METRIC VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS DEPENDING ON LOCAL SLOPES

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Abstract. We continue the study of viscosity solutions of Hamilton–Jacobi equations in metric spaces initiated in [37]. We present a more complete account of the theory of metric viscosity solutions based on local slopes. Several comparison and existence results are proved and the main techniques for such metric viscosity solutions are illustrated.

1. Introduction

The study of first order Hamilton–Jacobi–Bellman (HJB) equations in infinite dimensional Hilbert spaces or Banach spaces with the Radon-Nikodym property or a differentiable norm started several decades ago [7, 18, 19, 20, 22, 23, 25, 26]. However there is a need to go beyond these spaces as last decade has witnessed many studies connecting first order Hamilton–Jacobi equations on spaces of measures or more general metric spaces to several areas. Such equations arise for instance in statistical mechanics and large deviations [9, 10, 11, 12, 13, 14, 32, 33, 34], fluid mechanics [30, 33, 34, 35, 36, 38, 39, 40], Mean Field Games [16, 51, 52], optimal control [31, 33, 34], study of functional inequalities [6, 42, 43, 53] and other related areas [3, 4, 5, 17, 41]. Also the study of partially observed stochastic optimal control problems, rewritten in the form of optimal control of the Zakai equation, naturally leads to the investigation of second order HJB equations in the space of measures, and attempts in this direction have been made in [44, 46].

There is a substantial literature on HJB equations in Hilbert and Banach spaces. The reader may consult the book [7] for earlier results in Hilbert spaces covering mostly the case of regular solutions and connections with optimal control. M. G. Crandall and P-L. Lions introduced the theory of viscosity solutions in Hilbert and Banach spaces in a series of papers [18, 19, 20, 22, 23, 25, 26] for equations with bounded and unbounded terms, and later other notions of viscosity solution appeared, see for instance [15, 49, 56, 57], and the subject is relatively well established. There is also a well developed theory of viscosity solutions for second order HJB equations in Hilbert spaces.

Much less is known about equations in spaces of measures and general metric spaces. Several approaches and definitions of viscosity solution for special and more general HJB

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equations in the Wasserstein space have been proposed, see [1, 16, 17, 30, 31, 32, 33, 34, 36, 47, 52] and also a short discussion about them in the introduction of [37]. In [45] regular solutions for an equation in the space of probability measures were studied. As regards equations in metric spaces, Hamilton–Jacobi semigroups, and pointwise differential inequalities involving local slopes satisfied by them have been investigated in [3, 4, 5, 6, 42, 43, 53], mostly in connection with applications to various functional inequalities. A definition of metric viscosity solution was introduced in [41] for eikonal equation which can possibly be extended to a class equations which are similar in type to these considered in this paper. It looks at the behavior of functions along curves and it is substantially based on the sub- and super-optimality inequalities of dynamic programming. We introduced a different notion of metric viscosity solution in [37] which is based on local slopes and the use of appropriate test functions. The definition was influenced by the definition of strict viscosity solution in [20]. Similar in the spirit definition of metric viscosity solution appeared independently in [1] and well posedness of the equations was proved there. We compare our definition with that of [1] in Section 2. There is some overlap between [1] and this paper however the majority of the results here are different. During the revision of the manuscript we also learned about a preprint [54] in which another definition of a metric viscosity solution was introduced for a similar class of evolution equations. The definition of [54] is related to the definition of [41], it is based on the optimality inequalities of the dynamic programming principle and, like the one of [41], can be used in any metric space.

In this manuscript we have endeavored to develop a more complete theory of metric viscosity solutions began in [37] which encompasses a large class of Hamiltonians. We will consider time dependent problems

$$\partial_t u + H(t, x, u, |\nabla u|) = 0, \quad \text{in } (0,T)\times\Omega, \tag{1.1}$$

$$\begin{cases} u(t,x) = f(t,x) & \text{on } (0,T)\times\partial\Omega, \\ u(0,x) = g(x) & \text{on } \Omega, \end{cases} \tag{1.2}$$

and stationary equations

$$H(x, u, |\nabla u|) = 0, \quad \text{in } \Omega, \tag{1.3}$$

$$u(x) = f(x) \quad \text{on } \partial\Omega, \tag{1.4}$$

where $H : [0,T]\times\Omega\times\mathbb{R}\times[0,+\infty) \to \mathbb{R}$ is continuous, Ω is an open subset of a geodesic metric space, and $|\nabla u|$ is the local slope of u (see Definition 2.1). In [37] results were proved for equation (1.1)-(1.2) with special Hamiltonians of the form $H(|\nabla u|) + f(x)$ even though the techniques developed there would apply for more general equations. In the current manuscript we prove a range of comparison and existence results that apply to a wide range of equations and we present a sample of techniques that the reader can apply in other cases. Comparison theorems are proved for equations with Hamiltonians that are sublinear and superlinear in the local slope variable and some other more special cases, for instance for equations of eikonal type. We also prove a domain of dependence theorem. Existence of metric viscosity solutions is established for general equations by Perron's method and Perron's method together with approximation. In a particular case of equations with convex Hamiltonians associated with variational problems, existence of metric viscosity solutions is also established by showing directly that the value function is a metric viscosity solution (cf. Subsection 4.2). The results of this paper, together with these of [37], create a foundation for a theory of metric viscosity solutions that can be applied and expanded in various directions. We point out that after the basic definitions and techniques are properly set up, the methods are inspired by these in finite dimensional spaces [24, 8] or in infinite dimensional Hilbert or Banach spaces [18, 19, 20, 22, 23, 25, 26, 49]. Although it should be easy for the readers familiar with viscosity solutions to make a transition to the metric case, we have made an effort to include computations other readers may be unfamiliar with.

