La limite quasineutre du système de Vlasov-Poisson

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The quasineutral limit of Vlasov-Poisson

\[
\begin{aligned}
\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + E_\varepsilon \cdot \nabla_v f_\varepsilon &= 0, \quad t \geq 0, \ (x, v) \in \mathbb{T}^d \times \mathbb{R}^d \\
E_\varepsilon &= -\nabla_x U_\varepsilon, \\
U_\varepsilon - \varepsilon^2 \Delta_x U_\varepsilon &= \int_{\mathbb{R}^d} f_\varepsilon \, dv - 1, \\
f_\varepsilon \big|_{t=0} &= f_{0, \varepsilon} \geq 0, \quad \int_{\mathbb{T}^d \times \mathbb{R}^d} f_{0, \varepsilon} \, dv \, dx = 1.
\end{aligned}
\]

- \( f_\varepsilon \) describes the dynamics of ions, in a background of massless electrons following a linearized Maxwell-Boltzmann law:

\[
n_e = e^{U_\varepsilon} \sim 1 + U_\varepsilon.
\]

- The parameter \( \varepsilon \in (0, 1] \) is the ratio between the Debye length and the observation length. In practice, \( \varepsilon \ll 1 \).

- Quasineutral limit: \( \varepsilon \to 0 \).
Assuming $f_{0,\varepsilon} \to f_0$ and taking $\varepsilon = 0$ yields

\[
\begin{aligned}
\partial_t f + v \cdot \nabla_x f - \nabla_x \rho \cdot \nabla_v f &= 0, \\
\rho &= \int_{\mathbb{R}^d} f \, dv - 1, \\
f \big|_{t=0} &= f_0 \geq 0, \quad \int_{\mathbb{R}^d} f_0 \, dv dx = 1.
\end{aligned}
\]

- ... a system called **Vlasov-Dirac-Benney** by Bardos.

- **Loss of derivative?** The force $-\nabla_x \rho$ is one derivative less regular than $f$.

- Is Vlasov-Dirac-Benney a good approximation of Vlasov-Poisson when $\varepsilon \to 0$?
More on Vlasov-Dirac-Benney

\[
\begin{cases}
    \partial_t f + v \cdot \nabla_x f - \nabla_x \rho \cdot \nabla_v f = 0, \\
    \rho = \int_{\mathbb{R}^d} f \, dv - 1, \\
    f\big|_{t=0} = f_0 \geq 0, \quad \int_{\mathbb{R}^d} f_0 \, dvdx = 1.
\end{cases}
\]

(Local) Existence of solutions is known for

- **analytic** initial data (Cauchy-Kowalevski type result);
- in \(d = 1\), Sobolev initial data that, for all \(x\), have the shape of one bump [Bardos, Besse 2013], through a water-bag representation;
- **Penrose stable Sobolev** initial data [DHK, Rousset, to appear].

More to come in a few minutes!
There are unstable equilibria around which the linearized equations have unbounded unstable spectrum [Bardos, Nouri 2012]. This reflects the singularity of the equation.

In [DHK, T. Nguyen, preprint], we lead a detailed study of the consequences of this (designing an abstract framework to prove ill-posedness properties, which we also applied to the hydrostatic Euler equations).

In particular, we prove that the solution map $f_0 \mapsto f(t)$ of Vlasov-Dirac-Benney cannot be $C^\alpha_{loc}(H^s, L^2)$ for any $\alpha \in (0, 1]$ and $s \geq 0$, even for $t \ll 1$ (we build a sequence of solutions making the Hölder norm blow up).

Similar to [Métivier 2005] for quasilinear symmetric non hyperbolic systems.
Quasineutral limit and large time behavior

• For all $\varepsilon \in (0, 1]$, the Cauchy theory for Vlasov-Poisson is very well understood (Arsenev, Ukai-Okabe, Pfaffelmoser, Schaeffer, Lions-Perthame, Batt-Rein,...), but does not provide useful uniform estimates.

Using conservation laws only yield a weak form of the limit with defect measures [Brenier, Grenier '94], [Grenier '95].

• The change of variables $(t, x, \nu) \mapsto (\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \nu)$ gives the unscaled Vlasov-Poisson system (that is, with $\varepsilon = 1$).

