Singular limits for reaction diffusion equations with fractional Laplacian

Sepideh Mirrahimi

CNRS, IMT, Toulouse, France

Joint work with Sylvie Méléard

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Fractional reaction-diffusion equations

We are interested in the **long time** behavior of

$$\partial_t n + (-\Delta)^{\frac{\alpha}{2}} n = n\, R(x, [n]).$$

In particular for the **KPP** reaction term

$$R = 1 - n,$$

but our motivation comes from **selection-mutation** models.

We first give some known results on the classical Fisher-KPP equation ($\alpha = 2$):

$$\partial_t n - d\Delta n = r\, n\,(1 - n)$$

Fisher-KPP equation (1937):

$$\partial_t n - d\Delta n = r\, n\,(1-n)$$

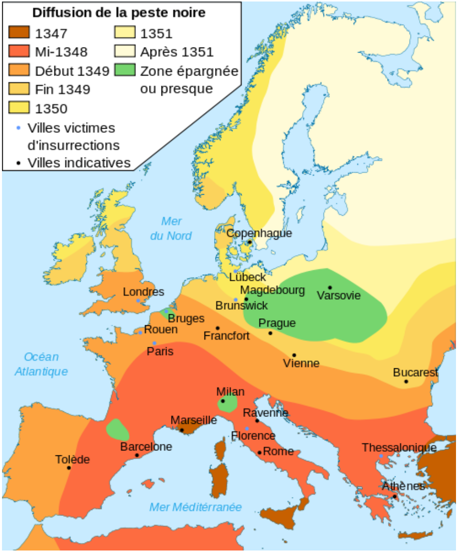
Diffusion operator : dispersion of individuals

Logistic growth : reproduction and saturation

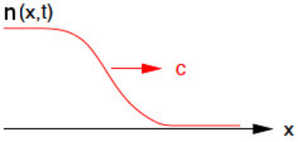
- Growth rate : $r > 0$
- Diffusion rate : $d > 0$

Examples of application: combustion waves, phase transitions, progress of an invasive species in a virgin environment, spread of a disease

Example: Propagation of the front of plague through Europe in the middle of 14th century.



Traveling front solutions: $n(x,t) = u(x - ct)$.



a **front** that travels with **constant speed** : c.

The minimal speed :$c_* = 2\sqrt{dr}$.

Starting with a **compactly supported initial data** : (Aronson and Weinberger 1978)

$$\begin{cases} n(x,t) \to 0 \quad \text{in } \{|x| \geq ct\} \text{ as } t \to +\infty, \quad \text{if } c > c_* \\ n(x,t) \to 1 \quad \text{in } \{|x| \leq ct\} \text{ as } t \to +\infty, \quad \text{if } c < c_* \end{cases}$$

Geometric optics approach

Freidlin 1986, Evans and Souganidis 1989:

A **Long range/Long time** asymptotic study:

$$x \to \frac{x}{\varepsilon}, \qquad t \to \frac{t}{\varepsilon}; \qquad n_\varepsilon(x,t) = n(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}).$$

The equation becomes

$$\varepsilon \partial_t n_\varepsilon - \varepsilon^2 d \Delta n_\varepsilon = r\, n_\varepsilon\, (1 - n_\varepsilon)$$

The behavior as $\varepsilon \to 0$?

The **fundamental solution** of the linearized equation :

$$K_\varepsilon(t,x) = \frac{1}{\sqrt{4\pi\varepsilon d t}} \exp\left(\frac{rt}{\varepsilon} - \frac{x^2}{4\varepsilon d\, t}\right).$$

$\Rightarrow$ **Hopf-Cole** transformation:

$$n_\varepsilon = \exp\left(\frac{u_\varepsilon}{\varepsilon}\right).$$

Replacing this in the equation on n_ε we obtain

$$\partial_t u_\varepsilon - \varepsilon d \Delta u_\varepsilon = d\,|\nabla u_\varepsilon|^2 + r\,(1 - n_\varepsilon).$$

