Filtered Poisson processes: some geometric properties and some applications for images

Agnès Desolneux

CNRS et CMLA (ENS Cachan)

Séminaire du LJLL, le vendredi 1er avril 2016.

Joint work with Hermine Biermé (Université de Poitiers) and Lionel Moisan (Université Paris Descartes)
1er avril oblige ....
A **filtered Poisson process** also called **shot noise random field** is a random function $X : \mathbb{R}^n \to \mathbb{R}$ given by

$$\forall x \in \mathbb{R}^n, \quad X(x) = \sum_{i \in I} g_{m_i}(x - x_i),$$

where

- \( \{x_i\}_{i \in I} \) is a Poisson point process of intensity \( \lambda > 0 \) in \( \mathbb{R}^n \),
- \( \{m_i\}_{i \in I} \) are independent « marks » with distribution \( F(dm) \) on \( \mathbb{R}^d \), and independent of \( \{x_i\}_{i \in I} \).
- The functions \( g_m \) are real-valued deterministic functions, called **kernel functions**, such that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^n} |g_m(y)| \, dy \, F(dm) < +\infty.$$
A filtered Poisson process also called shot noise random field is a random function $X : \mathbb{R}^n \to \mathbb{R}$ given by

$$\forall x \in \mathbb{R}^n, \quad X(x) = \sum_{i \in I} g_{m_i}(x - x_i),$$

where

- $\{x_i\}_{i \in I}$ is a Poisson point process of intensity $\lambda > 0$ in $\mathbb{R}^n$,
- $\{m_i\}_{i \in I}$ are independent « marks » with distribution $F(dm)$ on $\mathbb{R}^d$, and independent of $\{x_i\}_{i \in I}$.
- The functions $g_m$ are real-valued deterministic functions, called kernel functions, such that
  $$\int_{\mathbb{R}^d} \int_{\mathbb{R}^n} |g_m(y)| \, dy \, F(dm) < +\infty.$$

A Poisson point process $\Phi = \{x_i\}_{i \in I}$ of intensity $\lambda > 0$ in $\mathbb{R}^n$ is characterized by : for any region $B$ of finite Lebesgue measure, the number $N(B) = \#\{\Phi \cap B\}$ of points in $B$ follows a Poisson distribution of parameter $\lambda|B|$, that is

$$\forall k \in \mathbb{N}, \quad \mathbb{P}(N(B) = k) = e^{-\lambda|B|} \frac{(\lambda|B|)^k}{k!}.$$

And for any two disjoint regions $B_1$ and $B_2$, the number of points $N(B_1) = \#\{\Phi \cap B_1\}$ and $N(B_2) = \#\{\Phi \cap B_2\}$ are independent random variables.
Applications in Physics

Shot noise random fields can model many various situations in Physics where there are additive contributions caused by point sources, or point events in 1D:

- electric potential created by point charges,
- temperature created by heat point sources,
- gravitational potential created by point masses,
- total power received from a collection of antennas,
- substance concentration in blood
- etc.
Example 1: exponential kernel

The exponential kernel is given by:

\[ g(s) = e^{-s} \text{ if } s \geq 0, \text{ and } g(s) = 0 \text{ if } s < 0. \]

**Figure:** Shot noise process with an exponential kernel.

One of the questions is to compute the number of crossings of a given level value \( \alpha \).
Example 2: Yukawa potential

The «Yukawa potential» is a radial function given by

\[ g(x_1, x_2) = g(r) = \frac{1}{r + 1} e^{-r/\gamma}, \]

where \( \gamma \) is the screening length.

**Figure:** Left: a Poisson point process with an intensity \( \lambda(x_1, x_2) \) linearly varying with \( x_1 \) (convenient to «see» several \( \lambda \)'s at the same time). Right: shot noise random field with \( \gamma = 5 \).
**Figure:** Left: the shot noise random fields with $\gamma = 5$ and some of its level lines. Right: with $\gamma = 10$.

One of the goal is to understand the geometry of the level lines of these fields (as they are related to the electric field lines).
Example 3: the heat kernel

When the kernel function $g$ is the Gaussian kernel of parameter $\sigma$ given by

$$g_\sigma(r) = \frac{1}{2\pi \sigma^2} e^{-r^2 / 2\sigma^2},$$

then we get

**Figure:** $\sigma = 2$ on the left and $\sigma = 3$ on the right. ($t = \sigma^2$ represents time).
General Properties

In all the following, we consider a shot noise random field defined on $\mathbb{R}^n$ by

$$\forall x \in \mathbb{R}^n, \quad X(x) = \sum_{i \in I} g_{m_i}(x - x_i),$$

where $\{x_i, m_i\}_{i \in I}$ is a marked Poisson point process of intensity $\lambda dx F(dm)$ on $\mathbb{R}^n \times \mathbb{R}^d$.

There are many mathematical studies of this random field: S.O. Rice (1944), Papoulis (1971), Bar David and Nemirovsky (1972), Heinrich and Schmidt (1985), Baccelli and Blaszczyszyn (2001), etc.

- The random field is stationary: its statistics are invariant by translation.
General Properties

In all the following, we consider a shot noise random field defined on \( \mathbb{R}^n \) by

\[
\forall x \in \mathbb{R}^n, \quad X(x) = \sum_{i \in I} g_{m_i}(x - x_i),
\]

where \( \{x_i, m_i\}_{i \in I} \) is a marked Poisson point process of intensity \( \lambda dx F(dm) \) on \( \mathbb{R}^n \times \mathbb{R}^d \).

