}$ follows from Proposition 2.14 by noticing that $\text{reg}_{\text{LLE}}(\mathcal{D}_m)$ is controlled by $\theta_{\mathcal{T}_m}$. □
Proof of the property \( (P) \) for MPFA-O gradient discretisations. We drop the indices \( m \) for sake of legibility. We consider the polytopal mesh \( \mathcal{T} = (\mathcal{M}, \mathcal{E}', \mathcal{P}, \mathcal{V}') \) where the sets \( (\mathcal{M}, \mathcal{P}) \) are those of the original polytopal mesh, \( \mathcal{E}' = \{ \sigma_v ; \sigma \in \mathcal{E}, v \in \mathcal{V} \} \), and \( \mathcal{V}' \) is the set of all vertices of the elements of \( \mathcal{E}' \). We define a control of \( \mathcal{D} \) by \( \mathcal{T} \) (in the sense of Definition 2.29) as the isomorphism \( \Phi : X_{\mathcal{D},0} \rightarrow X_{\mathcal{T},0} \) given by \( \Phi(u)_K = u_K \) and \( \Phi(u)_{\sigma_v} = u_{(\sigma,v)} \). We observe that

\[
\int_K |\nabla_{\mathcal{D}} u(x)|^p dx \geq C_3 \sum_{\sigma \in \mathcal{E}_K} \sum_{v \in \mathcal{V}_\sigma} |\sigma_v| d_{K,\sigma} \left| \frac{u_{(\sigma,v)} - u_K}{d_{K,\sigma}} \right|^p,
\]

with \( C_3 = 1 \) for parallelepipedic meshes, and \( C_3 > 0 \) depends on an upper bound of the regularity of the mesh for simplicial meshes. Therefore \( \| \nabla_{\mathcal{D}} u \|_{L^p(\Omega)}^p \geq C_3 \| \Phi(u) \|_{\mathcal{T},0}^p \) and (2.34) is proved. Since \( \Pi_D u = \Pi_T \Phi(u) \), we get \( \omega^\Pi(D, T, \Phi) = 0 \), which proves (2.35). Finally, we have

\[
\int_K \nabla_{\mathcal{D}} u(x) dx = \sum_{\sigma \in \mathcal{E}_K} \sum_{v \in \mathcal{V}_\sigma} |\sigma_v|(u_{\sigma,v} - u_K)n_{K,\sigma} = \sum_{\sigma' \in \mathcal{E}_K'} |\sigma'|(\Phi(u)_{\sigma'} - \Phi(u)_K)n_{K,\sigma} = |K| \nabla_T \Phi(u)_K.
\]

This shows that \( \omega^\nabla(D, T, \Phi) = 0 \), which establishes (2.36). Proposition 2.31 therefore shows that \( (\mathcal{D}_m)_{m \in \mathbb{N}} \) is coercive in the sense of Definition 2.2, limit-conforming in the sense of Definition 2.4, and compact in the sense of Definition 2.5.

It is proved in [40,41] that the definitions of the approximation points \( S \) give the LLE property in both the Cartesian and simplicial cases. Hence, the consistency of \( (\mathcal{D}_m)_{m \in \mathbb{N}} \) follows from Proposition 2.14. \( \square \)
Proof of the property \((P)\) for HMM gradient discretisations. Let \((D_m)_{m \in \mathbb{N}}\) be HMM gradient discretisations built on polytopal meshes \((T_m)_{m \in \mathbb{N}}\), and let us define a control of \(D_m\) by \(T_m\) in the sense of Definition 2.29. We drop the index \(m\) from time to time. Since \(X_{D,0} = X_{T,0}\), we can take \(\Phi = \text{Id}\). Estimate (2.34) is given by (3.14). Relation (2.35) follows immediately since \(\omega^\Pi(D, T, \Phi) = 0\), owing to \(\Pi_D u = \Pi_T u = \Pi_T \Phi(u)\). Recalling that \(|D_{K,\sigma}| = \frac{|\sigma|d_{K,\sigma}}{d}\) we have

\[
\int_K \nabla_D u(x) \, dx = |K| \nabla_K u + \frac{1}{\sqrt{d}} \sum_{\sigma \in \mathcal{E}_K} |\sigma| [\mathcal{L}_K R_K(Q_K(u))]_{\sigma} n_{K,\sigma}.
\]  

(3.15)