Let's suppose that, as $\varepsilon \to 0$,

$$u_\varepsilon \to u, \qquad n_\varepsilon \rightharpoonup n.$$

Since n is bounded we obtain

$$u \leq 0.$$

Moreover

$$\text{if} \quad u(t,x) < 0, \qquad \text{then} \quad (t,x) \notin \operatorname{supp} n,$$

Convergence to a Hamilton-Jacobi equation with obstacle

Combining the above arguments we obtain that, as $\varepsilon \to 0$, u_ε converges to u which solves

$$\begin{cases} \max\left(\partial_t u - d|\nabla u|^2 - r, u\right) = 0 \\ u(x,0) = u_0(x). \end{cases}$$

u is indeed a **viscosity solution**.

We also obtain

$$\begin{cases} n_\varepsilon \to 0, & \text{in } \{u(x,t) < 0\}, \\ n_\varepsilon \to 1, & \text{in } \mathrm{Int}\{u(x,t) = 0\}. \end{cases}$$

Speed of propagation

Let's suppose

$$n_0(x) = \rho_0 \delta(x - \overline{x}),$$

$$u_0(x) = \begin{cases} 0 & \text{for } x = \overline{x}, \\ -\infty & \text{for } x \neq \overline{x}. \end{cases}$$

Then, solving the Hamilton-Jacobi equation we obtain

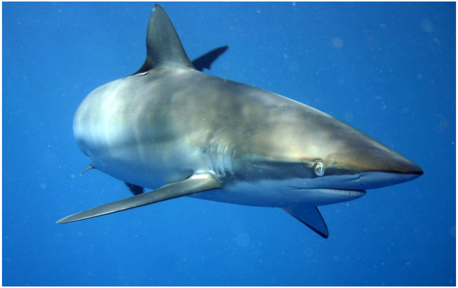
$$u(t,x) = \min\left(-\frac{(x-\overline{x})^2}{4dt} + rt, 0 \right).$$

The **speed of propagation**:

$$c_* = 2\sqrt{rd}.$$

This can be computed indeed as the speed of propagation for any **compactly supported initial data**.

The movement of individuals can have Lévy patterns instead of Brownian patterns



Humphries et al. Environmental context explains Lévy and Brownian movement patterns of marine predators, Nature 2010

How the dynamics are modified when the diffusion has heavy tails ?

Fractional Fisher-KPP :

$$\partial_t n + (-\Delta)^{\frac{\alpha}{2}} n = n(1-n)$$

Is it possible to do a rescaling to describe asymptotically the exponential propagation of the population ?

Fractional Fisher-KPP

Let $x \in \mathbb{R}$ and $0 \leq \alpha \leq 2$. The **fractional Fisher-KPP** equation:

$$\begin{cases} \partial_t n &= -(-\Delta)^{\frac{\alpha}{2}} n + n(1-n) \\ &= \displaystyle\int_0^\infty \left[n(x+h) + n(x-h) - 2n(x)\right] \frac{dh}{|h|^{1+\alpha}} + n(1-n) \\ n(x,0) &= n_0(x). \end{cases}$$

Fast propagation of the population: Berestycki, Cabré, Coulon, Englar, Garnier, Rossi, Roquejoffre....

Exponential propagation of the population

Cabré, Roquejoffre (Comm. Math. Phys. 2013):

For any **initial data** satisfying

$$0 \le n_0(x) \le \frac{C}{1+|x|^{1+\alpha}}$$

we have as $t \to \infty$,

$$\begin{cases} n(x,t) \to 0, & \text{in } \{|x| \ge e^{\sigma t}\}, \text{ if } \sigma > \frac{1}{1+\alpha}, \\ n(x,t) \to 1, & \text{in } \{|x| \le e^{\sigma t}\}, \text{ if } \sigma < \frac{1}{1+\alpha}. \end{cases}$$

More precise result on the level sets, in periodic media: Cabré, Coulon, Roquejoffre (C.R. Acad. Sci. 2012):

$$\{n(x,t) = \lambda\} \subset \left\{ c_\lambda \, e^{\frac{t}{1+\alpha}} \le |x| \le c_\lambda^{-1} \, e^{\frac{t}{1+\alpha}} \right\},$$

for t large enough.