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2. Definitions

Throughout this manuscript, we assume that $(\mathbb{S}, d)$ is a complete metric space which is a geodesic space. A metric space is a geodesic space if for every $x, y \in \mathbb{S}$ there exists a geodesic of constant speed $\sigma : [0,1] \to \mathbb{S}$ connecting y and x, i.e. a curve such that

$$\sigma(0) = y, \sigma(1) = x,\ d(\sigma(s), \sigma(t)) = |s-t| d(x,y),\ 0 \le t \le s \le 1.$$

Let $T > 0$ and let $\Omega \subset \mathbb{S}$ be open.

Definition 2.1. *([2, 3, 42, 53]). Let $v : (0,T) \times \Omega \to \mathbb{R}$. The upper and lower local slopes of v at (t,x) are defined respectively by*

$$|\nabla^+ v(t,x)| := \limsup_{y \to x} \frac{[v(t,y) - v(t,x)]_+}{d(y,x)},\ |\nabla^- v(t,x)| := \limsup_{y \to x} \frac{[v(t,y) - v(t,x)]_-}{d(y,x)}. \tag{2.1}$$

The local slope of v at (t,x) is defined by

$$|\nabla v(t,x)| := \limsup_{y \to x} \frac{|v(t,y) - v(t,x)|}{d(y,x)}.$$

It is easy to see that $|\nabla^- v| = |\nabla^+(-v)|$.

For a function f defined on a subset of $Q \subset [0,T] \times \mathbb{S}$ (or $Q \subset \mathbb{S}$) we will write f^* to denote its upper semicontinuous envelope, and f_* to denote its lower semicontinuous envelope, i.e.

$$f^*(t,x) = \limsup_{(s,y)\to(t,x)} f(s,y), \quad f_*(t,x) = \liminf_{(s,y)\to(t,x)} f(s,y).$$

We will say that a function f is locally bounded in Q (or just locally bounded) if it is bounded on bounded subsets of Q. A function $\rho : [0,+\infty) \to [0,+\infty)$ is called a modulus if ρ is continuous, nondecreasing, subadditive, and $\rho(0) = 0$. If $\lim_{s\to\infty} \rho(s)/s < +\infty$ we can also assume without loss of generality that ρ is concave, and for every $\epsilon > 0$ there is a constant C_ϵ such that $\rho(s) \le \epsilon + C_\epsilon s, s \ge 0$.

We recall the definitions of viscosity solution of (1.1)-(1.2) and (1.3)-(1.4) from [37].

Definition 2.2. *A function $\psi : (0,T) \times \Omega \to \mathbb{R}$ is a subsolution test function ($\psi \in \underline{\mathcal{C}}$) if $\psi(t,x) = \psi_1(t,x) + \psi_2(t,x)$, where ψ_1, ψ_2 are Lipschitz on every bounded and closed subset of $(0,T) \times \Omega$, $|\nabla\psi_1(t,x)| = |\nabla^-\psi_1(t,x)|$ is continuous, and $\partial_t\psi_1, \partial_t\psi_2$ are continuous. A function $\psi : (0,T) \times \Omega \to \mathbb{R}$ is a supersolution test function ($\psi \in \overline{\mathcal{C}}$) if $-\psi \in \underline{\mathcal{C}}$.*

If the equation is time independent the test functions in $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$ are assumed to be time independent.

The following lemma is a simplified version of Lemma 7.2 of [37].

Lemma 2.3. *Let $\psi_1(t,x) = k(t) + k_1(t)\varphi(d(x,y))$, where $y \in \mathbb{S}$, $\varphi \in C^1([0,+\infty))$, $\varphi'(0) = 0, \varphi' \ge 0$, $k, k_1 \in C^1((0,T)), k_1 \ge 0$. Then*

$$|\nabla^-\psi_1(t,x)| = |\nabla\psi_1(t,x)| = k_1(t)\varphi'(d(x,y)).$$

In particular $|\nabla\psi_1(t,x)|$ is continuous and thus the function can be used as the ψ_1 part of a test function.

Proof. We have

$$\psi_1(t,z) - \psi_1(t,x) = k_1(t)\varphi'(d(x,y))(d(z,y) - d(x,y)) + o(d(z,y) - d(x,y)).$$

Therefore by triangle inequality and the fact that $k_1(t)\varphi'(d(x,y)) \ge 0$,

$$|\nabla\psi_1(t,x)| \le \limsup_{z\to x} k_1(t)\varphi'(d(x,y))\frac{d(z,x)}{d(z,x)} = k_1(t)\varphi'(d(x,y)).$$

If $x = y$ we thus have $|\nabla\psi_1(t,x)| = 0$ and the proof is finished. If $x \ne y$, let σ be a geodesic of constant speed connecting y and x, i.e. a curve such that $\sigma(0) = y, \sigma(1) = x$, $d(\sigma(s),\sigma(t)) = |s-t|d(x,y)$, $0 \le t \le s \le 1$. Then $d(\sigma(s),y) = sd(x,y), d(\sigma(s),x) = (1-s)d(x,y)$. Therefore

$$|\nabla^-\psi_1(t,x)| \ge \limsup_{s\to1} k_1(t)\varphi'(d(x,y))\frac{d(x,y) - d(\sigma(s),y)}{d(\sigma(s),x)}$$

$$= \lim_{s\to1} k_1(t)\varphi'(d(x,y))\frac{(1-s)d(x,y)}{(1-s)d(x,y)} = k_1(t)\varphi'(d(x,y)). \tag{2.2}$$

This proves the claim since $|\nabla^-\psi_1(t,x)| \leq |\nabla\psi_1(t,x)|$. ∎

For notational purposes we extend H to $s<0$ by setting

$$H(t,x,r,s) = H(t,x,r,0) \quad \text{for } s<0. \tag{2.3}$$

However the definition of a metric viscosity solution works for any continuous extension of H. We define for $\eta \geq 0$

$$H_\eta(t,x,r,s) := \inf_{|\tau-s|\leq\eta} H(t,x,r,\tau), \quad H^\eta(t,x,r,s) := \sup_{|\tau-s|\leq\eta} H(t,x,r,\tau).$$