Quasineutral limit $\rightarrow$ Large Time Behavior problem

• The stability or instability of homogeneous equilibria play a decisive role in the derivation of Vlasov-Dirac-Benney in the quasineutral limit.
Plan of the talk

I. Invalidity of Vlasov-Dirac-Benney in the quasineutral limit
   • [Grenier ’99], [DHK, Hauray 2015]

II. Validity of Vlasov-Dirac-Benney in the quasineutral limit
   • Uniform analytic regularity [Grenier ’96]
   • Zero-temperature limit [Brenier 2000], [DHK 2011]
   • General Penrose stable data [DHK, Rousset, to appear]
Nonlinear instability

Penrose instability conditions ensure the linear spectral instability of homogeneous equilibria of Vlasov-Poisson (two-stream instability). In [Guo, Strauss '95], it is proved that spectral instability implies nonlinear instability as well.

**Theorem 1 ([DHK, Hauray 2015])**

Let $\mu(v)$ be a smooth Penrose unstable equilibrium. For all $n \geq 0$, there is $\theta > 0$ such that, for all $\delta > 0$, there is a solution $g$ of Vlasov-Poisson with

$$\|g(0) - \mu\|_{W_{x,v}^{n,1}} \leq \delta$$

but

$$\sup_{t \in [0, t_\delta]} \|g(t) - \mu\|_{W_{x,v}^{-n,1}} \geq \theta > 0$$

with $t_\delta = O(|\log \delta|)$ as $\delta \to 0$.

A non-derivation result in the quasineutral limit

Combining the previous **nonlinear instability theorem** and the change of variables $(t, x, v) \mapsto \left( \frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v \right)$ (**high frequency regime**), we obtain:

**Theorem 2** ([DHK, Hauray 2015])

Let $\mu(v)$ be a smooth Penrose unstable equilibrium. For all $n, k \geq 0$, there exists a sequence of solutions $(f_\varepsilon)$ such that

$$\|f_\varepsilon(0) - \mu\|_{W^{n,1}_{x,v}} \leq \varepsilon^k,$$

but

$$\liminf_{\varepsilon \to 0} \sup_{t \in [0, \varepsilon|\log \varepsilon|]} \|f_\varepsilon(t) - \mu\|_{W^{-n,1}_{x,v}} > 0.$$  

We deduce that the limit equation can not admit $\mu(v)$ as a stationary solution. **In particular Vlasov-Dirac-Benney is not a good approximation near such equilibria.** (Extended to 3D Vlasov-Maxwell in [DHK, T. Nguyen, preprint].)
Plan of the talk

I. Invalidity of Vlasov-Dirac-Benney in the quasineutral limit
   - [Grenier '99], [DHK, Hauray 2015]

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Derivation in Analytic regularity

- In [Grenier '96], it is shown that two-stream instabilities have no effect for solutions with uniform **analytic regularity**.

- Loosely speaking, the principle of his proof is to write the distribution function \( f_\epsilon \) as the **superposition of layers of fluids**.

For some fixed probability space \((M, \mu(d\theta))\), write the decomposition

\[
f_\epsilon(t, x, v) = \int_M \rho_\epsilon^\theta(t, x) \delta_{v = u_\epsilon^\theta(t, x)} \mu(d\theta),
\]
Derivation in Analytic regularity

This leads to the study of a system of coupled Burgers eq.:

\[
\begin{cases}
\partial_t \rho_\varepsilon^\theta + \nabla x \cdot (\rho_\varepsilon^\theta u_\varepsilon^\theta) = 0, \\
\partial_t u_\varepsilon^\theta + u_\varepsilon^\theta \cdot \nabla x u_\varepsilon^\theta = -\nabla x U_\varepsilon, \\
U_\varepsilon - \varepsilon^2 \Delta x U_\varepsilon = \int_M \rho_\varepsilon^\theta \mu(d\theta) - 1.
\end{cases}
\]

**Theorem 3 ([Grenier ’96])**

Assume that for \(f_0\) with analytic regularity (\(\| \cdot \|\) is a norm that is analytic in \(x\), continuous in \(v\))

\[\|f_\varepsilon,0 - f_0\| \to 0.\]

Then there is \(T > 0\) such that \(f_\varepsilon\) weakly converges on \([0, T]\) to a weak solution to Vlasov-Dirac-Benney with initial condition \(f_0\).

In [DHK, Iacobelli, preprints]: still true for exponentially small but rough perturbations of such data (\(d \leq 3\)). Uses quantitative Wasserstein stability estimates [Loeper 2006], [Hauray 2013].
Derivation in stable cases?

- Is it possible to say something under an assumption of Penrose stability on the initial condition?

- The first result in this direction is due to [Brenier 2000] where the Modulated Energy method was introduced (see also [Yau ’91], [Golse 2000]) .