An asymptotic approach

For simple representation, we consider only

$$x \in \mathbb{R}, \qquad n_0(x) = n_0(|x|).$$

A **long-time** rescaling (and **long-range**)

$$t \to \frac{t}{\varepsilon} \qquad \Longrightarrow \qquad |x| \to |x|^{\frac{1}{\varepsilon}}$$

$$n_\varepsilon(x,t) = n(|x|^{\frac{1}{\varepsilon}}, \frac{t}{\varepsilon})$$

$$\begin{cases} \varepsilon \partial_t n_\varepsilon(x,t) = \int_0^\infty \left(n_\varepsilon\left(\left||x|^{\frac{1}{\varepsilon}} + h\right|^\varepsilon, t\right) + n_\varepsilon\left(\left||x|^{\frac{1}{\varepsilon}} - h\right|^\varepsilon, t\right) - 2n_\varepsilon(x,t)\right) \frac{dh}{|h|^{1+\alpha}} \\ \qquad\qquad + n_\varepsilon(x,t)(1 - n_\varepsilon(x,t)) \\ \\ n_\varepsilon(x,0) = n_\varepsilon^0(x) \end{cases}$$

The fundamental solution

The **fundamental solution** p :

$$\begin{cases} \partial_t p + (-\Delta)^{\alpha/2} p = p, & \text{in } \mathbb{R} \times \mathbb{R}^+, \\ p(x,0) = \delta_0, & \text{in } \mathbb{R}, \end{cases}$$

satisfies

$$\frac{B_m\, e^t}{t^{\frac{1}{\alpha}}(1 + |t^{\frac{-1}{\alpha}}|x||^{1+\alpha})} \le p(x,t) \le \frac{B_M\, e^t}{t^{\frac{1}{\alpha}}(1 + |t^{\frac{-1}{\alpha}}|x||^{1+\alpha})},$$

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satisfies

$$\frac{B_m\, e^{\frac{t}{\varepsilon}}}{(\frac{t}{\varepsilon})^{\frac{1}{\alpha}}(1 + |(\frac{t}{\varepsilon})^{\frac{-1}{\alpha}}|x|^{\frac{1}{\varepsilon}}|^{1+\alpha})} \leq p(|x|^{\frac{1}{\varepsilon}}, \frac{t}{\varepsilon}) \leq \frac{B_M\, e^{\frac{t}{\varepsilon}}}{(\frac{t}{\varepsilon})^{\frac{1}{\alpha}}(1 + |(\frac{t}{\varepsilon})^{\frac{-1}{\alpha}}|x|^{\frac{1}{\varepsilon}}|^{1+\alpha})}$$

Being inspired from this we use again the **Hopf-Cole** transformation

$$n_\varepsilon(x,t) = \exp\left(\frac{u_\varepsilon}{\varepsilon}\right).$$

The main result

We assume

$$0 \le n_\varepsilon(x,0) \le \frac{C}{1+|x|^{\frac{1+\alpha}{\varepsilon}}}.$$

Theorem (Méléard, SM, CPDE 2014)

(i) As $\varepsilon \to 0$, $(u_\varepsilon)_\varepsilon$ converges locally uniformly to u defined as below

$$u(x,t) = \min(0, -(1+\alpha)\log|x| + t).$$

(ii) Moreover, as $\varepsilon \to 0$,

$$\begin{cases} n_\varepsilon \to 0, & \textit{locally uniformly in } \mathcal{A} = \{(x,t) \,|\, t < (1+\alpha)\log|x|\}, \\ n_\varepsilon \to 1, & \textit{locally uniformly in } \mathcal{B} = \{(x,t) \,|\, t > \max(0, (1+\alpha)\log|x|)\}. \end{cases}$$

proof – Lower and upper bounds

In view of the form of the fundamental solution we find

$$\frac{C_1}{1+|x|^{1+\alpha}} \leq n(x,1) \leq \frac{C_2}{1+|x|^{1+\alpha}}$$

proof – Lower and upper bounds

In view of the form of the fundamental solution we find

$$\frac{C_1}{1+|x|^{\frac{1+\alpha}{\varepsilon}}} \leq n_\varepsilon(x,\varepsilon) \leq \frac{C_2}{1+|x|^{\frac{1+\alpha}{\varepsilon}}}.$$

From now on, we assume that the initial datum has such decay.

