Régularite höldérienne des solutions d'équations hypoelliptiques avec des coefficients non réguliers

Collaboration avec François Golse, Cyril Imbert et Alexis Vasseur

Clément Mouhot

Université de Cambridge

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A motivation in nonlinear analysis of kinetic equations

- Oldest kinetic equation (Maxwell 1867, Boltzmann 1872): $\partial_t f + v \cdot \nabla_x f = Q(f,f)$
- Nonlinear (quadratic) PDE on $f = f(t,x,v) \geq 0$ with Q bilinear integral operator acting only along v
- For long-distance interactions Q has fractional ellipticity in v
- Global well-posedness major open mathematical problem
- Toy model: $\partial_t f + v \cdot \nabla_x f = \rho[f] \nabla_v \cdot (\nabla_v f + vf)$ (where $\rho[f] := \int_v f$) $\to$ quadratic nonlinearity and preserves mass $\int_{x,v} f$
- Surprisingly to my knowledge, global well-posedness and propagation of regularity still unknown for this model
- Follow De Giorgi and Nash's strategy when solving Hilbert's 19th problem: consider $\rho[f]$ as a given rough coefficient and study local regularity of weak solutions

The De Giorgi - Nash - Moser theory (1)

- Hilbert's 19th problem: whether minimizers u of an energy functional $\int_U L(\nabla u)\,\mathrm{d}x$ are analytic, where $L : \mathbb{R}^d \to \mathbb{R}$ Lagrangian satisfies growth, smoothness and convexity conditions
- Euler-Lagrange equations for the minimizers take the form

$$\nabla \cdot \Big[\nabla L(\nabla u)\Big] = 0 \quad \text{i.e.} \quad \sum_{ij} \underbrace{\big[(\partial_{ij} L)(\nabla u)\big]}_{b_{ij}} \partial_{ij} u = 0$$

- Dirichlet energy $L(p) = |p|^2$, minimal surfaces $L(p) = \sqrt{1 + |p|^2}$
- If L and domain convex, control of ∇u in terms of boundary data
- However existence-uniqueness-regularity requires more: if $u \in C^{1,\alpha}$ then $b_{ij} \in C^\alpha$ and Schaüder estimates imply $u \in C^{2,\alpha}$ (then bootstrap yields higher regularity...)

The De Giorgi - Nash - Moser theory (2)

- Equation on derivative $f := \partial_k u$ (divergence form):

$$\sum_{ij} \partial_i \left[\underbrace{(\partial_{ij} L)(\nabla u)}_{a_{ij}} \partial_j f \right] = 0$$

- De Giorgi 1956 – Nash (1958): with controls (but no regularity) on a_{ij} then $f = \nabla u$ is Hölder (Nash considered the parabolic case)
- Proof of Nash not used here
- Proof of De Giorgi: (1) iterative gain of integrability (2) "isoperimetric argument" to control oscillations
- Proof of Moser: (1) iterative gain of integrability (2) relating positive and negative Lebesgue norms by studying the equation on $g := \ln f$
- We use the iteration in the Moser form and the control of oscillations in the De Giorgi form, but hypoelliptic nature creates new difficulties
- (Non-divergence theory by Krylov-Safonov not considered here)

Hörmander's theory of hypoellipticity (1)

- Starting point: 3 pages note of Kolmogorov Annals of Math. 1934 "*Zufällige Bewegungen (Zur Theorie der Brownschen Bewegung)*"
- Kolmogorov considered dimension $d = 1$ transport with constant drift and diffusion (thus sometimes called "Kolmogorov equation")

$\partial_t f + v \cdot \partial_x f \, (+b\partial_v f) = a\partial_v^2 f$ and fundamental solutions from δ_{x_0,v_0}

$$\frac{1}{3a\pi^2 t^2} \exp\left\{ -\frac{1}{\pi^2 a}\left(\frac{|v-v_0|^2}{t} + \frac{3|x-x_0-tv_0|^2}{t^3} - \frac{3(x-x_0-tv_0)\cdot(v-v_0)}{t^2} \right) \right\}$$

Hörmander's theory of hypoellipticity (2)