There are many mathematical studies of this random field: S.O. Rice (1944), Papoulis (1971), Bar David and Nemirovsky (1972), Heinrich and Schmidt (1985), Baccelli and Blaszczyszyn (2001), etc.

- The random field is stationary: its statistics are invariant by translation.
- The expectation (mean value) of \( X \) is given by

\[
\mathbb{E}X(x) = \mathbb{E}X(0) = \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} g_{m}(y) \, dy \, F(dm).
\]
General Properties

In all the following, we consider a shot noise random field defined on \( \mathbb{R}^n \) by

\[
\forall x \in \mathbb{R}^n, \quad X(x) = \sum_{i \in I} g_{m_i}(x - x_i),
\]

where \( \{x_i, m_i\}_{i \in I} \) is a marked Poisson point process of intensity \( \lambda \, dx \, F(dm) \) on \( \mathbb{R}^n \times \mathbb{R}^d \).

There are many mathematical studies of this random field: S.O. Rice (1944), Papoulis (1971), Bar David and Nemirovsky (1972), Heinrich and Schmidt (1985), Baccelli and Blaszczyszyn (2001), etc.

- The random field is stationary: its statistics are invariant by translation.
- The expectation (mean value) of \( X \) is given by

\[
\mathbb{E} X(x) = \mathbb{E} X(0) = \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} g_m(y) \, dy \, F(dm).
\]

- Assume \( N \) is random variable following a Poisson distribution of parameter \( \lambda_0 \), then the characteristic function of \( N \) is defined for all \( u \in \mathbb{R} \) by:

\[
\phi_N(u) := \mathbb{E}(e^{iuN}) = \sum_{k=0}^{+\infty} e^{iuk} e^{-\lambda_0} \frac{\lambda_0^k}{k!} = \exp(\lambda_0(e^{iu} - 1)).
\]
The characteristic function of $X(0)$ (or any $X(x)$) is given by:

\[
\forall u \in \mathbb{R}, \quad \psi(u) := \mathbb{E}(e^{iuX(0)}) = \exp \left( \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} [e^{iugm(y)} - 1] \, dy \, F(dm) \right).
\]
The characteristic function of $X(0)$ (or any $X(x)$) is given by:

$$
\forall u \in \mathbb{R}, \quad \psi(u) := \mathbb{E}(e^{iuX(0)}) = \exp \left( \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} [e^{iug_m(y)} - 1] \, dy \, F(dm) \right).
$$

If moreover $\int \|g_m\|_2^2 \, F(dm) < +\infty$, then $X$ has second-order moments and $\forall x \in \mathbb{R}^n$,

$$
\text{Cov}(X(0), X(x)) = \mathbb{E}(X(0)X(x)) - \mathbb{E}(X(0))\mathbb{E}(X(x)) = \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} g_m(y)g_m(y-x) \, dy \, F(dm).
$$

In particular

$$
\text{Var}(X(0)) = \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} g_m(y)^2 \, dy \, F(dm).
$$
The characteristic function of $X(0)$ (or any $X(x)$) is given by:

$$\forall u \in \mathbb{R}, \quad \psi(u) := \mathbb{E}(e^{iuX(0)}) = \exp \left( \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} [e^{iug_m(y)} - 1] \, dy \, F(dm) \right).$$

If moreover $\int \|g_m\|^2_2 F(dm) < +\infty$, then $X$ has second-order moments and for all $x \in \mathbb{R}^n$,

$$\text{Cov}(X(0), X(x)) = \mathbb{E}(X(0)X(x)) - \mathbb{E}(X(0))\mathbb{E}(X(x)) = \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} g_m(y)g_m(y-x) \, dy \, F(dm).$$

In particular

$$\text{Var}(X(0)) = \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} g_m(y)^2 \, dy \, F(dm).$$

When the intensity $\lambda$ goes to $+\infty$, the normalized random field

$$Z(x) = \frac{X(x) - \lambda \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} g_m(y) \, dy \, F(dm)}{\sqrt{\lambda}}$$

converges (f.d.d. sense) to a stationary centered Gaussian field with covariance

$$\Gamma(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} g_m(y)g_m(y-x) \, dy \, F(dm).$$
Geometry of excursion sets

**Figure:** Left: a sample of a shot noise random field with indicator functions of random disks. Right: excursion set for some level $t = 2, 4$ and $7$ respectively.

We would like to analyze some mean geometric features of the excursion sets: perimeter, Euler characteristic, etc.
Geometry of excursion sets

**Figure:** Left: a sample of a shot noise random field with indicator functions of random disks. Right: excursion set for some level $t = 2, 4$ and $7$ respectively.

We would like to analyze some mean geometric features of the excursion sets: perimeter, Euler characteristic, etc.
Geometry of excursion sets

Figure: Left: a sample of a shot noise random field with indicator functions of random disks. Right: excursion set for some level $t = 2, 4$ and $7$ respectively.

We would like to analyze some mean geometric features of the excursion sets: perimeter, Euler characteristic, etc.
Perimeter of excursion sets

Lots of results for Gaussian random fields (Adler and Taylor, Azaïs and Wschebor, etc.). But the usual techniques do not apply here: shot noise random fields are not necessarily smooth, they don’t always admit a probability density, etc.