Remark 2.4. *Suppose $a,b\geq 0$, $r_0,r_1\geq 0$ and $|b-a|\leq r_0+r_1$. Then there exists a number c between a and b such that $|c-a|\leq r_1$ and $|b-c|\leq r_0$. As a consequence*

$$H^{r_1}(t,x,r,a) \geq H(t,x,r,c) \geq H_{r_0}(t,x,r,b).$$

Definition 2.5. *A locally bounded upper semicontinuous function $u:[0,T)\times\overline{\Omega}\to\mathbb{R}$ is a metric viscosity subsolution of (1.1)-(1.2) if $u(t,x)\leq f(t,x)$ on $(0,T)\times\partial\Omega$, $u(0,x)\leq g(x)$ on Ω, and whenever $u-\psi$ has a local maximum at (t,x) for some $\psi\in\underline{\mathcal{C}}$, then*

$$\partial_t\psi(t,x) + H_{|\nabla\psi_2(t,x)|^*}(t,x,u(t,x),|\nabla\psi_1(t,x)|) \leq 0. \tag{2.4}$$

A locally bounded lower semicontinuous function $u:[0,T)\times\overline{\Omega}\to\mathbb{R}$ is a metric viscosity supersolution of (1.1)-(1.2) if $u(t,x)\geq f(t,x)$ on $(0,T)\times\partial\Omega$, $u(0,x)\geq g(x)$ on Ω, and whenever $u-\psi$ has a local minimum at (t,x) for some $\psi\in\overline{\mathcal{C}}$, then

$$\partial_t\psi(t,x) + H^{|\nabla\psi_2(t,x)|^*}(t,x,u(t,x),|\nabla\psi_1(t,x)|) \geq 0. \tag{2.5}$$

A continuous function $u:[0,T)\times\overline{\Omega}\to\mathbb{R}$ is a metric viscosity solution of (1.1)-(1.2) if it is both a metric viscosity subsolution and a metric viscosity supersolution of (1.1)-(1.2).

Definition 2.6. *A locally bounded upper semicontinuous function $u:\overline{\Omega}\to\mathbb{R}$ is a metric viscosity subsolution of (1.3)-(1.4) if $u(x)\leq f(x)$ on $\partial\Omega$, and whenever $u-\psi$ has a local maximum at x for some $\psi\in\underline{\mathcal{C}}$, then*

$$H_{|\nabla\psi_2(x)|^*}(x,u(x),|\nabla\psi_1(x)|) \leq 0. \tag{2.6}$$

A locally bounded lower semicontinuous function $u:\overline{\Omega}\to\mathbb{R}$ is a metric viscosity supersolution of (1.3)-(1.4) if $u(x)\geq f(x)$ on $\partial\Omega$, and whenever $u-\psi$ has a local minimum at x for some $\psi\in\overline{\mathcal{C}}$, then

$$H^{|\nabla\psi_2(x)|^*}(x,u(x),|\nabla\psi_1(x)|) \geq 0. \tag{2.7}$$

A continuous function $u:\overline{\Omega}\to\mathbb{R}$ is a metric viscosity solution of (1.3)-(1.4) if it is both a metric viscosity subsolution and a metric viscosity supersolution of (1.3)-(1.4).

Compared to the definition in [37] we added the requirement that metric viscosity subsolution/supersolutions be locally bounded. This is not essential but it allows to weaken some of the conditions on the Hamiltonian H. We will mostly work with $\Omega = \mathbb{S}$. We stated the definition of viscosity solution for equation (1.1)-(1.2) defined in $(0,T) \times \Omega$, however we may often need to use the notion of viscosity subsolution/supersolution in an open subset Q of $(0,T) \times \mathbb{S}$ without reference to initial and boundary conditions. We will then say that an upper/lower semicontinuous function $u : Q \to \mathbb{R}$ is a metric viscosity subsolution/supersolution of (1.1) in Q if (2.4)/(2.5) is satisfied whenever $u - \psi$ has a local maximum/minimum in Q. Initial and boundary conditions are disregarded in this case.

There are many similarities and several differences between our definition of metric viscosity solution and the one in [1]. The basic principle is similar however the approach is different. The authors in [1] take a very restrictive class of test functions, basically enough functions to prove comparison principle, and then define upper and lower versions of the Hamiltonian for the subsolution and the supersolution test functions. This seems to be rather cumbersome from the practical point of view, for instance if one wants to use a different perturbed optimization technique or use a different cut-off function one has to redefine the upper and lower Hamiltonians. Having few test functions is also restrictive. For instance the sum of two test functions in [1] is not a test function, and the product of a test function and a smooth function of time is not a test function. It would be impossible to prove the results of Section 3 without using test functions which are more general than these allowed in [1]. Many viscosity solution techniques require a wider class of test functions. This issue is rather technical but we think it is better to have a definition that allows for more flexibility in the use of test functions and is not tied to any specific form of them as in [1]. Another difference between the results of [1] and these in our paper is that in [1] the assumptions were introduced through the Lagrangian. Here, apart from Subsection 4.2, we introduce the assumptions directly through the Hamiltonian. This makes the conditions more transparent from the PDE point of view.