The definition of \(R_K\) and the property \(\sum_{\sigma \in \mathcal{E}_K} |\sigma| n_{K,\sigma}(\overline{x}_\sigma - x_K)^T = |K| \text{Id}\) (a consequence of Stokes’ formula) show that for any \(\eta \in \text{Im}(R_K)\) we have \(\sum_{\sigma \in \mathcal{E}_K} |\sigma| \eta_\sigma n_{K,\sigma} = 0\). Hence, since \(\text{Im}(\mathcal{L}_K) = \text{Im}(R_K)\), (3.15) gives

\[
\int_K \nabla_D u(x) \, dx = |K| \nabla_K u = |K| \nabla_T \Phi(u)|_K,
\]

which shows that \(\omega^\nabla(D, T, \Phi) = 0\), and thus that (2.36) holds. The coercivity, limit-conformity and compactness of \((D_m)_{m \in \mathbb{N}}\) therefore follow from Proposition 2.31. Since HMM gradient discretisations are LLE gradient discretisations, the consistency of \((D_m)_{m \in \mathbb{N}}\) readily follows from Proposition 2.14, after noticing that the regularity assumption on \((D_m)_{m \in \mathbb{N}}\) gives a bound on \(\text{reg}_{\text{LLE}}(D_m))_{m \in \mathbb{N}}\). 

\[\square\]
Proof of the property (P) for the nMFD gradient discretisation. As in previous proofs, we drop indices $m$ from time to time. We define a control $\Phi$ of $D$ by $T$, in the sense of Definition 2.29, by

$$\forall K \in M, \Phi(u)_K = u_K = \frac{1}{|K|} \sum_{v \in V_K} \omega^v_K u_v \quad \text{and} \quad \forall \sigma \in E, \Phi(u)_\sigma = \frac{1}{|\sigma|} \sum_{v \in V_\sigma} \omega^v_\sigma u_v. \quad (3.28)$$

Let us prove (2.34). Since $\sum_{v \in V_\sigma} \omega^v_\sigma = |\sigma|$ we have $\Phi(u)_\sigma - \Phi(u)_K = \frac{1}{|\sigma|} \sum_{v \in V_\sigma} \omega^v_\sigma (u_v - u_K)$. Therefore, using Jensen’s inequality and the fact that $\frac{1}{d_{K,\sigma}} \leq \frac{\theta_T}{h_K}$ we find

$$\begin{align*}
\sum_{\sigma \in E_K} |\sigma| d_{K,\sigma} \left| \Phi(u)_\sigma - \Phi(u)_K \right|^p &\leq \sum_{\sigma \in E_K} d_{K,\sigma} \sum_{v \in V_\sigma} \omega^v_\sigma \left| \frac{u_v - u_K}{d_{K,\sigma}} \right|^p \leq \theta_T^p \sum_{\sigma \in E_K} d_{K,\sigma} \sum_{v \in V_\sigma} \omega^v_\sigma \left| \frac{u_v - u_K}{h_K} \right|^p \\
&\leq \theta_T^p \sum_{v \in V_K} \left( \sum_{\sigma \in E_{K,v}} d_{K,\sigma} \omega^v_\sigma \right) \left| \frac{u_v - u_K}{h_K} \right|^p = \theta_T^p d \sum_{v \in V_K} |V_{K,v}| \left| \frac{u_v - u_K}{h_K} \right|^p.
\end{align*}$$

We conclude the proof of (2.34) thanks to (3.26). Since $\Pi_D u = \Pi_T \Phi(u)$, we have $\omega^\Pi(D,T,\Phi) = 0$ and (2.35) follows. For $K \in M$ we have $\nabla K u = (\nabla_T \Phi(u))|_K$. Therefore

$$\int_K \nabla_D u(x) dx = |K| (\nabla_T \Phi(u))|_K + \frac{1}{d} \sum_{v \in V_K} [\mathcal{L}_K R_K(Q_K(v))]|_v \sum_{\sigma \in E_{K,v}} \omega^v_\sigma n_{K,\sigma}. \quad (3.29)$$

Similarly as for the HMM method, for any $\eta \in \text{Im}(R_K)$ we have $\sum_{v \in V_K} \eta_v \sum_{\sigma \in E_{K,v}} \omega^v_\sigma n_{K,\sigma} = 0$. Hence, the last term in (3.29) vanishes and (2.36) holds since $\omega^{\nabla}(D,T,\Phi) = 0$. Hence the hypotheses of Proposition 2.31 are verified, which shows that $(D_m)_{m \in N}$ is coercive, limit-conforming and compact.