- Hörmander 1967's seminal paper starts from observing the regularisation of this fundamental solution and builds a general theory based on commutator estimates
- Regularisation Gevrey instead of analytic as for parabolic equations
- Simpler case when no first order part and missing directions of diffusion ("Hörmander type I"): DGNM theory already extended
- Hörmander original theory is local but global estimates derived under the impulsion of hypocoercivity
- Example of commutator estimates in a (very) simple case:

$$\partial_t f + Bf + A^* A f = 0, \quad B = v \cdot \partial_x, \quad A = \partial_v$$

$$[A, B] = C = \partial_x, \quad \frac{\mathrm{d}}{\mathrm{d}t} \langle Af, Bf \rangle = -\| Cf \|^2 + \dots$$

An hypoelliptic extension of the NDGM theory

Theorem (Golse-Imbert-CM-Vasseur)

$$\textit{Equation} \quad \partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A(t,x,v) \nabla_v f)$$

where the $d \times d$ symmetric matrix A satisfies the ellipticity condition $0 < \lambda Id \leq A \leq \Lambda Id$ but is, besides that, merely measurable.
We define for $z = (t,x,v)$ the cube $Q_r(z) = B_{r^3}(x) \times B_r(v) \times (t - r^2, t]$.
Then for $0 < r_1 < r_0$, if f is a solution in $Q_{r_0(z_0)}$ then

$$\|f\|_{L^\infty(Q_{r_1}(z_0))} + \|f\|_{C^\alpha(Q_{r_1}(z_0))} \leq C \|f\|_{L^2(Q_{r_0}(z_0))}$$

where C depends on $z_0, r_0, r_1, \lambda, \Lambda, d$ and $\alpha \in (0,1)$ depends on λ, Λ, d.

Gain of L^∞ in [Pascucci-Polidoro 2004]
See also [Wang-Zhang 2011]

Gain of integrability - The elliptic case (following Moser)

- We consider, with $f = f(t, v)$ and g source term nicely behaved:

$$\nabla_v \left(A(t, v) \nabla_v f \right) = g$$

- Central energy estimate (valid for subsolutions):

$$\|f\|_{H^1(Q_{r_1})} \lesssim \frac{1}{(r_0 - r_1)^2} \|f\|_{L^2(Q_{r_0})} + \|g\|_{L^2(Q_{r_1})}$$

- Sobolev embedding translates the gain H^1 into L^p, $p > 2$
- Iteration by applying the argument to any subsolution $f^{p/2}$, $p \geq 2$, for a sequence of radii $r_n \to r_\infty > 0$, to get finally L^∞ in Q_{r_∞}
- Uses the ellipticity of the operator in all directions $v \in \mathbb{R}^d$

The parabolic case (following Moser)

- Parabolic case (one step closer to our setting) with $f = f(t,v)$:

$$\partial_t f = \nabla_v \left(A(t,v) \nabla_v f \right)$$

- Central energy estimate:

$$\left(\int_{v \in B_{r_1}} f^2 \, \mathrm{d}v \right)_{t=T} + \int_{T-r_1^2}^{T} \int_{v \in B_{r_1}} |\nabla_v f|^2 \, \mathrm{d}v \, \mathrm{d}t \lesssim \frac{1}{(r_0 - r_1)^2} \int_{T-r_0^2}^{T} \int_{v \in B_{r_0}} f^2 \, \mathrm{d}v \, \mathrm{d}t$$

- Similar iteration argument in both variables t, v
- Again uses ellipticity of the operator in all directions $v \in \mathbb{R}^d$

Difficulties in the hypoelliptic case

- Coming back to our equation $\partial_t f + v \cdot \nabla_x f \leq \nabla_v (A \nabla_v f)$ we derive the corresponding energy estimate:

$$\left(\int_{x \in B_{r_1^3}} \int_{v \in B_{r_1}} f^2 \, \mathrm{d}x \, \mathrm{d}v \right)_{t=T} + \int_{T-r_1^2}^{T} \int_{x \in B_{r_0^3}} \int_{v \in B_{r_0}} |\nabla_v f|^2 \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t$$
$$\lesssim \frac{1}{(r_0 - r_1)^2} \int_{T-r_0^2}^{T} \int_{x \in B_{r_0^3}} \int_{v \in B_{r_0}} f^2 \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t$$

- Problem 1: control only on v-gradients, not x-gradients
- Key tool in kinetic theory to remedy this: averaging lemma [Golse-Perthame-Sentis 1985]
- Problem 2: the iteration requires to work on subsolutions ($f^{p/2}$, $p > 2$) for which averaging lemma do not hold in general