Definition 2.7. *We say that ψ is a strong metric subsolution of (1.1) in an open subset Q of $(0,T) \times \mathbb{S}$ if $\psi = \psi_1 + \psi_2 \in \overline{\mathcal{C}}$ and*

$$\partial_t \psi(t,x) + H^{|\nabla \psi_2(t,x)|^*}\big(t, x, \psi(t,x), |\nabla \psi_1(t,x)|\big) \le 0 \quad \textit{for } (t,x) \in Q. \tag{2.8}$$

We say that ψ is a strong metric supersolution of (1.1) in an open subset Q of $(0,T) \times \mathbb{S}$ if $\psi = \psi_1 + \psi_2 \in \underline{\mathcal{C}}$ and

$$\partial_t \psi(t,x) + H_{|\nabla \psi_2(t,x)|^*}\big(t, x, \psi(t,x), |\nabla \psi_1(t,x)|\big) \ge 0 \quad \textit{for } (t,x) \in Q. \tag{2.9}$$

Strong metric viscosity sub/super-solutions of time independent equations are defined similarly. We want to alert the reader that strong metric subsolutions are supersolution test functions and strong metric supersolutions are subsolution test functions, $H^{|\nabla\psi_2(t,x)|^*}$ is used in (2.8) instead of $H_{|\nabla\psi_2(t,x)|^*}$ in (2.4), and $H_{|\nabla\psi_2(t,x)|^*}$ is used in (2.9) instead of $H^{|\nabla\psi_2(t,x)|^*}$ in (2.5). Thus inequalities (2.4) and (2.5) are weaker than (2.8) and (2.9). Strong metric subsolutions and supersolutions will be used in Perron's method to construct metric viscosity subsolutions and supersolutions with the help of the following lemma.

Lemma 2.8. *If ψ is a strong metric subsolution (respectively, supersolution) of (1.1) in an open set $Q \subset (0,T) \times \mathbb{S}$ then it is a metric viscosity subsolution (respectively, supersolution) of (1.1) in Q.*

Proof. We will only do the proof for the subsolution case. Suppose that $\psi = \psi_1 + \psi_2 \in \overline{\mathcal{C}}$ satisfies (2.8) in Q and let $\psi - \tilde{\psi}$ have a local maximum at $(t,x) \in Q$ for some $\tilde{\psi} = \tilde{\psi}_1 + \tilde{\psi}_2 \in \underline{\mathcal{C}}$. Then $\partial_t \tilde{\psi}(t,x) = \partial_t \psi(t,x)$ and

$$|\nabla^+(\psi_1 - \tilde{\psi}_1)(t,x)| \le |\nabla^+(\tilde{\psi}_2 - \psi_2)(t,x)| \le |\nabla(\tilde{\psi}_2 - \psi_2)(t,x)|. \tag{2.10}$$

Therefore

$$\begin{aligned}|\nabla^+(\tilde{\psi}_2 - \psi_2)(t,x)| &\ge \limsup_{y\to x} \frac{[(\psi_1 - \tilde{\psi}_1)(t,y)) - (\psi_1 - \tilde{\psi}_1)(t,x))]_+}{d(y,x)} \\ &\ge \limsup_{y\to x} \frac{[\psi_1(t,y)) - \psi_1(t,x)]_+}{d(y,x)} - \limsup_{y\to x} \frac{|\tilde{\psi}_1(t,y) - \tilde{\psi}_1(t,x)|}{d(y,x)} \\ &= |\nabla^+\psi_1(t,x)| - |\nabla\tilde{\psi}_1(t,x)| = |\nabla\psi_1(t,x)| - |\nabla\tilde{\psi}_1(t,x)|.\end{aligned}$$

Likewise we obtain

$$\begin{aligned}|\nabla^+(\tilde{\psi}_2 - \psi_2)(t,x)| &\ge |\nabla^+(-\tilde{\psi}_1)(t,x)| - |\nabla\psi_1(t,x)| \\ &= |\nabla^-\tilde{\psi}_1(t,x)| - |\nabla\psi_1(t,x)| \\ &= |\nabla\tilde{\psi}_1(t,x)| - |\nabla\psi_1(t,x)|.\end{aligned}$$

It thus follows from the above two inequalities and (2.10) that

$$\Big|\, |\nabla\tilde{\psi}_1(t,x)| - |\nabla\psi_1(t,x)| \,\Big| \le |\nabla(\tilde{\psi}_2 - \psi_2)(t,x)| \le |\nabla\psi_2(t,x)| + |\nabla\tilde{\psi}_2(t,x)|$$

We apply Remark 2.4 with

$$a := |\nabla\psi_1(t,x)|, \quad b := |\nabla\tilde{\psi}_1(t,x)|, \quad r_1 := |\nabla\psi_2(t,x)|, \quad r_0 := |\nabla\tilde{\psi}_2(t,x)|$$

and use (2.8) to obtain

$$\partial_t\tilde{\psi}(t,x) + H_{|\nabla\tilde{\psi}_2(t,x)|^*}(t,x,\psi(t,x),|\nabla\tilde{\psi}_1(t,x)|) \le 0$$

which completes the proof. ∎

The following result is a combination of Ekeland's lemma [27] and a smooth perturbed optimization technique of [28, 55]. It follows from the proof of the Theorem on page 82, in [50]. The result in [50] was stated in a Banach space, however the proof is the same if a Banach space is replaced by a complete metric space. The statement of the lemma below is simpler than the statement of the corresponding Lemma 7.4 in [37] which was based on Borwein-Preiss variational principle, however each can be used to give the same results.

Lemma 2.9. *Let K be a real Hilbert space with norm $|\cdot|_K$, and let D be a bounded, closed subset of $\mathbb{S}\times K$. Let $\Phi : D \to [-\infty, +\infty)$ be upper semicontinuous, bounded from above and not be identically equal to $-\infty$. Then for every $\epsilon > 0$ there exist $(\bar{x}, \bar{y}) \in D$ and $p_\epsilon \in K, |p_\epsilon|_K < \epsilon$, such that*

$$\Phi(x, y) - \epsilon d(x, \bar{x}) + \langle p_\epsilon, y\rangle$$

has a maximum over D at $(\bar{x}, \bar{y})$, and

$$\Phi(\bar{x}, \bar{y}) > \sup_{\Omega} \Phi - \epsilon.$$

Remark 2.10. *We remark how the theory of metric viscosity solutions and the results of this paper can be extended to metric spaces which are length spaces. We only show how to obtain Lemma 2.3 which gives enough test functions. Having Lemma 2.3, all the proofs in Sections 3, 5 and Subsection 4.1 are the same. Only proofs in Subsection 4.2 need some small technical changes, similar in the spirit to the changes needed to adjust the proof of Lemma 2.3 explained below.*

Let $(\mathbb{S}, d)$ be a complete metric space. Recall that the length of a curve $\sigma : [a, b] \to \mathbb{S}$ is defined by

$$l(\sigma) := \sup \sum_{i=1}^{k} d(\sigma(t_i), \sigma(t_{i-1})),$$

where the supremum is taken over all $k \in \mathbb{N}$ and $a = t_0 \le t_1 \le ... \le t_k = b$. The metric space $(\mathbb{S}, d)$ is called a length space if for every $x, y \in \mathbb{S}$

$$d(x, y) = \inf l(\sigma),$$

where the infimum is taken over all curves of finite length $\sigma : [0, 1] \to \mathbb{S}$ such that $\sigma(0) = y, \sigma(1) = x$.