For monokinetic data

\[ f(t, x, v) = \rho(t, x) \delta_{v=u(t,x)}, \]

note that \( f \) satisfies Vlasov-Dirac-Benney iff \((\rho, u)\) satisfies the Shallow Water equations (isentropic compressible Euler with \( \gamma = 2 \)) :

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho u) &= 0, \\
\partial_t u + u \cdot \nabla_x u + \nabla_x \rho &= 0, \\
(\rho, u)|_{t=0} &= (\rho_0, u_0).
\end{align*}
\]
Derivation in the zero–temperature regime

• Consider

\[ f_{0,\varepsilon} \rightarrow \rho_0(x)\delta_{v=u_0(x)} \]

(zero-temperature limit, extremal case of a Maxwellian (stable)).

• Following [Brenier 2000], introduce

\[
\mathcal{H}_\varepsilon(t) := \frac{1}{2} \int f_\varepsilon |v - u(t, x)|^2 \, dv\,dx \\
+ \frac{1}{2} \int (U_\varepsilon - \rho(t, x))^2 \, dx + \frac{\varepsilon^2}{2} \int |E_\varepsilon(t, x)|^2 \, dx.
\]

where \((\rho, u)\) solves the Shallow Water system on \([0, T]\).

• One proves [DHK 2011] that

\[
\frac{d}{dt} \mathcal{H}_\varepsilon(t) \lesssim \mathcal{H}_\varepsilon(t) + o(1)
\]

so that roughly

\[ f_{0,\varepsilon} \rightarrow \rho_0(x)\delta_{v=u_0(x)} \implies \forall t \in [0, T], f_\varepsilon(t) \rightarrow \rho(t, x)\delta_{v=u(t,x)}. \]

Derivation of Shallow Water.
May one generalize the modulated energy method?

- A natural idea would be to adapt this method to handle other stable initial conditions.
- \[\text{[DHK, Hauray 2015]}\] : works for stationary \( \mu(v) \) satisfying

\[ \uparrow \text{ on } (-\infty, 0], \quad \downarrow \text{ on } [0, +\infty) \text{ and even} \]

- Fails to handle other stable initial data one would like to consider, for symmetry and rigidity reasons.

The modulated energy method requires that the solution of the limit system is the minimizer of some entropy and thus satisfies

\[ f \equiv g(t, x, -|v - v(t, x)|^2). \]

We prove that such solutions of Vlasov-Dirac-Benney are necessarily of the form \( g(-|v - \bar{v}|^2) \)...

- Implies one can not hope to use the modulated energy method...
Derivation result for stable data

- Another method is needed to handle general stable data.

- We say that $f(\nu)$ satisfies the $c_0$ Penrose stability condition if

$$\inf_{(\gamma, \tau, \eta) \in \mathbb{R}^+_* \times \mathbb{R} \times \mathbb{R}^d} \left| 1 - \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{i\eta}{1 + |\eta|^2} \cdot \int_{\mathbb{R}^d} e^{-is\eta \cdot \nu} (\nabla_{\nu} f)(\nu) \, d\nu \, ds \right| \geq c_0.$$  

Recall that this also appears for Landau Damping [Mouhot, Villani 2011]. In particular, this is satisfied by functions with the shape of one bump.

- Introduce also for $k \in \mathbb{N}$, $r \in \mathbb{R}$, the weighted Sobolev norms

$$\|f\|_{\mathcal{H}_r^k} := \left( \sum_{|\alpha| + |\beta| \leq k} \int_{T^d} \int_{\mathbb{R}^d} (1 + |\nu|^2)^r |\partial_x^\alpha \partial_{\nu}^\beta f|^2 \, dv dx \right)^{1/2}$$

and the regularity indices

$$k_0 = 4 + d, \quad r_0 = \max(d, 2 + \frac{d}{2}).$$
Derivation result for stable data

**Theorem 4 ([DHK, Rousset, to appear])**

Let \(2m > k_0, 2r > r_0\). Let \(M_0 > 0, c_0 > 0\). Assume that for all \(\varepsilon \in (0, 1]\), \(\|f_{0,\varepsilon}\|_{H^{2m}} \leq M_0\) and for all \(x \in \mathbb{T}^d\), \(f_{0,\varepsilon}(x, \cdot)\) satisfies the \(c_0\) Penrose stability condition. Assume that \(f_{0,\varepsilon} \to f_0\) in \(L^2\). Then there is \(T > 0\) such that

\[
\sup_{[0,T]} \|f_{\varepsilon}(t) - f(t)\|_{L^2} \to 0,
\]

where \(f(t)\) satisfies Vlasov-Dirac-Benney with initial data \(f_0\).