→ work in a « weak way »
Perimeter of excursion sets

Lots of results for Gaussian random fields (Adler and Taylor, Azaïs and Wschebor, etc.). But the usual techniques do not apply here: shot noise random fields are not necessarily smooth, they don’t always admit a probability density, etc.

→ work in a « weak way »

**General deterministic framework**

Let $U$ be an open subset of $\mathbb{R}^n$. A function $f \in L^1(U)$ is a *function of bounded variation in* $U$ if its differential in the sense of distributions is a finite Radon measure in $U$, and we then define the *total variation of* $f$ in $U$ as

$$V(f, U) := \|Df\|(U).$$
Perimeter of excursion sets

Lots of results for Gaussian random fields (Adler and Taylor, Azaïs and Wschebor, etc.). But the usual techniques do not apply here: shot noise random fields are not necessarily smooth, they don’t always admit a probability density, etc.

→ work in a « weak way »

**General deterministic framework**

Let $U$ be an open subset of $\mathbb{R}^n$. A function $f \in L^1(U)$ is a *function of bounded variation in $U$* if its differential in the sense of distributions is a finite Radon measure in $U$, and we then define the *total variation of $f$ in $U$* as

$$V(f, U) := \|Df\|(U).$$

We define the *excursion set* or *level set* of level $t$ of $f$ in $U$ by

$$E_f(t, U) := \{x \in U \text{ such that } f(x) > t\}.$$
The coarea formula

- The perimeter $L_f(t, U)$ of the excursion set $E_f(t, U)$ in $U$ is defined by
  \[ L_f(t, U) := V(\chi_{E_f(t, U)}, U), \]
  where $\chi_E$ is the indicator function of a set $E$. 
The coarea formula

- The *perimeter* $L_f(t, U)$ of the excursion set $E_f(t, U)$ in $U$ is defined by
  \[ L_f(t, U) := V(\chi_{E_f(t, U)}, U), \]
  where $\chi_E$ is the indicator function of a set $E$.

- When the boundary of $E_f(t, U)$ is piecewise $C^1$ we find again the usual notion of perimeter, that is
  \[ L_f(t, U) = \mathcal{H}^{n-1}(\partial E_f(t, U) \cap U). \]
  When $n = 1$, it is the number of crossings and when $n = 2$, it is the length of the level lines.
The coarea formula

- The perimeter $L_f(t, U)$ of the excursion set $E_f(t, U)$ in $U$ is defined by

$$L_f(t, U) := V(\chi_{E_f(t, U)}, U),$$

where $\chi_E$ is the indicator function of a set $E$.

- When the boundary of $E_f(t, U)$ is piecewise $C^1$ we find again the usual notion of perimeter, that is

$$L_f(t, U) = \mathcal{H}^{n-1}(\partial E_f(t, U) \cap U).$$

When $n = 1$, it is the number of crossings and when $n = 2$, it is the length of the level lines.

- The link between the total variation and the perimeters is given by the coarea formula:

$$\|Df\|(U) = \int_{\mathbb{R}} L_f(t, U) \, dt.$$
Generalized coarea formula

In all the following we assume that $f \in SBV(U)$ (special function of bounded variation), which means that its differential in the sense of distributions is of the form

$$Df = \nabla f \mathcal{L}^n + (f^+ - f^-) \nu_f \mathcal{H}^{n-1} \angle J_f.$$ 

The coarea formula then becomes

$$\int_{\mathbb{R}} L_f(t, U) dt = \int_U \|\nabla f(x)\| dx + \int_{J_f \cap U} (f^+(y) - f^-(y)) \mathcal{H}^{n-1}(dy).$$
Generalized coarea formula

In all the following we assume that \( f \in SBV(U) \) (special function of bounded variation), which means that its differential in the sense of distributions is of the form

\[
Df = \nabla f \mathcal{L}^n + (f^+ - f^-) \nu_f \mathcal{H}^{n-1} \angle J_f.
\]

The coarea formula then becomes

\[
\int_\mathbb{R} L_f(t, U) dt = \int_U \|\nabla f(x)\| \, dx + \int_{J_f \cap U} (f^+(y) - f^-(y)) \, \mathcal{H}^{n-1}(dy).
\]

More generally, for any continuous bounded function \( h \) on \( \mathbb{R} \), we have:

\[
\int_\mathbb{R} h(t)L_f(t, U) dt = \int_U h(f(x)) \|\nabla f(x)\| \, dx + \int_{J_f \cap U} \left( \int_{f^-(y)}^{f^+(y)} h(s) \, ds \right) \mathcal{H}^{n-1}(dy).
\]

We are going to use this formula with functions \( h \) of the form

\[
\forall t \in \mathbb{R}, \quad h(t) = e^{iut}.
\]
Back to the shot noise random fields

We assume that the kernel functions $g_m$ satisfy: for $F$-almost every $m \in \mathbb{R}^d$ the function $g_m$ is in $SBV(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^d} \|g_m\|_{BV(\mathbb{R}^n)} F(dm) < +\infty.$$ 