So let $(\mathbb{S}, d)$ be a length space, $x, y \in \mathbb{S}$ and ψ_1 be as in Lemma 2.3. To prove Lemma 2.3 in this case we need to explain how to obtain (2.2). Let then $x \neq y$ and let σ^n be a sequence of paths such that $\sigma^n(0) = y, \sigma^n(1) = x$, and

$$d(x, y) \le l(\sigma^n) \le d(x, y) + \frac{1}{n^2}.$$

We can assume that all paths σ^n have constant speed parametrization, i.e. that

$$l\left(\sigma^n_{|[t,s]}\right) = (s-t)l(\sigma^n) \quad \textit{for all } 0 \le t < s \le 1.$$

Set $z_n = \sigma^n(1-1/n)$. Then

$$d(z_n, x) \le l\left(\sigma^n_{|[1-1/n,1]}\right) = \frac{1}{n}l(\sigma^n) \le \frac{1}{n}d(x,y) + \frac{1}{n^3}, \tag{2.11}$$

$$d(z_n, y) \le l\left(\sigma^n_{|[0,1-1/n]}\right) = \left(1-\frac{1}{n}\right)l(\sigma^n) \le \left(1-\frac{1}{n}\right)d(x,y) + \frac{n-1}{n^3}. \tag{2.12}$$

Therefore, using (2.11) and (2.12), we now have in place of (2.2)

$$\begin{aligned} |\nabla^-\psi_1(t,x)| &\ge \limsup_{n\to\infty} k_1(t)\varphi'(d(x,y))\frac{d(x,y)-d(z_n,y)}{d(z_n,x)} \\ &\ge \lim_{n\to\infty} k_1(t)\varphi'(d(x,y))\frac{d(x,y)-(1-\frac{1}{n})d(x,y)-\frac{n-1}{n^3}}{\frac{1}{n}d(x,y)+\frac{1}{n^3}} = k_1(t)\varphi'(d(x,y)). \end{aligned} \tag{2.13}$$

3. General Hamiltonians with Sublinear Growth

We will only study the case $\Omega = \mathbb{S}$. Let x_0 be a fixed element of $\mathbb{S}$. The Hamiltonian H was initially defined on $(0,T)\times\mathbb{S}\times\mathbb{R}\times[0,+\infty)$ and then extended by (2.3). Here we assume from the beginning that H is defined on $(0,T)\times\mathbb{S}\times\mathbb{R}\times\mathbb{R}$. Thus all the results are true for possibly other extensions of the original H. We make the following assumptions.

- $(A1)$ H is uniformly continuous on bounded subsets of $(0,T)\times\mathbb{S}\times\mathbb{R}\times\mathbb{R}$.
- $(A2)$ There exists $\nu \ge 0$ such that for every $(t,x,r_1,r_2,s) \in (0,T)\times\mathbb{S}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$
$$H(t,x,r_1,s) - H(t,x,r_2,s) \ge \nu(r_1-r_2) \quad \text{if } r_1 \ge r_2.$$
- $(A3)$ For every $R>0$ there is a modulus ω_R such that for every $(t,r,s) \in (0,T)\times\mathbb{R}\times\mathbb{R}, x,y\in\mathbb{S}$
$$|H(t,x,r,s)-H(t,y,r,s)| \le \omega_R\Big(d(x,y)(1+|s|)\Big) \quad \text{if } \max(d(x,x_0),d(y,x_0),|r|) \le R.$$
- $(A4)$ There is $L \ge 0$ such that for every $(t,x,r,s,\tau) \in (0,T)\times\mathbb{S}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$
$$|H(t,x,r,s)-H(t,x,r,\tau)| \le L\Big(1+d(x,x_0)\Big)|s-\tau|.$$
- $(A5)$ For every $R>0$ there is a modulus σ_R such that
$$|g(x)-g(y)| \le \sigma_R(d(x,y)) \quad \text{if } \max(d(x,x_0),d(y,x_0)) \le R.$$

Remark 3.1. *Observe that by $(A4)$*

$$|H(t,x,r,s)-H_\eta(t,x,r,s)|,\ |H(t,x,r,s)-H^\eta(t,x,r,s)| \le L\Big(1+d(x,x_0)\Big)\eta.$$

3.1. **Preliminaries.** We recall a technique that will be used in many proofs (see also [49]). Suppose $(A1-A5)$ are satisfied. Let u be a metric viscosity subsolution of (1.1) and v be a metric viscosity supersolution of (1.1) such that there are constants $c, \bar{k} \geq 0$ such that

$$u(t,x), -v(t,x) \leq c(1+d^{\bar{k}}(x,x_0)), \quad \text{for all } t \in [0,T), x \in \mathbb{S}. \tag{3.1}$$

Fix $\mu > 0$, $k > \bar{k}$, $k \geq 1$ and set $M = Lk+1$. We define

$$\Psi_0(t,s,x,y) = u(t,x) - v(s,y) - \frac{\mu}{T-t} - \frac{\mu}{T-s},$$

for $t, s \in [0,T)$ and $x, y \in \mathbb{S}$. For these variables and for $\delta, \epsilon, \beta > 0$ we define

$$\Psi_\delta(t,s,x,y) = \Psi_0(t,s,x,y) - \delta\Big(e^{Mt} d^k(x,x_0) + e^{Ms} d^k(y,x_0)\Big),$$

$$\Psi_{\delta,\epsilon}(t,s,x,y) = \Psi_\delta(t,s,x,y) - \frac{d^2(x,y)}{2\epsilon}, \quad \Psi_{\delta,\epsilon,\beta}(t,s,x,y) = \Psi_{\delta,\epsilon}(t,s,x,y) - \frac{(t-s)^2}{2\beta}.$$