An example of admissible initial data is given by smooth local Maxwellians

\[
M(x, v) = \frac{\rho(x)}{(2\pi T(x))^{d/2}} \exp\left(-\frac{|v - u(x)|^2}{T(x)}\right).
\]

As a by-product we get well-posedness (i.e. existence + uniqueness) in the class of such data for Vlasov-Dirac-Benney.
Sketch of the proof

We have \( \| f_{0,\epsilon} \|_{\mathcal{H}_{2r}^{2m}} \leq M_0 \). Introduce

\[
\mathcal{N}_{2m, 2r}(t, f_{\epsilon}) := \| f_{\epsilon} \|_{L^\infty((0,t),\mathcal{H}_{2r}^{2m-1})} + \| \rho_{\epsilon} \|_{L^2((0,t),H^{2m})}
\]

with \( \rho_{\epsilon} = \int_{\mathbb{R}^d} f_{\epsilon} \, dv \). The main task is to find \( T > 0, R > 0 \) such that

\[
\forall \epsilon \in (0, 1], \sup_{[0,T]} \mathcal{N}_{2m, 2r}(t, f_{\epsilon}) \leq R.
\]

The proof is based on a bootstrap argument. By a standard energy estimate, we see that the key quantity to be controlled is actually \( \| \rho_{\epsilon} \|_{L^2((0,t),H^{2m})} \).
Sketch of the proof

- **Natural idea**: up to commutators, $\partial^2_x f_\varepsilon$ evolves according to the linearized equation around $f_\varepsilon$, that is

$$
\partial_t \partial^2_x f_\varepsilon + \nu \cdot \nabla_x \partial^2_x f_\varepsilon + \partial^2_x E_\varepsilon \cdot \nabla_\nu f_\varepsilon + E_\varepsilon \cdot \nabla_\nu \partial^2_x f_\varepsilon = S,
$$

where $S$ should involve remainder terms only.

- When $f_\varepsilon \equiv \mu(\nu)$ does not depend on $t$ and $x$, then $E_\varepsilon = 0$ and the linearized equation reduces to

$$
(\partial_t + \nu \cdot \nabla_x) g + E_g \cdot \nabla_\nu \mu(\nu) = S,
$$

$$
E_g = -\nabla_x (I - \varepsilon^2 \Delta)^{-1} \rho_g.
$$

This case was studied in [Mouhot, Villani 2011] in view of Landau Damping.
Sketch of the proof

By the method of characteristics,

\[ g(t, x, v) = g_0(x - tv, v) - \int_0^t E_g(x - (t - s)v) \cdot \nabla_v \mu(v) \, ds + S \]

and thus, integrating in \( v \), we obtain an integral equation for \( \rho_g = \int_{\mathbb{R}^d} g \, dv \):

\[ \rho_g(t, x) = \int_0^t \int_{\mathbb{R}^d} \nabla_x (I - \varepsilon^2 \Delta)^{-1} \rho_g(x - (t - s)v) \cdot \nabla_v \mu(v) \, dv \, ds + S + S_0 \]

By Fourier analysis one may estimate \( \rho_g \) in \( L^2_{t,x} \), under a Penrose stability condition for \( \mu(v) \).
Sketch of the proof

• **However**, when applying this strategy, there are subprincipal terms which involve $2m$ derivatives of $f$:

$$\partial_x E_\epsilon \cdot \nabla_v \partial_x^{2m-1} f_\epsilon.$$  

• Applying more general vector fields would also generate bad subprincipal terms. Instead: Consider powers of relevant **second order operators** (with coefficients depending on $f_\epsilon$ itself), yielding a family $(f_{I,J})_{I,J \in \{1,\ldots,d\}^m}$ that satisfy two key properties.

  • We have the control

$$\|\rho_\epsilon\|_{H^{2m}} \lesssim \sum_{I,J} \left\| \int_{\mathbb{R}^d} f_{I,J} \, dv \right\|_{L^2} + R$$

  where $R$ is a good remainder.

  • $f_{I,J}$ roughly satisfies

$$\partial_t f_{I,J} + v \cdot \nabla_x f_{I,J} + E_\epsilon \cdot \nabla_v f_{I,J} + E_{f_{I,J}} \cdot \nabla_v f_\epsilon + (\text{zero order terms}) = S_{I,J},$$

  where $S_{I,J}$ is a good remainder.
Sketch of the proof

In dimension one, only one operator is needed :

\[ L := \partial_{xx} + \varphi \partial_x \partial_v + \psi \partial_{vv}, \]

with \((\varphi, \psi)\) satisfying the system

\[
\begin{cases}
(\partial_t + \nu \cdot \nabla_x + E \cdot \nabla_v) \varphi = \partial_x E + \text{(zero order terms),} & \varphi|_{t=0} = 0 \\
(\partial_t + \nu \cdot \nabla_x + E \cdot \nabla_v) \psi = \varphi \partial_x E + \text{(zero order terms),} & \psi|_{t=0} = 0
\end{cases}
\]

this system being designed to kill the bad subprincipal term.