We prove that

$$f_m = \frac{C_m}{1+e^{-\frac{t(1-\varepsilon^2)}{\varepsilon}}|x|^{\frac{1+\alpha}{\varepsilon}}} \leq n_\varepsilon(x,t) \leq \frac{C_M}{1+e^{-\frac{t(1+\varepsilon^2)}{\varepsilon}}|x|^{\frac{1+\alpha}{\varepsilon}}} = f_M$$

Proof – f_m and f_M are sub and supersolutions

f_m and f_M are indeed **sub and super-solutions** of the following ODE:

$$\varepsilon\partial_t f_m \leq f_m(1-\varepsilon^2 - f_m), \qquad \varepsilon\partial_t f_M \geq f_M(1+\varepsilon^2 - f_M)$$

and the term from **the fractional Laplacian is small**:

$$|\Delta_\varepsilon^{\frac{\alpha}{2}} f_{M,m}(x,t)| \leq \varepsilon^2 f_{M,m}(x,t)$$

It follows that

$$\varepsilon\partial_t f_m \leq \Delta_\varepsilon^{\frac{\alpha}{2}} f_m + f_m(1-f_m), \qquad \varepsilon\partial_t f_M \geq \Delta_\varepsilon^{\frac{\alpha}{2}} f_M + f_M(1-f_M)$$

It then follows from the comparison principle that for all t

$$f_m(x,t) \leq n_\varepsilon(x,t) \leq f_M(x,t).$$

Proof – the convergence of u_ε

We proved that

$$\frac{C_m}{1+e^{-\frac{t(1-\varepsilon^2)}{\varepsilon}}|x|^{\frac{1+\alpha}{\varepsilon}}} \le n_\varepsilon(x,t) \le \frac{C_M}{1+e^{-\frac{t(1+\varepsilon^2)}{\varepsilon}}|x|^{\frac{1+\alpha}{\varepsilon}}}$$

$$\Downarrow$$

$$u_\varepsilon(x,t) \longrightarrow u(x,t) = \min\left(0, -(1+\alpha)\log|x| + t\right), \qquad \text{as } \varepsilon \to 0.$$

It then follows from the Hopf-Cole transformation, $n_\varepsilon = \exp(\frac{u_\varepsilon}{\varepsilon})$, that

$$n_\varepsilon \to 0, \text{ locally uniformly in } \mathcal{A} = \{(x,t) \,|\, t < (1+\alpha)\log|x|\}.$$

Proof – Convergence of n_ε to 1

We prove that $n_\varepsilon \to 1$, uniformly in K,
with K a compact set such that

$$K \subset \mathcal{B} = \{(x,t) \in \mathbb{R} \times (0,\infty) \mid t > (1+\alpha)\log|x|\}.$$

To this end, let $(x_0, t_0) \in K$. Define

$$\varphi(x,t) = \min\left(0, -(1+\alpha)\log|x| + t_0 - \delta\right) - (t-t_0)^2,$$

with δ small enough s.t. for all $(y,s) \in K$, $s \geq 2\delta$ and s. t.

$$(1+\alpha)\log|x_0| < t_0 - \delta.$$

$u - \varphi$ **attains a local (and strict) in** t **and global in** x **minimum at** (x_0, t_0).