By noticing that $\text{reg}_{\text{LLE}}(D_m)$ remains bounded by regularity assumption on $(D_m)_{m \in N}$, the consistency of $(D_m)_{m \in N}$ is an immediate consequence of Proposition 2.14 since nMFD gradient discretisations are LLE gradient discretisations. \qed
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A catch-all model

\[
\begin{cases}
\partial_t \beta(\bar{u}) - \text{div}(a(x, \nabla \zeta(\bar{u}))) = f & \text{in } \Omega \times (0, T), \\
\zeta(\bar{u}) = 0 & \text{on } \partial\Omega \times (0, T), \\
\beta(\bar{u}) = \beta(u_{\text{ini}}) & \text{at } t = 0.
\end{cases}
\]

- \(a(x, \xi) = |\xi|^{p-2} \xi, \ \beta(s) = \zeta(s) = s\): transient \textbf{\textit{p-Laplace}} (also for generic Leray–Lions operator),
- \(a(x, \xi) = \Lambda(x)\xi, \ \beta(s) = s, \ \zeta \text{ non-decreasing:} \textbf{Stefan’s model} \) of melting material (\(\bar{u} = \text{enthalpy}, \ \zeta(\bar{u}) = \text{temperature}\),
- \(a(x, \xi) = \Lambda(x)\xi, \ \zeta(s) = s, \ \beta \text{ non-decreasing:} \textbf{Richards’ equation} \) of underground water flow (\(\bar{u} = \text{pressure}, \ \beta(\bar{u}) = \text{water content}\)).
Application of the GDM

Weak formulation: find $\overline{u}$ in the proper space s.t.

$$\int_0^T \langle \partial_t \beta(\overline{u}), \nu \rangle + \int_0^T \int_\Omega a(\mathbf{x}, \nabla \zeta(\overline{u})) \cdot \nabla \nu = \int_0^T \int_\Omega f \nu, \quad \forall \nu.$$
Application of the GDM

Weak formulation: find $\bar{u}$ in the proper space s.t.

$$\int_0^T \langle \partial_t \beta(\bar{u}), v \rangle + \int_0^T \int_\Omega a(x, \nabla \zeta(\bar{u})) \cdot \nabla v = \int_0^T \int_\Omega f v, \quad \forall v.$$ 

Consider $\mathcal{D}$, time steps $0 = t_1 < t_2 < \cdots < t_N = T$, and interpolator $l_D : L^2(\Omega) \to X_{\mathcal{D},0}$:

Find $u_D = (u^n)_{n=0, \ldots, N} \in X_{\mathcal{D},0}^{N+1}$ s.t. $u^0 = l_D u_{\text{ini}}$ and, for all $n = 0, \ldots, N - 1$ and all $v_D \in X_{\mathcal{D},0}$,

$$\int_\Omega \delta_{\mathcal{D}}^{n+\frac{1}{2}} \beta(u_D) \Pi_D v_D + \int_\Omega a(x, \nabla_D [\zeta(u_D^{n+1})]) \cdot \nabla_D v_D$$

$$= \frac{1}{\delta t} \int_{n\delta t}^{(n+1)\delta t} \int_\Omega f \Pi_D v_D$$
Convergence analysis tools

▶ Space–time Kolmogorov with (discrete) varying spaces,
▶ Discrete Aubin–Simon theorem with (discrete) varying spaces,
▶ Discontinuous weak Ascoli–Arzela.
▶ Discrete “compensated compactness” theorem (convergence of $\int f_n g_n$ from time-derivatives estimates on $f_n$ and space-derivative estimates on $g_n$).

All these results already fully developed for GD, can be used off-the-shelf.
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A few of them

- Unified analysis framework for host of methods (no need to re-invent the wheel).
- Proper generic definition and treatment of mass-lumping.
- Generic treatment of barycentric elimination of DOFs.
- Development of generic discrete functional analysis results, for the GDM and more.
- Uniform-in-time strong $L^2(\Omega)$ convergence results for degenerate parabolic equations, without regularity assumptions.
A few of them

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- Development of generic discrete functional analysis results, for the GDM and more.
- Uniform-in-time strong $L^2(\Omega)$ convergence results for degenerate parabolic equations, without regularity assumptions.
- **Super-convergence for TPFA finite volumes.**
Improved $L^2$ estimate for GS

\[- \text{div}(A \nabla \bar{u}) = f \text{ in } \Omega, \quad \bar{u} = 0 \text{ on } \partial \Omega\]

\[- \text{div}(A \nabla \bar{w}) = \frac{\Pi_D u - \bar{u}}{\|\Pi_D u - \bar{u}\|_2} \text{ in } \Omega, \quad \bar{w} = 0 \text{ on } \partial \Omega.\]