For $\eta, \gamma, R > 0$ we consider the sets

$$\begin{aligned} \mathcal{E}_\eta &= \{(t,s,x,y) \in [0,T)^2 \times \mathbb{S}^2 : \ \ |t-s| < \eta\} \\ \mathcal{E}_{\eta,\gamma} &= \{(t,s,x,y) \in \mathcal{E}_\eta : \ d(x,y) < \gamma\} \\ \mathcal{E}_{\eta,\gamma,R} &= \{(t,s,x,y) \in \mathcal{E}_{\eta,\gamma} : \ d(x,x_0) + d(y,x_0) \leq R\}. \end{aligned}$$

We set

$$\begin{aligned} m &:= \lim_{R\to+\infty} \lim_{\gamma\to 0} \lim_{\eta\to 0} \sup \Big\{ \Psi_0(t,s,x,y) : (t,s,x,y) \in \mathcal{E}_{\eta,\gamma,R} \Big\} \in (\infty, \infty] \\ m_\delta &:= \lim_{\gamma\to 0} \lim_{\eta\to 0} \sup_{\mathcal{E}_{\eta,\gamma}} \Psi_\delta(t,s,x,y) \\ m_{\delta,\epsilon} &:= \lim_{\eta\to 0} \sup_{\mathcal{E}_\eta} \Psi_{\delta,\epsilon}, \qquad m_{\delta,\epsilon,\beta} := \sup_{[0,T)^2\times\mathbb{S}^2} \Psi_{\delta,\epsilon,\beta}. \end{aligned} \tag{3.2}$$

Remark 3.2. *The following hold:*

$$m = \lim_{\delta\to 0} m_\delta, \tag{3.3}$$

$$m_\delta = \lim_{\epsilon\to 0} m_{\delta,\epsilon}, \tag{3.4}$$

$$m_{\delta,\epsilon} = \lim_{\beta\to 0} m_{\delta,\epsilon,\beta}. \tag{3.5}$$

Proof. We only prove (3.3) and (3.4) and omit the proof of (3.5) which is similar to that of (3.4). We have

$$\sup_{\mathcal{E}_{\eta,\gamma}} \Psi_\delta \geq \sup_{\mathcal{E}_{\eta,\gamma,R}} \Psi_\delta \geq \sup_{\mathcal{E}_{\eta,\gamma,R}} \Psi_0 - 2R^k \delta e^{MT}.$$

Thus,

$$\lim_{\delta\to 0} m_\delta \geq \lim_{\delta\to 0} \lim_{\gamma\to 0} \lim_{\eta\to 0} \sup_{\mathcal{E}_{\eta,\gamma,R}} (\Psi_0 - 2R^k \delta e^{MT}) = \lim_{\gamma\to 0} \lim_{\eta\to 0} \sup_{\mathcal{E}_{\eta,\gamma,R}} \Psi_0. \tag{3.6}$$

Letting R tend to $+\infty$ in (3.6) we obtain $m \leq \lim_{\delta\to 0} m_\delta$. Moreover, because of (3.1), for every $\delta > 0$ there is R^δ such that $\lim_{\delta\to 0} R^\delta = +\infty$ and

$$\sup_{\mathcal{E}_{\eta,\gamma}} \Psi_\delta = \sup_{\mathcal{E}_{\eta,\gamma,R^\delta}} \Psi_\delta.$$

Therefore

$$\lim_{\delta\to 0} m_\delta = \lim_{\delta\to 0}\lim_{\gamma\to 0}\lim_{\eta\to 0} \sup_{\mathcal{E}_{\eta,\gamma,R^\delta}} \Psi_\delta \leq \lim_{\delta\to 0}\lim_{\gamma\to 0}\lim_{\eta\to 0} \sup_{\mathcal{E}_{\eta,\gamma,R^\delta}} \Psi_0 = m.$$

We have

$$\sup_{\mathcal{E}_{\eta,\gamma}} \Psi_\delta = \sup_{\mathcal{E}_{\eta,\gamma}} \Big\{ \frac{d^2(x,y)}{2\epsilon} + \Psi_{\delta,\epsilon} \Big\} \leq \frac{\gamma^2}{2\epsilon} + \sup_{\mathcal{E}_{\eta,\gamma}} \Psi_{\delta,\epsilon} \leq \frac{\gamma^2}{2\epsilon} + \sup_{\mathcal{E}_\eta} \Psi_{\delta,\epsilon}.$$

Hence,

$$m_\delta = \lim_{\gamma\to 0}\lim_{\eta\to 0}\sup_{\mathcal{E}_{\eta,\gamma}} \Psi_\delta \leq \lim_{\gamma\to 0}\lim_{\eta\to 0}\Big(\frac{\gamma^2}{2\epsilon} + \sup_{\mathcal{E}_\eta} \Psi_{\delta,\epsilon}\Big) = m_{\delta,\epsilon}, \tag{3.7}$$

which implies $m_\delta \leq \lim_{\epsilon\to 0} m_{\delta,\epsilon}$.