Let $U$ be an open bounded subset of $\mathbb{R}^n$. Then, one can prove that $X$ is, almost surely, in $SBV(U)$, that its jump set is $J_X = \bigcup_{i \in I} \left(x_i + J_{g_{m_i}}\right)$, and that its gradient is

$$\nabla X(x) = \sum_{i \in I} \nabla g_{m_i}(x - x_i).$$
We assume that the kernel functions $g_m$ satisfy: for $F$-almost every $m \in \mathbb{R}^d$ the function $g_m$ is in $SBV(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^d} \|g_m\|_{BV(\mathbb{R}^n)} F(dm) < +\infty.$$  

Let $U$ be an open bounded subset of $\mathbb{R}^n$. Then, one can prove that $X$ is, almost surely, in $SBV(U)$, that its jump set is $J_X = \bigcup_{i \in I} (x_i + J_{g_{m_i}})$, and that its gradient is

$$\nabla X(x) = \sum_{i \in I} \nabla g_{m_i}(x - x_i).$$  

For $t \in \mathbb{R}$, let us denote the mean crossing function

$$C_X(t) := \mathbb{E}(L_X(t, (0, 1)^n)).$$  

Then the function $t \mapsto C_X(t)$ belongs to $L^1(\mathbb{R})$ and its Fourier transform is given for all $u \in \mathbb{R}, u \neq 0$ by

$$\widehat{C_X}(u) = \mathbb{E}(e^{iuX(0)} \|\nabla X(0)\|) + \mathbb{E}(e^{iuX(0)}) \frac{\lambda}{iu} \int_{\mathbb{R}^d} \int_{J_{gm}} (e^{iug_m^+(y)} - e^{iug_m^-(y)}) \mathcal{H}^{n-1}(dy) F(dm).$$
Example 1: disks with random radius

When $n = 2$ and $g_r$ is the indicator function of the disk of random radius $r$ with distribution $F(dr)$, then one can compute

$$\widehat{C}_X(u) = \lambda \exp(\lambda \pi \overline{r}^2 (e^{iu} - 1)) \cdot 2\pi \overline{r} \frac{e^{iu} - 1}{iu},$$

where $\overline{r} = \int_{\mathbb{R}_+} r F(dr)$, $\overline{r}^2 = \int_{\mathbb{R}_+} r^2 F(dr)$. And thus,

$$\forall k \in \mathbb{N}, \quad C_X(\alpha) = 2\pi \overline{r} e^{-\lambda \pi \overline{r}^2} \frac{(\lambda \pi \overline{r}^2)^k}{k!} \quad \text{for all } \alpha \in (k, k + 1).$$
Example 2: exponential kernel

The kernel functions are here \( g_\beta(s) = \beta e^{-s} \mathbb{I}_{s \geq 0} \), where \( \beta \) follows an exponential distribution of parameter \( \mu \), that is \( F(d\beta) = \mu e^{-\mu \beta} \mathbb{I}_{\beta > 0} d\beta \).

One can compute

\[
\hat{C}_X(u) = \frac{2\lambda \mu^\lambda}{(\mu - iu)^{\lambda+1}}.
\]

And then get:

\[
C_X(\alpha) = \frac{2\lambda \mu^\lambda \alpha^\lambda e^{-\mu \alpha}}{\Gamma(\lambda + 1)} \mathbb{I}_{\{\alpha \geq 0\}} \text{ for all } \alpha \in \mathbb{R}.
\]
We consider here the case of shot noise random fields $X$ in **dimension 2**.

Let us denote $\psi_X(t)$ the Euler Characteristic (EC) of the excursion set $E_X(t, U) := \{x \in U \text{ such that } f(x) > t\}$ for $t \in \mathbb{R}$:

$$\psi_X(t) = \chi(E_X(t, U)),$$

where $\chi(A)$ denotes the Euler Characteristic of a set $A$. 
Euler Characteristic of excursion sets

We consider here the case of shot noise random fields $X$ in **dimension 2**.

Let us denote $\psi_X(t)$ the Euler Characteristic (EC) of the excursion set $E_X(t, U) := \{ x \in U \text{ such that } f(x) > t \}$ for $t \in \mathbb{R}$:

$$\psi_X(t) = \chi(E_X(t, U)),$$

where $\chi(A)$ denotes the Euler Characteristic of a set $A$.

Despite its global definition (\# connected components – \# holes), the EC is in fact a *local feature* that can be computed thanks to the total curvature.
Theorem (Gauss-Bonnet Theorem)

Let \( R \subset U \) be a regular region such that its boundary \( \partial R \) is formed by \( n \) closed, simple and piecewise regular curves \( C_1, \ldots, C_n \). Suppose that each \( C_i \) is positively oriented and let \( \beta_1, \ldots, \beta_p \) be the set of all turning angles of the curves \( C_1, \ldots, C_n \). Then

\[
TC(\partial R) := \sum_{i=1}^{n} \int_{C_i} \kappa(s) \, ds + \sum_{j=1}^{p} \beta_j = 2\pi \chi(R),
\]

where \( s \) denotes the arc length of \( C_i \), the integral over \( C_i \) means the sum of integrals in every regular arc of \( C_i \) and \( \chi(R) \) is the Euler characteristic of \( R \).
Deterministic framework:
Let $h$ be a bounded continuous function on $\mathbb{R}$, and let $H$ be a primitive of $h$ (for instance $H(t) = \int_0^t h(u) \, du$). Then we have:

- If $f$ is a piecewise constant function taking two values $f^+ > f^-$ and with a discontinuity curve $J_f$, that is a simple curve of finite total curvature, then

$$2\pi \int_{\mathbb{R}} h(t) \psi_f(t) \, dt = \int_{f^-}^{f^+} h(t) \text{TC}(J_f) \, dt = [H(f^+) - H(f^-)] \times \text{TC}(J_f).$$
• If $f$ and $g$ are two piecewise constant functions taking respectively two values $f^+ > f^-$ and $g^+ > g^-$, and with respective discontinuity curves $J_f$ and $J_g$, intersecting in generic position (i.e., not at a corner point, and not tangentially), then

\[
2\pi \int_R h(t)\psi_{f+g}(t)\,dt = \sum_{p \in J_f \cap J_g} |\theta_f(p) - \theta_g(p)| \sum_k \left[ H(f^+ + g^+) + H(f^- + g^-) - H(f^- + g^+) - H(f^+ + g^-) \right] \sum_{j=1}^k \left[ H(f^+ + g(\gamma_f(s_j))) - H(f^- + g(\gamma_f(s_j))) \right] \alpha_{J_f}^j
\]

\[
+ \int_{J_f} \left[ H(f^+ + g(\gamma_f(s))) - H(f^- + g(\gamma_f(s))) \right] \kappa_f(s) \,ds + \sum_{j=1}^k \left[ H(f^+ + g(\gamma_f(s_j))) - H(f^- + g(\gamma_f(s_j))) \right] \alpha_{J_f}^j
\]

\[
+ \int_{J_g} \left[ H(g^+ + f(\gamma_g(s))) - H(g^- + f(\gamma_g(s))) \right] \kappa_g(s) \,ds + \sum_{j=1}^k \left[ H(f^+ + g(\gamma_f(s_j))) - H(f^- + g(\gamma_f(s_j))) \right] \alpha_{J_g}^j
\]

where $s$ is a parametrization of $J_f$ (resp. $J_g$) by arc length, $\kappa_f(s)$ (resp. $\kappa_g(s)$) is the signed curvature at $s$, and $\alpha_{J_f}^j, \ldots, \alpha_{J_f}^k$ are the (possible) turning angles of $J_f$ (resp. $J_g$); and $\theta_f(p)$ (resp. $\theta_g(p)$) are the angles of the tangent to $J_f$ (resp. $J_g$) at a point $p$. 
Theorem

Let $X$ be a simple shot noise random field, meaning that the functions $g_m$ are piecewise constant. Then, the Fourier transform of the mean EC of the excursion sets of $X$ is given, for all $u \in \mathbb{R}$ by

$$2\pi \int_{\mathbb{R}} e^{iut} \mathbb{E}(\psi_X(t)) \, dt = \lambda \mathbb{E}(e^{iuX(0)}) \int_{\mathbb{R}^d} \int_{J_m} \kappa_m(z) \frac{e^{iug_m^+(z)} - e^{iug_m^-(z)}}{iu} \mathcal{H}^1(dz) F(dm)$$

$$+ \lambda^2 \mathbb{E}(e^{iuX(0)}) \times \frac{1}{2} \int_{\mathbb{R}^d} \int \sum_{z \in \tau_x J_m \cap \tau_y J_{m'} \cap U} |\theta_m(z+x)-\theta_{m'}(z+y)| \Delta_{m,m'}(u, z; x, y) \, dx \, dy F(dm) F(dm')$$

where

$$\Delta_{m,m'}(u, z; x, y) :=$$

$$\frac{e^{iu(g_m^+(z+x)+g_m^-(z+y))} + e^{iu(g_m^+(z+x)+g_{m'}^-(z+y))} - e^{iu(g_m^-(z+x)+g_{m'}^-(z+y))} - e^{iu(g_m^-(z+x)+g_{m'}^+(z+y))}}{iu}$$
Example : mean EC for random disks

If $g_m$ and $g_m'$ are the indicator functions of a disk of radius $r$ (resp. $r'$), then one can compute the following *kinematic formula*

$$\int \sum_{z \in \tau_x J_m \cap \tau_y J_m' \cap U} |\theta_m(z + x) - \theta_m'(z + y)| \, dx \, dy = |U| \times 2\pi r \times 2\pi r'.$$

We again denote $\bar{r} = \int_{\mathbb{R}^+} r \, F(dr)$ and $\bar{r}^2 = \int_{\mathbb{R}^+} r^2 \, F(dr)$, then

$$\mathbb{E} \left( \int_R e^{iut} \psi X(t) \, dt \right) = \lambda |U| \mathbb{E}(e^{iux(0)}) \frac{e^{iu} - 1}{iu} \left( 1 + \pi \bar{r}^2 \lambda (e^{iu} - 1) \right).$$

One can invert this Fourier transform, using the fact that $\mathbb{E}(e^{iux(0)})$ is the Fourier transform of a Poisson distribution of parameter $\lambda \pi \bar{r}^2$, and obtain :

$$\forall k \in \mathbb{N}, \text{ and a.e. } t \in [k, k + 1), \quad \mathbb{E}(\psi_X(t)) = \lambda |U| e^{-\lambda \pi \bar{r}^2} \frac{(\lambda \pi \bar{r}^2)^k}{k!} (k + 1 - \pi \bar{r}^2 \lambda).$$
Example: mean EC for random disks