**Theorem (Improved $L^2$ estimate for GS)**

For all $P_D \bar{u} \in X_{D,0}$,

\[\|\Pi_D u - \bar{u}\|_2 \lesssim \left[ h^{-1} I_h(\bar{u}, P_D \bar{u}) \right]^2 + \left[ S_D(\bar{u}) + W_D(A \nabla \bar{u}) \right]^2 + I_h(\bar{u}, P_D \bar{u}) + \left| \widetilde{W}_D(A \nabla \bar{u}, P_D \bar{w}) \right| + \text{ symmetric in } \bar{w}\]

where

\[I_h(\phi, P_D \phi) = \|\Pi_D(P_D \phi) - \phi\|_2 + h \|\nabla_D(P_D \phi) - \nabla \phi\|_2,\]

\[\widetilde{W}(\psi, P_D \phi) = \int_{\Omega} \nabla_D(P_D \phi) \cdot \psi + \Pi_D(P_D \phi) \text{div} \psi.\]
Application to Hybrid Mimetic Mixed schemes

- HMM = family of schemes gathering Hybrid FV, hybrid Mimetic Finite Differences, and Mixed FV.
- HMM is a GDM, based on a polytopal mesh $\mathcal{T}$ and such that
  \[
  X_{D,0} = X_{\mathcal{T},0}, \quad \Pi_D v = \Pi_{\mathcal{T}} v \text{ (piecewise constant)},
  \]
  \[
  \nabla_D v = \nabla_{\mathcal{T}} v + \text{stabilisation}.
  \]

**Theorem (Super-convergence for HMM)**

Assume optimal $H^2$ regularity for the PDE. If the cell “centers” $P$ are, on average on local patches of cells, close to the centers of mass of the cells, then, for $u_D$ solution to HMM,

\[
\|\Pi_D u_D - \bar{u}\|_2 = O(h^2 \| f \|_{H^1}).
\]
Application to Hybrid Mimetic Mixed schemes

- HMM = family of schemes gathering Hybrid FV, hybrid Mimetic Finite Differences, and Mixed FV.
- HMM is a GDM, based on a polytopal mesh $\mathcal{K}$ and such that

$$X_{D,0} = X_{\mathcal{K},0}, \quad \Pi_D v = \Pi_{\mathcal{K}} v \text{ (piecewise constant)},$$
$$\nabla_D v = \nabla_{\mathcal{K}} v + \text{stabilisation}.$$

Theorem (Super-convergence for HMM)

Assume optimal $H^2$ regularity for the PDE. If the cell “centers” $P$ are, on average on local patches of cells, close to the centers of mass of the cells, then, for $u_D$ solution to HMM,

$$\|\Pi_D u_D - \bar{u}\|_2 = O(h^2\|f\|_{H^1}).$$

- Starts by a modified HMM scheme that always super-converges.

$$\forall x \in K, \quad \Pi^*_D v(x) = \Pi_D v(x) + \nabla_D v(x) \cdot (x - x_K).$$
Super-convergence for TPFA

On triangular meshes with $A(x) = a(x)\text{Id}$, TPFA is an HMM, with $x_K = \text{circumcenter of } K$. 
Super-convergence for TPFA

- On triangular meshes with $A(x) = a(x)\text{Id}$, TPFA is an HMM, with $x_K = $ circumcenter of $K$.

- For TPFA on triangular meshes as used in benchmarks, local compensation always occurs (up to a small portion).
Theorem (Super-convergence for TPFA on triangles)

In all classical triangular meshes used in benchmarking, with $x_K$ circumcenter of $K$, under optimal $H^2$ regularity,

$$\|u_h - \bar{u}_P\|_2 = O(h^{2-\varepsilon} \|f\|_{H^1}),$$

where $u_h$ is the solution to the TPFA FV scheme and $\bar{u}_P$ is the piecewise constant function equal to $\bar{u}(x_K)$ on $K$. 
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**Theorem (Super-convergence for TPFA on triangles)**

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Long-standing (20+ years) conjecture, only obtained by first abstracting from the specificities of the schemes.
Conclusion

- GDM = generic analysis framework for many numerical methods and many diffusion models (linear and non-linear).

- Easy proof that a given method fits into the GDM.

- Proof of convergence based on 3-5 properties.

- Discrete functional analysis results readily usable in the GDM.

- Led to novel results, thanks to a level of abstraction that liberates from the specificities of each scheme.

https://hal.archives-ouvertes.fr/hal-01382358


Thanks.