We fix $\delta > 0$. For $\eta, \epsilon > 0$, let $(t^\epsilon_\eta, s^\epsilon_\eta, x^\epsilon_\eta, y^\epsilon_\eta)$ be points in $[0,T)^2 \times \mathbb{S}^2$ such that $|t^\epsilon_\eta - s^\epsilon_\eta| < \eta$ and

$$\sup_{\mathcal{E}_\eta} \Psi_{\delta,\epsilon} \leq \eta + \Psi_{\delta,\epsilon}(t^\epsilon_\eta, s^\epsilon_\eta, x^\epsilon_\eta, y^\epsilon_\eta). \tag{3.8}$$

Because of (3.1) and (3.8) it is clear that we must have $d(x^\epsilon_\eta, y^\epsilon_\eta) < C_\delta\sqrt{\epsilon}$ for some constant C_δ. Therefore, by (3.8),

$$\begin{aligned} m_\delta &= \lim_{\gamma\to 0}\lim_{\eta\to 0}\sup_{\mathcal{E}_{\eta,\gamma}} \Psi_\delta = \lim_{\epsilon\to 0}\lim_{\eta\to 0}\sup_{\mathcal{E}_{\eta,C_\delta\sqrt{\epsilon}}} \Psi_\delta \\ &\geq \lim_{\epsilon\to 0}\lim_{\eta\to 0}\Psi_{\delta,\epsilon}(t^\epsilon_\eta, s^\epsilon_\eta, x^\epsilon_\eta, y^\epsilon_\eta) \geq \lim_{\epsilon\to 0}\lim_{\eta\to 0}(\sup_{\mathcal{E}_\eta}\Psi_{\delta,\epsilon} - \eta) = \lim_{\epsilon\to 0} m_{\delta,\epsilon}. \end{aligned}$$

∎

3.2. **Time dependent problems.**

Proposition 3.3 (A comparison principle)**.** *Let $(A1-A5)$ be satisfied. Let u be a metric viscosity subsolution of (1.1) and v be a metric viscosity supersolution of (1.1)-(1.2) such that (3.1) holds and*

$$\lim_{t\to 0}([u(t,x) - g(x)]_+ + [v(t,x) - g(x)]_-) = 0 \tag{3.9}$$

uniformly on bounded subsets of $\mathbb{S}$. Then for every $0 < T_1 < T$

$$\lim_{R\to+\infty}\lim_{\gamma\to 0}\lim_{\eta\to 0}\sup\Big\{u(t,x) - v(s,y) : (t,s,x,y) \in \mathcal{E}_{\eta,\gamma,R},\ 0 \leq t,s < T_1\Big\} \leq 0. \tag{3.10}$$

In particular $u \leq v$.

Proof. Let $k > \bar{k}, k \geq 1$, and let $M = Lk + 1$. Suppose that (3.10) is not true. Then for sufficiently small $\mu > 0$, defining m as in (3.2), we have $m > 0$.

We notice that there is a constant c_1 (depending only on L, k) such that

$$Mr^k - Lk(1+r)r^{k-1} \geq c_1, \quad \text{for all } r \geq 0. \tag{3.11}$$

Moreover if $\tilde{\psi}(t,x) = e^{Mt}d^k(x,x_0)$, we have

$$|\nabla\tilde{\psi}(t,x)|^* = ke^{Mt}d^{k-1}(x,x_0). \tag{3.12}$$

In the sequel, we use the function Ψ in place of $\Psi_{\delta,\epsilon,\beta}$ and we set $\Psi(t,s,x,y) = -\infty$ if either $t = T$ or $s = T$. Since $\Psi(t,s,x,y) \to -\infty$ as $\min(d(x,x_0), d(y,x_0)) \to +\infty$, uniformly for $t, s \in [0,T]$ and ϵ, β, using Lemma 2.9 in $[0,T] \times [0,T] \times \bar{B}_{R_\delta}(x_0) \times \bar{B}_{R_\delta}(x_0)$ for big enough R_δ, for every $n \geq 1$ there are $a_n, b_n \in \mathbb{R}, |a_n| + |b_n| \leq \frac{1}{n}$, and $(\bar{t}, \bar{s}, \bar{x}, \bar{y}) \in [0,T] \times [0,T] \times \bar{B}_{R_\delta}(x_0) \times \bar{B}_{R_\delta}(x_0)$ such that

$$\Psi(t,s,x,y) + a_n t + b_n s - \frac{1}{n}(d(x,\bar{x}) + d(y,\bar{y}))$$

has a global maximum over $[0,T] \times [0,T] \times \mathbb{S} \times \mathbb{S}$ at $(\bar{t}, \bar{s}, \bar{x}, \bar{y})$. Thus if δ is fixed, for some constant $\tilde{R}_\delta$, independent of ϵ, β, n, we have

$$d(\bar{x},x_0),\ d(\bar{y},x_0),\ |u(\bar{t},\bar{x})|,\ |v(\bar{s},\bar{y})| \leq \tilde{R}_\delta. \tag{3.13}$$

Moreover we have

$$m_{\delta,\epsilon,\beta} \leq \Psi(\bar{t},\bar{s},\bar{x},\bar{y}) + \frac{1}{n}. \tag{3.14}$$

We can then conclude

$$m_{\delta,\epsilon,\beta} + \frac{(\bar{t}-\bar{s})^2}{4\beta} \leq \Psi(\bar{t},\bar{s},\bar{x},\bar{y}) + \frac{(\bar{t}-\bar{s})^2}{4\beta} + \frac{1}{n} \leq m_{\delta,\epsilon,2\beta} + \frac{1}{n} \tag{3.15}$$

and

$$m_{\delta,\epsilon,\beta} + \frac{d^2(\bar{x},\bar{y})}{4\epsilon} + \frac{(\bar{t}-\bar{s})^2}{4\beta} \leq m_{\delta,2\epsilon,2\beta} + \frac{1}{n}. \tag{3.16}$$

It thus follows from (3.15) and (3.5) that

$$\lim_{\beta\to 0} \limsup_{n\to+\infty} \frac{(\bar{t}-\bar{s})^2}{\beta} = 0 \quad \text{for every } \delta, \epsilon > 0. \tag{3.17}$$

Moreover (3.4), (3.16) and (3.17) imply

$$\lim_{\epsilon\to 0} \limsup_{\beta\to 0} \limsup_{n\to\infty} \frac{d^2(\bar{x},\bar{y})}{\epsilon} = 0 \quad \text{for every } \delta > 0. \tag{3.18}$$

It now follows from (3.3), (3.4), (3.5), (3.14), (3.17) and (3.18) that there are $\tilde{m} > 0, \delta_0 > 0$ such that for $\delta < \delta_0$

$$\liminf_{\epsilon\to 0} \liminf_{\beta\to 0} \liminf_{n\to\infty} (u(\bar{t},\bar{x}) - v(\bar{s},\bar{y})) \geq \tilde{m} > 0. \tag{3.19}$$

Thus (3.19), together with (3.9), (3.17), (3.18) and $(A5)$, implies that if δ, ϵ, β are small enough and n is sufficiently big we must have $0 < \bar{t}, \bar{s} < T$.