We then define

$$\varphi_\varepsilon(x,t) = -\varepsilon \log\left(1 + e^{-\frac{t_0-\delta}{\varepsilon}} |x|^{\frac{1+\alpha}{\varepsilon}}\right) - (t-t_0)^2.$$

$(\varphi_\varepsilon)_\varepsilon$ converges locally uniformly to φ.

$$\Downarrow$$

there exist points $(x_\varepsilon, t_\varepsilon)$ s. t. $u_\varepsilon - \varphi_\varepsilon$ **has a local in** t **and global in** x **minimum at** $(x_\varepsilon, t_\varepsilon)$ and s. t.

$$t_\varepsilon \to t_0, \qquad (u_\varepsilon - \varphi_\varepsilon)(x_\varepsilon, t_\varepsilon) \to 0, \quad \text{as } \varepsilon \to 0.$$

Using the Hopf-Cole transformation we write

$$\partial_t u_\varepsilon(x,t) = \int_0^\infty \left(e^{\frac{u_\varepsilon\left(\left||x|^{\frac{1}{\varepsilon}}+h\right|^\varepsilon,t\right)-u_\varepsilon(x,t)}{\varepsilon}} + e^{\frac{u_\varepsilon\left(\left||x|^{\frac{1}{\varepsilon}}-h\right|^\varepsilon,t\right)-u_\varepsilon(x,t)}{\varepsilon}} - 2\right)\frac{dh}{|h|^{1+\alpha}} + 1 - n_\varepsilon(x,t).$$

Since $u_\varepsilon - \varphi_\varepsilon$ has a local in t and global in x minimum at $(x_\varepsilon, t_\varepsilon)$,

$$\Rightarrow \qquad \partial_t u_\varepsilon(x_\varepsilon, t_\varepsilon) = \partial_t \varphi_\varepsilon(x_\varepsilon, t_\varepsilon) = -2(t_\varepsilon - t_0),$$

$$\int_0^\infty \left(e^{\frac{u_\varepsilon\left(\left||x_\varepsilon|^{\frac{1}{\varepsilon}}+h\right|^\varepsilon,t_\varepsilon\right)-u_\varepsilon(x_\varepsilon,t_\varepsilon)}{\varepsilon}} + e^{\frac{u_\varepsilon\left(\left||x_\varepsilon|^{\frac{1}{\varepsilon}}-h\right|^\varepsilon,t_\varepsilon\right)-u_\varepsilon(x_\varepsilon,t_\varepsilon)}{\varepsilon}} - 2\right)\frac{dh}{|h|^{1+\alpha}}$$
$$\geq \int_0^\infty \left(e^{\frac{\varphi_\varepsilon\left(\left||x_\varepsilon|^{\frac{1}{\varepsilon}}+h\right|^\varepsilon,t_\varepsilon\right)-\varphi_\varepsilon(x_\varepsilon,t_\varepsilon)}{\varepsilon}} + e^{\frac{\varphi_\varepsilon\left(\left||x_\varepsilon|^{\frac{1}{\varepsilon}}-h\right|^\varepsilon,t_\varepsilon\right)-\varphi_\varepsilon(x_\varepsilon,t_\varepsilon)}{\varepsilon}} - 2\right)\frac{dh}{|h|^{1+\alpha}}.$$

The last term is again negligible $\Rightarrow$ $n_\varepsilon(x_\varepsilon, t_\varepsilon) \geq 1 + o(1)$.

Since $u_\varepsilon - \varphi_\varepsilon$ **has a local minimum in** $(x_\varepsilon, t_\varepsilon)$,

$$u_\varepsilon(x_\varepsilon, t_\varepsilon) - \varphi(x_\varepsilon, t_\varepsilon) \leq u_\varepsilon(x_0, t_0) - \varphi_\varepsilon(x_0, t_0).$$

Moreover, by definition, we have

$$\varphi_\varepsilon(x_\varepsilon, t_\varepsilon) \leq \varphi_\varepsilon(x_\varepsilon, t_0).$$

Combining the above inequalities we find

$$\frac{n_\varepsilon(x_\varepsilon, t_\varepsilon)}{n_\varepsilon(x_0, t_0)} \leq \frac{1 + e^{-\frac{t_0-\delta}{\varepsilon}} |x_0|^{\frac{1+\alpha}{\varepsilon}}}{1 + e^{-\frac{t_0-\delta}{\varepsilon}} |x_\varepsilon|^{\frac{1+\alpha}{\varepsilon}}} \leq 1 + o(1),$$