In dimension \(d\), we obtain in a similar way some relevant operators \((L_{i,j})_{1 \leq i,j \leq d}\), and we define

\[ f_{I,J} := L_{i_1,j_1} \cdots L_{i_m,j_m} f. \]
Sketch of the proof

We thus study

\[(\partial_t + v \cdot \nabla_x + E_\varepsilon \cdot \nabla \nu) g + E_g \cdot \nabla \nu f_\varepsilon = S.\]

As \(f_\varepsilon\) depends on \(x\), the force field \(E_\varepsilon\) is not trivial.

However, we can use a near identity change of variables to straighten the vector field and come down to the equation

\[(\partial_t + \Phi(t, x, v) \cdot \nabla_x) g + E_g \cdot \nabla \nu f_\varepsilon = S\]

where \(\Phi(t, x, v)\) satisfies the Burgers equation

\[\partial_t \Phi + \Phi \cdot \nabla_x \Phi = E, \quad \Phi|_{t=0} = v,\]

and is close to \(v\) for small times.

Integrating along characteristics and integrating in \(v\), we end up with the study, for small times, of...
Sketch of the proof

...the integral equation

\[ h = K_{\nabla v f_0, \varepsilon} (I - \varepsilon^2 \Delta)^{-1} h + R, \]

with

\[ K_{\nabla v f_0, \varepsilon} (G) = \int_0^t \int_{\mathbb{R}^d} (\nabla x G)(s, x - (t - s)v) \cdot \nabla v f_0, \varepsilon(x, v) \, dv \, ds. \]

Note that \( K_{\nabla v f_0, \varepsilon} \) may seem to feature a loss of derivative. However, we have

**Proposition 1**

\( K_{\nabla v \mu} \) is a bounded operator on \( L^2 \) if \( \mu \) is smooth and decaying in \( v \).

This is an effect in the spirit of **averaging lemmas** ([Golse, Lions, Perthame, Sentis '88]).
Sketch of the proof

\[ h = K_{\nabla v f_0, \varepsilon} \left( I - \varepsilon^2 \Delta \right)^{-1} h + R \]

Let \( \gamma > 0 \) to be chosen. We can relate \( e^{-\gamma t} K_{\nabla v f_0, \varepsilon} (e^{\gamma t}(I - \varepsilon^2 \Delta)^{-1} \cdot) \) to a semi-classical pseudodifferential operator, of symbol

\[
a(\gamma, \tau, x, \eta) := \int_0^{+\infty} e^{-(\gamma + i\tau)s} \frac{i\eta}{1 + |\eta|^2} \cdot \int_{\mathbb{R}^d} e^{-is\eta \cdot v} \nabla v f_{0, \varepsilon}(x, v) \, dv \, ds,
\]

and we rewrite the integral equation as

\[ Op_{(1-a)}^{\gamma, \varepsilon}(e^{-\gamma t} h) = \mathcal{R}. \]

The \( c_0 \) Penrose condition is precisely \( \inf |1 - a| \geq c_0 \) and therefore implies the ellipticity of the symbol associated to the operator we want to invert.
Sketch of the proof

\[ Op_{1-a}^{\gamma,\varepsilon}(e^{-\gamma t} h) = R. \]

We can finally use a semi-classical pseudodifferential calculus with parameter \( \gamma \) in order to invert \( Op_{1-a}^{\gamma,\varepsilon} \) up to a small remainder. More precisely we have the general estimate

\[ \| Op_b^{\gamma,\varepsilon} Op_c^{\gamma,\varepsilon} u - Op_{bc}^{\gamma,\varepsilon} u \|_{L^2} \lesssim \frac{1}{\gamma} \| b \| \| c \| \| u \|_{L^2}. \]

that we apply to

\[ b = \frac{1}{1-a}, \quad c = 1-a, \quad u = e^{-\gamma t} h, \]

which roughly gives

\[ \| h \|_{L^2} \leq \| Op_{1-a}^{\gamma,\varepsilon} R \|_{L^2} + \frac{1}{\gamma} \left\| \frac{1}{1-a} \right\| \| 1-a \| \| h \|_{L^2} \]

Choosing \( \gamma \gg 1 \) yields an estimate for \( h \), in \( L^2_{t,x} \).

This allows to close the bootstrap argument.
Merci pour votre attention!