Strategy

- Averaging lemma: solutions to (with $p > 1$)

$$\partial_t f + v \cdot \nabla_x f = (1 - \Delta_{t,x})^{\beta} \nabla_v^k g, \quad f, g \in L^p_{t,x,v}, \quad k \geq 0, \ \beta \in (0, 1/2)$$

have some regularity on the v-averages: $\int_v f \, \mathrm{d}v \in W^{s,p}_{t,x}$ ($s > 0$ small)

- Averages "transversal" to cancellations of symbol of the hyperbolic transport operator (gain of regularity limited by order 1 of operator)
- It degenerates if RHS g not controlled $\Rightarrow$ problem for subsolutions

$$\partial_t f + v \cdot \nabla_x f \leq \nabla_v \cdot H_0 + H_1 \text{ with } H_0, H_1 \in L^2$$

- Comparison principle: $0 \leq f \leq F$ with a true solution F of previous equation on which energy estimate $L^2_{t,x} H^1_v$ plus averaging lemma $H^s_{t,x} L^1_v$ imply some $H^{s'}_{t,x,v}$ ($0 < s' < s$) and thus some gain of integrability $L^{p>2}$ by Sobolev embedding, inherited by f

The iteration

- The previous argument proves: there is $\kappa > 1$ such that for all $q > 1$:

$$\|(f^q)^\kappa\|^2_{L^2(Q_{r_1})} \le C \left(\frac{1}{(r_0 - r_1)^2} + \frac{1}{r_0(r_1 - r_0)} \right)^\kappa \|f^q\|^{2\kappa}_{L^2(Q_{r_0})}$$

- Choose $q = q_n = 2\kappa^n$ and $r_{n+1} = r_n - \frac{1}{a(n+1)^2}$ (a large enough)
- We obtain

$$\|f\|_{L^{q_{n+1}}(Q_{n+1})} \le C_{n+1}^{\frac{1}{q_{n+1}}} \|f\|_{L^{q_n}(Q_n)} \quad \text{with} \quad C_n \sim c(a^2 n^4 + bn^2)^\kappa$$

and

$$\prod_{n=0}^{+\infty} C_n^{\frac{1}{2\kappa^n}} < +\infty, \qquad r_n \to r_\infty > 0$$

which proves the convergence of the iteration

Regularity of non-negative subsolutions revisited (1)

- With the $L^2 \to L^\infty$ gain at hand we return to the regularity of subsolutions to $\partial_t f + v \cdot \nabla_x f \leq \nabla_v \cdot H_0 + H_1$:

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot H_0 + H_1 - \mu$$

  with a measure $\mu \geq 0$ and $H_0, H_1 \in L^2$ and $0 \leq f \in L^\infty_{t,x,v} \cap L^2_{t,x} H^1_v$

- We perform an estimate on the mass rather than quadratic: integrating $\int_{t,x,v} f \phi$ (for a cutoff function ϕ) yields

$$\|\mu\|_{M^1(Q_{r_1})} \lesssim \|f\|_{L^2(Q_{r_0})} + \|H_0\|_{L^2(Q_{r_0})} + \|H_1\|_{L^2(Q_{r_0})}$$

  which gives a control on the unknown error μ

Regularity of non-negative subsolutions revisited (2)

- We then write $-\mu = (1-\Delta_{t,x})^{1/4}(1-\Delta_v)g$ with $g \in L^p$, $p \in (1,2)$ by ellipticity of the fractional Laplacian

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot H_0 + H_1 + (1-\Delta_{t,x})^{1/4}(1-\Delta_v)g$$

  Refined averaging lemma in $L^p \Longrightarrow W^{s,p}_{t,x} L^1_v$ regularity for a small s
- Interpolate with L^∞ to deduce $H^{s'}_{t,x} L^1_v$ regularity $(0 < s' < s)$
- Finally we combine it with the energy estimate $L^2_{t,x} H^1_v$ to get $f \in H^{s''}_{t,x,v}$ for $0 < s'' < s' < s$
- Note that because of the interpolation with L^∞ to "bring back" the regularity obtained in L^2 this argument does not supersede the previous comparison principle in the $L^2 \to L^\infty$ iteration, which is still needed

Gain of integrability on $\nabla_v f$ (1)

- Another interesting property on subsolutions of $\partial_t f + v \cdot \nabla_x f \leq \nabla_v (A \nabla_v f)$ resembling the parabolic case