We deduce that

$$1 \leq \liminf_{\varepsilon \to 0} n_\varepsilon(x_0, t_0)$$

$$\Rightarrow \qquad n_\varepsilon(x_0, t_0) \to 1, \quad \text{uniformly in } K, \text{ as } \varepsilon \to 0.$$

The fractional Laplacian disappears at the limit

It follows as before

$$\begin{cases} n_\varepsilon \to 0, & \text{in } \{u(x,t) < 0\} = \{(x,t) \mid t < (1+\alpha)\log|x|\}, \\ n_\varepsilon \to 1, & \text{in } \mathrm{Int}\{u(x,t) = 0\} = \{(x,t) \mid t > \max(0, (1+\alpha)\log|x|)\}. \end{cases}$$

The equation verified by u:

$$\max\left(\partial_t u - 1, u\right) = 0,$$

since the term from **the fractional Laplacian is small** and disappears at the limit.

The only effect of the fractional Laplacian is on the initial tails.

A difference with the classical Fisher-KPP

The **classical** Fisher-KPP:

$$\begin{cases} \partial_t m - \delta \Delta m = m(1-m), & \delta \in \{0,1\}, \\ m(x,0) = \exp\left(-\frac{x^2}{2}\right). \end{cases}$$

For $\delta = 0$: the invasion front scales as $x \sim \sqrt{2t}$
For $\delta = 1$: the invasion front scales as $x \sim 2t$.

The **fractional** Fisher-KPP :

$$\begin{cases} \partial_t m + \delta \left(-\Delta\right)^{\frac{\alpha}{2}} m = m(1-m), & \delta \in \{0,1\}, \\ m(x,0) = \frac{C}{1+|x|^{1+\alpha}}. \end{cases}$$

For $\delta = 0$ and $\delta = 1$: the invasion front scales as $x \sim e^{\frac{t}{1+\alpha}}$

Possible extensions

- **The non-symmetric and multi-d case** [Méléard, SM, CPDE 2014]: we consider the following rescaling

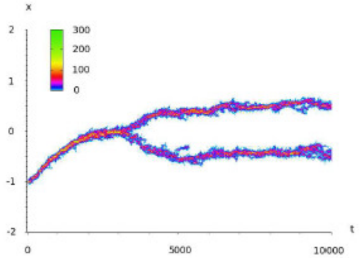
$$x \mapsto |x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}, \qquad t \mapsto \frac{t}{\varepsilon}, \qquad n_\varepsilon(x,t) = n\Big(|x|^{\frac{1}{\varepsilon}} \frac{x}{|x|}, \frac{t}{\varepsilon}\Big).$$

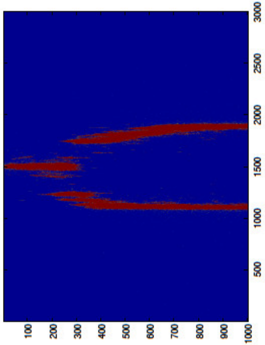
- **More general integral terms**

- **More general reaction terms** of KPP type

- **Periodic media** (homogenization with uncommon rescaling)

Selection-mutation; adaptive evolution





Basic mechanisms: Heredity, mutation and competition

Outcome: Selection of fittest traits and their evolution

Selection-mutation models; mutations with large steps

Fractional selection-mutation model:

$$\partial_t n + (-\Delta)^{\frac{\alpha}{2}} n = n\,(1 - I(t))$$

$$I(t) = \int n\,dx$$

Is it possible to do a rescaling to describe the selection of the fittest traits ?