**Figure:** Left: a sample of a shot noise random field with indicator functions of random disks. Right: excursion set for some level $t = 2, 4$ and $7$ respectively.
Example: mean EC for random disks

**Figure:** Left: a sample of a shot noise random field with indicator functions of random disks. Right: excursion set for some level $t = 2, 4$ and $7$ respectively.
Example: mean EC for random disks

**Figure:** Left: a sample of a shot noise random field with indicator functions of random disks. Right: excursion set for some level $t = 2, 4$ and $7$ respectively.
Example: mean EC for random disks

**Figure:** Left: a sample of a shot noise random field with indicator functions of random disks. Right: excursion set for some level $t = 2, 4$ and $7$ respectively.
What is the Texture Synthesis problem?

You have a sample image of some texture (grass, wall, wood, sky, etc) and you want to create an arbitrary large image of the same texture.

One of the difficulties is that texture images are not mathematically well-defined. They all contain the idea of a more or less randomly repeated «pattern».

→ Shot noise random fields are a way to model some texture images.
Examples of Shot noise texture images

**Figure**: Left: the kernel function. Middle: shot noise with $n = 100$. Right: with $n = 5000$.

Here, the problem of texture synthesis becomes an **inverse problem**: from a sample of a texture image, recover the kernel function $g$. 
The discrete framework

The images are defined on a finite discrete domain $\Omega = I_N \times I_N$ where $N$ is an odd integer and $I_N = \left[-\frac{N-1}{2}, \ldots, -1, 0, 1, \ldots, \frac{N-1}{2}\right]$.

Let $g$ be a kernel function defined on $\Omega$, and we assume that the mean value of $g$ on $\Omega$ is 0. Then consider the shot noise random field $X$ on $\Omega$, defined by

$$
\forall x \in \Omega, \quad X(x) = \sum_{k=1}^{n} g(x - x_k),
$$

where the $x_k$ are independent uniformly distributed on $\Omega$ (the differences $x - x_k$ are taken modulo $\Omega$) and $n$ plays the role of the intensity.
The discrete framework

The images are defined on a finite discrete domain $\Omega = I_N \times I_N$ where $N$ is an odd integer and $I_N = [-\frac{N-1}{2}, \ldots, -1, 0, 1, \ldots, \frac{N-1}{2}]$.

Let $g$ be a kernel function defined on $\Omega$, and we assume that the mean value of $g$ on $\Omega$ is 0. Then consider the shot noise random field $X$ on $\Omega$, defined by

$$\forall x \in \Omega, \quad X(x) = \sum_{k=1}^{n} g(x - x_k),$$

where the $x_k$ are independent uniformly distributed on $\Omega$ (the differences $x - x_k$ are taken modulo $\Omega$) and $n$ plays the role of the intensity.

As $n \to +\infty$, the normalized field $X/\sqrt{n}$ converges to a periodic Gaussian centered stationary random field $U$ with covariance

$$\Gamma(x) := \text{Cov}(U(0), U(x)) = \frac{1}{N^2} \sum_{y \in \Omega} g(y - x)g(y),$$

that is the autocorrelation of the kernel function $g$. 
The **Discrete Fourier Transform** (DFT) of an image $u : \Omega \mapsto \mathbb{R}$ is given by

$$\forall \xi \in \Omega, \ \hat{u}(\xi) = \sum_{x=(x_1,x_2)\in\Omega} u(x) e^{-\frac{2i\pi}{N} (x_1\xi_1 + x_2\xi_2)} = |\hat{u}(\xi)| e^{i\Phi(\xi)}.$$

Considering a shot noise with kernel $g$ and its Gaussian limit $U$, using the DFT we obtain that:

- The random variables $\hat{U}(\xi)$, $\xi \in \Omega_+$ are **independent** complex Gaussian random variables with mean 0 and variance $|\hat{g}(\xi)|^2$.

- In an equivalent way it means that: the phases $\Phi(\xi)$ are independent uniformly distributed on $[0, 2\pi)$ and the amplitudes $|\hat{U}(\xi)|$ are independent following a Rayleigh law of parameter $|\hat{g}(\xi)|$.

**Conclusion 1**: From $U$ we cannot hope to recover $g$, we can only recover $|\hat{g}(\xi)|$.

**Conclusion 2**: If we randomize the phases of $U$, we obtain a new image that has the same law as $U$. 
The **Discrete Fourier Transform** (DFT) of an image \( u : \Omega \mapsto \mathbb{R} \) is given by

\[
\forall \xi \in \Omega, \quad \hat{u}(\xi) = \sum_{x=(x_1,x_2) \in \Omega} u(x) e^{-\frac{2i\pi}{N}(x_1 \xi_1 + x_2 \xi_2)} = |\hat{u}(\xi)| e^{i\Phi(\xi)}.
\]

Considering a shot noise with kernel \( g \) and its Gaussian limit \( U \), using the DFT we obtain that:

- The random variables \( \hat{U}(\xi), \xi \in \Omega_+ \) are **independent** complex Gaussian random variables with mean 0 and variance \( |\hat{g}(\xi)|^2 \).
- In an equivalent way it means that: the phases \( \Phi(\xi) \) are independent uniformly distributed on \([0, 2\pi)\) and the amplitudes \( |\hat{U}(\xi)| \) are independent following a Rayleigh law of parameter \( |\hat{g}(\xi)| \).