Theorem

There is $\varepsilon > 0$ universal so that

$$\int_{Q_{r_1}} |\nabla_v f|^{2+\varepsilon} \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}v \lesssim_{r_0, r_1, \lambda, \Lambda, d} \left(\int_{Q_{r_0}} |\nabla_v f|^2 \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}v \right)^{\frac{2+\varepsilon}{2}}$$

- It follows (iteration) from (Gehring lemma): given $q > 1$ there is θ small enough s.t. if for all $z \in \Omega$ ("almost reversed Hölder inequality")

$$⨍_{Q_r(z)} g^q \leq C_\theta \left(⨍_{Q_{8r}(z)} g \, \mathrm{d}z \right)^q + \theta ⨍_{Q_{8r}(z)} g^q \, \mathrm{d}z$$

then $(⨍_{Q_r} g^{q+\varepsilon} \, \mathrm{d}z)^{1/(q+\varepsilon)} \lesssim (⨍_{Q_{4r}} g^q \, \mathrm{d}z)^{1/q}$ for some $\varepsilon > 0$

Gain of integrability on $\nabla_v f$ (2)

- Proof of Gehring lemma is based on the following inequalities

$$(1) \qquad \int_{\mathcal{Q}_r} |\nabla_v f|^2 \,\mathrm{d}z \le \frac{C}{r^2} \int |f - \tilde{f}_{2r}|^2 \,\mathrm{d}z$$

$$(2) \qquad \sup_{t \in (T-r^2, T]} \int_{\mathcal{Q}_r^t} |f - \tilde{f}_r|^2 \,\mathrm{d}z \le C r^2 \int_{Q_{3r}} |\nabla_v f|^2 \,\mathrm{d}z$$

$$(3) \qquad \left(\int |f - \tilde{f}|^{q+\varepsilon} \,\mathrm{d}z \right)^{1/(2+\varepsilon)} \lesssim \left(\int |\nabla_v f|^2 \,\mathrm{d}z \right)^{1/2}$$

  proved by the energy estimate (written removing the x, v-average $\tilde{f}_{\dots}$), fractional Poincaré in x, v, Sobolev embedding and the $H^s_{x,v}$ regularity for subsolutions (averages $\tilde{f}_{\dots}$ and cubes $\mathcal{Q}_{\dots}$ along free flow)

- Boostrap the estimate on $\nabla_v f$ using $\int |f - \tilde{f}|^2$ as pivot: gain of integrability on $f - \tilde{f}$ reason for smallness of θ

Control of oscillation: the classical theory (1)

- Cannot differentiate PDE as coefficients non regular: relate local suprema and infima (oscillation), and control this difference
- Gain of integrability suggest the "L^∞" setting: Hölder regularity
- In the parabolic case, it takes time for the diffusive effect to manifest $\to$ time delays when comparing suprema and infima (cf. cubes)
- Moser's strategy: gains $L^\epsilon \to L^\infty$ and $L^{-\infty} \to L^{-\epsilon}$ and then compare L^ϵ and $L^{-\epsilon}$ by studying the equation for $\ln f$ and using Poincaré inequality
- Moser manages to compare suprema and infima which is an independent property called Harnack inequality
- Harnack inequality implies Hölder regularity but reverse not true

Control of oscillation: the classical theory (2)

- De Giorgi's strategy: always consider oscillation as a whole without separating controls on suprema and infima, and control decrease of oscillation when reducing the size of the cube considered
- Main Lemma of decrease of oscillations: for f solution in Q_2 with $|f| \leq 1$ then $\mathrm{osc}_{Q_{1/2}} f \leq 2 - \delta$ for some $\delta > 0$
- It implies Hölder regularity at the point at which cubes shrink
- It is implied by the following Lemma of decrease of supremum bound: for f solution in Q_2 with $|f| \leq 1$ and $|\{f \leq 0\} \cap Q_1| \geq (1/2)|Q_1|$ then $\sup_{Q_{1/2}} f \leq 1 - \delta$
- This decrease of the supremum bound follows from the *isoperimetric argument of De Giorgi*

De Giorgi's isoperimetric argument (1)

- Original statement is proved by constructive proof (direct calculation):