Derivation of the model : Jourdain, Méléard, Woyczynski, 2012

A frist rescaling for fractional selection-mutation models

If we make the same rescaling as before:

$$\varepsilon \partial_t n_\varepsilon(x,t) = |x|^{\frac{-\alpha}{\varepsilon}} \int_0^\infty \Big(n_\varepsilon(|x| \cdot e^{\varepsilon k}, t) + n_\varepsilon(|x| \cdot e^{\varepsilon \log|2-e^k|}, t) - 2n_\varepsilon(x,t)\Big) \frac{e^k}{|e^k-1|^{1+\alpha}} dk + n_\varepsilon(x,t)(1 - I_\varepsilon(t)).$$

Rescaling of the integral kernel, with $h = |x|(e^k - 1)$:

$$M(x,k,dk) = \frac{|x|^{-\alpha} e^k dk}{|e^k - 1|^{1+\alpha}} \mapsto M_\varepsilon(x,k,dk) = |x|^{-\frac{\alpha}{\varepsilon}} \frac{e^{\frac{k}{\varepsilon}} \frac{dk}{\varepsilon}}{|e^{\frac{k}{\varepsilon}} - 1|^{1+\alpha}}.$$

This rescaling is **heterogeneous** in x; the mutation steps are rescaled differently at different points.

A homogeneous rescaling to consider small mutation steps

Let $h = e^k - 1$, then we suggest the following rescaling

$$M(k, dk) = \frac{e^k dk}{|e^k - 1|^{1+\alpha}} \mapsto M_\varepsilon(k, dk) = \frac{e^{\frac{k}{\varepsilon}} \frac{dk}{\varepsilon}}{|e^{\frac{k}{\varepsilon}} - 1|^{1+\alpha}}.$$

The equation becomes

$$\varepsilon \partial_t n_\varepsilon(x,t) = \int_0^\infty \left(n_\varepsilon(x + e^{\varepsilon k} - 1, t) + n_\varepsilon(x - e^{\varepsilon k} + 1, t) - 2n_\varepsilon(x,t)\right) \frac{e^k dk}{|e^k - 1|^{1+\alpha}} + n_\varepsilon(x,t)\left(1 - I_\varepsilon(t)\right)$$

Assumptions

We use again the **Hopf-Cole** transformation

$$n_\varepsilon = \exp\left(\frac{u_\varepsilon}{\varepsilon}\right)$$

We assume, with $A < \alpha$ a positive constant,

$$u_\varepsilon^0 \longrightarrow u^0, \qquad \text{as } \varepsilon \to 0$$

$$u_\varepsilon^0(x) \leq -A\log(|x|+1) + B$$

$$u_\varepsilon^0(x+h) \leq u_\varepsilon^0(x) + A\log\left(1+|h|\right)$$

Convergence to a Hamilton-Jacobi equation

Theorem (Méléard, SM, 2014)

(i) As $\varepsilon \to 0$, $(u_\varepsilon)_\varepsilon$ ***converges locally uniformly*** *to u, the unique viscosity solution to*

$$\begin{cases} \partial_t u - \int_0^\infty \left(e^{D_x u \cdot k} + e^{-D_x u \cdot k} - 2 \right) \frac{e^k dk}{|e^k - 1|^{1+\alpha}} = 0, \\ u(x,0) = u^0(x), \end{cases}$$

and

$$\|D_x u\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)} \le A, \qquad \max_{x\in\mathbb{R}} u(x,t) = 0.$$

Moreover, as $\varepsilon \to 0$, $(I_\varepsilon)_\varepsilon$ converges to 1.

(ii) Finally, along subsequences as $\varepsilon \to 0$, $n_\varepsilon \rightharpoonup n$, with

$$supp\, n \subset \{(x,t) \,|\, u(x,t) = 0\}.$$

The qualitative behavior of the population density

Let

$$u(x,0) = -A\log\left(1+|x|\right),$$

$$\Downarrow$$

$$\sup_{y\in\mathbb{R}}\{-A\log(1+|y|)-\frac{|x-y|^2}{4\underline{C}t}\} \le u(x,t) \le \sup_{y\in\mathbb{R}}\{-A\log(1+|y|)-\frac{|x-y|^2}{4\overline{C}t}\}$$

$$\Downarrow$$

u becomes **more and more flat** as time goes by, but

$$n(x,t) = I_0\,\delta(x=0).$$

No dependence on x for R $\Rightarrow$ No reason for the population to move

Perspective : for the Dirac mass **to evolve** in time one should consider $R(x,I)$

Thank you for your attention !