**Conclusion 1**: From \( U \) we cannot hope to recover \( g \), we can only recover \( |\hat{g}| \).
Discrete Fourier Transform

The Discrete Fourier Transform (DFT) of an image $u : \Omega \rightarrow \mathbb{R}$ is given by

$$\forall \xi \in \Omega, \quad \hat{u}(\xi) = \sum_{x=(x_1,x_2) \in \Omega} u(x) e^{-\frac{2i\pi}{N}(x_1 \xi_1 + x_2 \xi_2)} = |\hat{u}(\xi)| e^{i\Phi(\xi)}.$$  

Considering a shot noise with kernel $g$ and its Gaussian limit $U$, using the DFT we obtain that:

- The random variables $\hat{U}(\xi), \xi \in \Omega_+$ are independent complex Gaussian random variables with mean 0 and variance $|\hat{g}(\xi)|^2$.

- In an equivalent way it means that: the phases $\Phi(\xi)$ are independent uniformly distributed on $[0, 2\pi)$ and the amplitudes $|\hat{U}(\xi)|$ are independent following a Rayleigh law of parameter $|\hat{g}(\xi)|$.

**Conclusion 1**: From $U$ we cannot hope to recover $g$, we can only recover $|\hat{g}|$.

**Conclusion 2**: If we « randomize » the phases of $U$, we obtain a new image that has the same law as $U$. 
The Random Phase Algorithm

The Random Phase Algorithm has been introduced by B. Galerne, Y. Gousseau and J.-M. Morel in 2011 for texture synthesis. It is very simple:

1. Start from an original texture image \( u \).
2. Compute its discrete Fourier transform \( \hat{u} \).
3. Take independent random phases \( \phi(\xi) \) and define \( \hat{v}(\xi) = \hat{u}(\xi)e^{i\phi(\xi)} \).
4. Obtain a new image \( v \) by taking the inverse Fourier transform of \( \hat{v} \).

\[ \rightarrow \text{This algorithm works } \ll \text{very well} \rr \text{ on } \ll \text{micro-textures} \rr . \]

\[ \text{FIGURE: Left : original image. Right : after phase randomization.} \]
Other examples

**Figure:** Top: original image. Bottom: after phase randomization.
Extension to color images

A color image $\mathbf{u}$ has three channels (Red, Green, Blue):

$$\mathbf{u} = (u_r, u_g, u_b).$$

To keep the correlation between the channels, the Random Phase algorithm for color images adds the same random phase to each channel.

**Figure**: Left: original image. Right: the same random phase is added to the phase of each channel.
Otherwise:

**Figure:** Left: original image. Right: independent phase randomization of the three color channels.
Another example of result of the Random Phase Algorithm:

**Figure:** Left: original image. Right: the same random phase is added to the phase of each channel.
Failures

This algorithm « doesn’t work » (in the sense that the result is perceptually different from the original image) on all *the macro-textures images*.

**Figure:** Left : original image. Right : after applying the Phase Randomization Algorithm.
**Figure:** Left: original image. Right: after applying the Phase Randomization Algorithm.

**Rk:** There is a huge literature on texture synthesis methods: copy-paste methods (ex: Efros and Leung), or higher-order statistical methods (ex: Zhu, Wu and Mumford; Portilla and Simoncelli). These methods work better for macro-textures but are mathematically less tractable.
On the importance of the phase

It is well-know that phases are very informative for structured images. One can check this by a very simple experiment: take two images $u_1$ and $u_2$ (of same size), compute their Fourier transform $\hat{u}_1 = R_1 e^{i\phi_1}$ and $\hat{u}_2 = R_2 e^{i\phi_2}$, and create two new images $I_1$ and $I_2$ by

$$\hat{I}_1 = R_1 e^{i\phi_2} \text{ and } \hat{I}_2 = R_2 e^{i\phi_1}.$$  

What do $I_1$ and $I_2$ look like?

\[\text{FIGURE: Left : image } u_1. \text{ Right : image } u_2.\]
On the importance of the phase

It is well-know that phases are very informative for structured images.

One can check this by a very simple experiment: take two images $u_1$ and $u_2$ (of same size), compute their Fourier transform $\hat{u}_1 = R_1 e^{i\phi_1}$ and $\hat{u}_2 = R_2 e^{i\phi_2}$, and create two new images $I_1$ and $I_2$ by

$$\hat{I}_1 = R_1 e^{i\phi_2} \text{ and } \hat{I}_2 = R_2 e^{i\phi_1}.$$ 

What do $I_1$ and $I_2$ look like?