Theorem

Consider $f \in H^1$ on Q_2 with $f \leq 1$ and

$$\left| \left\{ f \geq \frac{1}{2} \right\} \cap Q_1 \right| \geq \delta_1 > 0 \quad \text{and} \quad |\{f \leq 0\} \cap Q_1| \geq \delta_2 > 0$$

then there is $\alpha > 0$ depending on δ_1, δ_2 and the H^1 norm so that

$$\left| \left\{ 0 < f < \frac{1}{2} \right\} \cap Q_1 \right| \geq \alpha$$

- Here no H^1 bound in all variables, and H^s with small $0 < s < 1/2$ seems insufficient
- We argue by contradiction for solutions to the equation

De Giorgi's isoperimetric argument (2)

Theorem ("Hypoelliptic version")

For all $\delta_1, \delta_2 > 0$ and $f \leq 1$ solution of our equation on Q_2 and

$$\left| \left\{ f \geq \frac{1}{2} \right\} \cap Q_1 \right| \geq \delta_1 \quad \textit{and} \quad |\{f \leq 0\} \cap Q_1| \geq \delta_2$$

there is $\alpha > 0$ depending on δ_1, δ_2 and the bounds on A so that

$$\left| \left\{ 0 < f < \frac{1}{2} \right\} \cap Q_1 \right| \geq \alpha$$

We consider a contradiction sequence f_k, A_k (the diffusion matrix must be let depending on k, in order to prove something universal and independent of scaling-zooming)

De Giorgi's isoperimetric argument (3)

- The sequence satisfies $\lambda \mathrm{Id} \le A_k \le \Lambda \mathrm{Id}$ and $f_k \le 1$ and

$$\left| \left\{ f_k \ge \frac{1}{2} \right\} \cap Q_1 \right| \ge \delta_1 \quad \text{and} \quad \left| \{ f_k \le 0 \} \cap Q_1 \right| \ge \delta_2$$

$$\left| \left\{ 0 < f_k < \frac{1}{2} \right\} \cap Q_1 \right| \xrightarrow[k \to \infty]{} 0$$

- The positive part f_k^+ is a subsolution bounded in L^2 and thus bounded in $L^2_{t,x} H^1_v \cap L^\infty_{t,x,v} \cap H^s_{t,x,v}$ with $\nabla_v f_k^+ \in L^2_{t,x} L^{2+\varepsilon}_v$ (previous results)
- Consider ζ smooth, $\zeta(z) = 0$ in $z \le 0$, $\zeta(z) = 1/2$ in $z \in [1/2, 1]$, then $g_k = \zeta(f_k)$ satisfies $0 \le g_k \le 1/2$ and $\left| \left\{ 0 < g_k < \frac{1}{2} \right\} \cap Q_1 \right| \to 0$
- g_k converges weakly $L^2_{t,x} H^1_v \cap H^s_{t,x,v}$ and strongly $L^2_{t,x,v}$ in Q_1
- And ζ- error term $\displaystyle\int \zeta''(f_k) A_k \nabla_v f_k \cdot \nabla_v f_k \phi \xrightarrow[k \to \infty]{} 0$ by the gain of integrability on $\nabla_v f_k^+ = 1_{f_k \ge 0} \nabla_v f_k$

De Giorgi isoperimetric argument (4)

- We have built a solution g valued in $\{0, 1/2\}$ with

$$\left| \left\{ g \geq \frac{1}{2} \right\} \cap Q_1 \right| \geq \delta_1 \quad \text{and} \quad |\{g \leq 0\} \cap Q_1| \geq \delta_2$$

- For almost every x, by the classical isoperimetric lemma in v (using the H^1_v control) it is constant in v, and $\nabla_v g = 0$
- The equation on g_k is

$$\partial_t g_k + v \cdot \nabla_x g_k = \nabla_v h_k \quad \text{with} \quad h_k := A_k \nabla_v g_k$$

- The equation on g is

$$\partial_t g + v \cdot \nabla_x g = \nabla_v h \quad \text{where} \quad h_k = A_k \nabla_v g_k \rightharpoonup_{L^2(Q_1)} h$$

- It remains to identify the product of weak limits $h_k = A_k \nabla_v g_k \rightharpoonup h$

De Giorgi isoperimetric argument (5)

- Integrating the equation on g_k against $g_k \phi$ we have

$$\lim_{k \to \infty} \int h_k \cdot \nabla_v (g_k \phi) = \int \frac{g^2}{2} (\partial_t \phi + v \cdot \nabla_x \phi)$$

- Integrating the limit equation on g against $g\phi$ we have

$$\int h \nabla_v (g\phi) = \int \frac{g^2}{2} (\partial_t \phi + v \cdot \nabla_x \phi)$$

- Moreover since $g_k \to g$ strongly in L^2 we have

$$\int g_k \left(h_k \cdot \nabla_v \phi \right) \to \int g \left(h \nabla_v \phi \right)$$

- Since $\nabla_v g = 0$ we deduce that $\int h_k \cdot (\nabla_v g_k) \phi \to 0$

De Giorgi isoperimetric argument (5)

- We have at the same time the coercivity

$$\int h_k(\nabla_v g_k)\phi = \int A_k \nabla_v g_k \cdot \nabla_v g_k \phi \geq \frac{1}{\Lambda} \int |h_k|^2 \phi$$

  and therefore $h = 0$
- Finally we end up with $\partial_t g + v \cdot \nabla_x g = 0$ with some mass at 0 and 1/2
- The free transport equation and $\nabla_v g = 0$ implies $\nabla_x g = 0$ and $\partial_t g = 0$, which reaches a contradiction
- Note that this proof is not quantitative and a quantitative proof would be interesting *per se*: tentative statement

$$\left| \left\{ 0 < f < \frac{1}{2} \right\} \cap Q_1 \right| \geq C \left| \{ f \leq 0 \} \cap Q_1 \right|^2 \left| \{ f \geq 1/2 \} \cap Q_1 \right|^{2 - \frac{2}{1+2d}}$$

  where C depends on $\|\nabla_v f\|_{L^2_{t,x,v}}$ and $\|\partial_t + v \cdot \nabla_x f\|_{L^2_{t,x} H^{-1}_v}$

From the isoperimetric estimate to the supremum bound (1)

- The $L^2 \to L^\infty$ gain implies: if the L^2 mass locally is below a certain threshold related to the constant in the gain $L^2 - L^\infty$, then the supremum has to be lower than $1/2$
- We zoom to a smaller cube and in the upper values as long as the mass there is not below the threshold: each time it is so, the intermediate region must contain some mass by the isoperimetric argument and the number of iteration is limited by the total volume of the cube
- In a finite number of iteration we must reach a smaller cube where the threshold condition is satisfied, and therefore where we diminish the upper bound

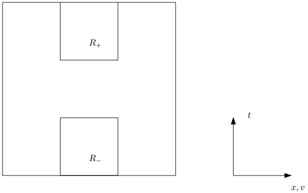
From the isoperimetric estimate to the supremum bound (2)

- Iteration: $f_1 = f$ and $f_{k+1} = 2f_k - 1$ which preserves $f_k \leq 1$
- We have $|\{f_1 \leq 0\} \cap Q_1| \geq \delta_1$ with $\delta_1 = |Q_1|/2$ and it propagates: $\{f_k \leq 0\} \subset \{f_{k+1} \leq 0\}$
- We prove that for some k_0 large enough (finite) then $|\{f_{k_0} \geq 0\} \cap Q_1| \leq \delta_2$ is small enough to get $\|f_{k_0}^+\|_{L^\infty(Q_{1/2})} \leq 1/2$ by applying the gain of integrability to $f_{k_0}^+$
- This means back on f: $f \leq 1 - 2^{-1-k_0}$ on $Q_{1/2}$
- k_0 exists since an amount α of mass necessary each time:
- As long as $|\{f_{k+1} \geq 0\} \cap Q_1| \geq \delta_2$ we have $|\{f_k \geq 1/2\} \cap Q_1| \geq \delta_2$
- With $|\{f_k \leq 0\} \cap Q_1| \geq \delta_1$ this implies $|\{0 \leq f_k \leq 1/2\} \cap Q_1| \geq \alpha$
- $|Q_1| \geq |\{f_{k+1} \leq 0\} \cap Q_1| \geq |\{f_k \leq 0\} \cap Q_1| + \underbrace{|\{0 \leq f_k \leq 1/2\} \cap Q_1|}_{\geq \alpha}$

Harnack inequality (in progress)

- Harnack inequalities were first inspired from observing fundamental solutions in simple cases (non-negative solutions)
- In the elliptic case: $\sup_B f \leq \gamma \min_B f$ for some universal constant γ
- In the parabolic case a time delay must be taken into account:

$$\sup_{R_+} f \leq \gamma \min_{R_-} f \qquad \text{for some universal constant } \gamma$$



- Two strategies: (1) Moser by "Poincaré inequality" (2) Equation + Hölder + Isoperimetric argument