**Figure:** Left: image $I_1$. Right: image $I_2$. 
Estimation of the « texton »

Back to micro-textures: we assume that they come from the phase randomization of a kernel image, and we would like to recover this kernel. Let \( u \) be a micro-texture image defined on \( \Omega \). Let us denote

\[
\Theta_u = \{ v \text{ such that } \forall \xi \in \Omega, |\hat{v}(\xi)| = |\hat{u}(\xi)| \}.
\]

**Question**: among all images \( v \) in \( \Theta_u \), which is « the most concentrated » one?

Variational formulation: Among the images \( v \in \Theta_u \), which one minimizes

\[
\|\nabla \hat{v}\|_p = \sum_{\xi = (\xi_1, \xi_2) \in \Omega} (|\hat{v}(\xi_1 + 1, \xi_2) - \hat{v}(\xi_1, \xi_2)|^p + |\hat{v}(\xi_1, \xi_2 + 1) - \hat{v}(\xi_1, \xi_2)|^p)^{1/p}.
\]

(Fast decrease in space \( \leftrightarrow \) regularity in the Fourier domain)

Simple answer: take \( v \) such that all phases of \( \hat{v} \) are equal to 0. The image thus obtained is denoted by \( T(u) \) and is called the texton of the image \( u \).

**Rk**: The notion of texton was introduced by Julesz in 1981 to describe « the putative units of pre-attentive human texture perception ».
Back to micro-textures: we assume that they come from the phase randomization of a kernel image, and we would like to recover this kernel. Let $u$ be a micro-texture image defined on $\Omega$. Let us denote

$$\Theta_u = \{ v \text{ such that } \forall \xi \in \Omega, |\hat{v}(\xi)| = |\hat{u}(\xi)| \}.$$ 

**Question**: among all images $v$ in $\Theta_u$, which is « the most concentrated » one?

**Variational formulation**:

Among the images $v \in \Theta_u$, which one minimizes

$$\| \nabla \hat{v} \|^p_p = \sum_{\xi=(\xi_1, \xi_2)\in\Omega} (|\hat{v}(\xi_1 + 1, \xi_2) - \hat{v}(\xi_1, \xi_2)|^p + |\hat{v}(\xi_1, \xi_2 + 1) - \hat{v}(\xi_1, \xi_2)|^p)?$$

(Fast decrease in space $\leftrightarrow$ regularity in the Fourier domain)
Estimation of the « texton »

Back to micro-textures: we assume that they come from the phase randomization of a kernel image, and we would like to recover this kernel. Let $u$ be a micro-texture image defined on $\Omega$. Let us denote

$$\Theta_u = \{v \text{ such that } \forall \xi \in \Omega, |\hat{v}(\xi)| = |\hat{u}(\xi)|\}.$$ 

**Question**: among all images $v$ in $\Theta_u$, which is « the most concentrated » one?

**Variational formulation**:
Among the images $v \in \Theta_u$, which one minimizes

$$\|\nabla \hat{v}\|_p = \sum_{\xi = (\xi_1, \xi_2) \in \Omega} (|\hat{v}(\xi_1 + 1, \xi_2) - \hat{v}(\xi_1, \xi_2)|^p + |\hat{v}(\xi_1, \xi_2 + 1) - \hat{v}(\xi_1, \xi_2)|^p)?$$

(Fast decrease in space $\leftrightarrow$ regularity in the Fourier domain)

**Simple answer**: take $v$ such that all phases of $\hat{v}$ are equal to 0.

The image thus obtained is denoted by $T(u)$ and is called the **texton** of the image $u$.

**Rk**: The notion of texton was introduced by Julesz in 1981 to describe « the putative units of pre-attentive human texture perception ». 
The texton $T(u)$ is characterized in the Fourier domain by:

\[ \forall \xi \in \Omega, \quad \hat{T}(u)(\xi) = |\hat{u}(\xi)| \]

and, in an equivalent way, in the spatial domain by:

\[ \forall x \in \Omega, \quad T(u)(x) = \frac{1}{N^2} \sum_{\xi \in \Omega} |\hat{u}(\xi)| e^{\frac{2i\pi}{N} \langle x, \xi \rangle}. \]

- The texton $T(u)$ also minimizes, under the constraint $|\hat{v}| = |\hat{u}|$, the following functional (that is a kind of measure of spatial concentration in 0):

\[ E(v) = \sum_{(x_1, x_2) \in \Omega} \left( \sin\left(\frac{\pi}{N} x_1 \right)^2 + \sin\left(\frac{\pi}{N} x_2 \right)^2 \right) v(x_1, x_2)^2. \]

- $T(u)$ is also the unique solution of the following optimization problem:

Find $v : \Omega \to \mathbb{R}$ that maximizes $v(0)$ under the constraint $|\hat{v}| = |\hat{u}|$. 
Examples: synthetic textures

**Figure**: Top: synthetic texture images obtained by randomizing the phases of a kernel image (resp. a square, two disks and a “snake”). Bottom: the textons.
Examples: real textures
Les modèles shot noise se retrouvent dans de nombreux domaines.

Par la formule de la coaire, on a accès à la fonction périmètre $t \mapsto L_f(t, U)$. Ce qui permet de calculer explicitement $t \mapsto \mathbb{E}(L_X(t, U))$ dans le cas de $X$ champ de type shot noise.

Par le théorème de Gauss-Bonnet, on peut aussi calculer $t \mapsto \mathbb{E}(\psi_X(t, U))$ la caractéristique d’Euler moyenne des ensembles d’excursion.

Mais calcul des moments d’ordre supérieur des périmètres et des EC ? Avoir des TCL ?

Sur les images de texture, le texton semble, d’après nos premières expériences, un bon outil de synthèse et d’analyse de texture.

Estimation du noyau $g$ et de l’intensité $\lambda$ dans le cadre non-asymptotique ?