Setting $\psi = \psi_1 + \psi_2$ where

$$\psi_1(t,x) = \frac{d^2(x,\bar{y})}{2\epsilon}, \quad \psi_2(t,x) = -a_n t + \frac{1}{n} d(x,x_0) + \frac{\mu}{T-t} + \delta e^{Mt} d^k(x,x_0) + \frac{(t-\bar{s})^2}{2\beta},$$

we have $\psi \in \underline{\mathcal{C}}$ and

$$|\nabla \psi_1|(t,x) = \frac{d(x,\bar{y})}{\epsilon}, \quad |\nabla \psi_2|^*(t,x) \leq \delta |\nabla \tilde{\psi}_2(t,x)|^* + \frac{1}{n} = \delta k e^{Mt} d^{k-1}(x,x_0) + \frac{1}{n}. \tag{3.20}$$

Thus, by the definition of metric viscosity subsolution and the maximality property of $u - \psi$ at $(\bar{t}, \bar{x})$,

$$\frac{\mu}{(T-\bar{t})^2} + \frac{\bar{t}-\bar{s}}{\beta} + \delta M e^{M\bar{t}} d^k(\bar{x},x_0) - a_n + H_{|\nabla \psi_2(\bar{t},\bar{x})|^*}\left(\bar{t}, \bar{x}, u(\bar{t},\bar{x}), \frac{d(\bar{x},\bar{y})}{\epsilon}\right) \leq 0.$$

This, together with Remark 3.1, (3.11) and (3.20) yields

$$\begin{aligned}
&\frac{\mu}{T^2} + \frac{\bar{t}-\bar{s}}{\beta} + H\left(\bar{t}, \bar{x}, \bar{u}, \frac{\bar{d}}{\epsilon}\right) \\
&\quad \leq H\left(\bar{t}, \bar{x}, \bar{u}, \frac{\bar{d}}{\epsilon}\right) - H_{\delta|\nabla \tilde{\psi}(\bar{t},\bar{x})|^* + \frac{1}{n}}\left(\bar{t}, \bar{x}, \bar{u}, \frac{\bar{d}}{\epsilon}\right) - \delta M e^{M\bar{t}} d^k(\bar{x},x_0) + a_n \\
&\quad \leq L\big(1 + d(\bar{x},x_0)\big)\left(\delta k e^{M\bar{t}} d^{k-1}(\bar{x},x_0) + \frac{1}{n}\right) - \delta M e^{M\bar{t}} d^k(\bar{x},x_0) + a_n \\
&\quad = a_n + \frac{L}{n}\big(1 + d(\bar{x},x_0)\big) + \delta e^{M\bar{t}} d^{k-1}(\bar{x},x_0)(Lk(1 + d(\bar{x},x_0)) - M d(\bar{x},x_0)) \\
&\quad \leq \frac{1}{n} + \frac{L}{n}\big(1 + \tilde{R}_\delta\big) + |c_1| \delta e^{MT} =: \rho_1(\delta, n),
\end{aligned} \tag{3.21}$$

where, we have set

$$\bar{u} := u(\bar{t},\bar{x}), \ \bar{d} = d(\bar{x},\bar{y}).$$

We notice that

$$\lim_{\delta \to 0} \lim_{n \to +\infty} \rho_1(\delta; n) = 0.$$

Similarly, since v is a metric viscosity subsolution,

$$\frac{\mu}{T^2} - \frac{\bar{t}-\bar{s}}{\beta} - H\left(\bar{s}, \bar{y}, \bar{v}, \frac{\bar{d}}{\epsilon}\right) \leq \rho_1(\delta, n). \tag{3.22}$$

It follows from $(A1)$ and (3.13) that

$$\left| H\left(\bar{s}, \bar{y}, \bar{v}, \frac{\bar{d}}{\epsilon}\right) - H\left(\bar{t}, \bar{y}, \bar{v}, \frac{\bar{d}}{\epsilon}\right) \right| \leq \rho_2(\delta, \epsilon; \beta, n), \tag{3.23}$$

where

$$\lim_{\beta \to 0} \limsup_{n \to +\infty} \rho_2(\delta, \epsilon; \beta, n) = 0.$$

Therefore, (3.19), (3.22) and (3.23), give

$$\frac{\mu}{T^2} - \frac{\bar{t}-\bar{s}}{\beta} - H\left(\bar{t},\bar{y},\bar{u},\frac{\bar{d}}{\epsilon}\right) \le \rho_1(\delta,n)+\rho_2(\delta,\epsilon;\beta,n) \tag{3.24}$$

if $\delta<\delta_0$, and $\epsilon,\beta,1/n$ are small enough.

Adding (3.21) to (3.24) and using $(A3)$ we obtain for such δ, ϵ, β, n

$$\begin{aligned} \frac{2\mu}{T^2} &\le H\left(\bar{t},\bar{y},\bar{u},\frac{\bar{d}}{\epsilon}\right) - H\left(\bar{t},\bar{x},\bar{u},\frac{\bar{d}}{\epsilon}\right) + 2\rho_1(\delta,n)+\rho_2(\delta,\epsilon;\beta,n) \\ &\le \omega_{\tilde{R}_\delta}\left(d(\bar{x},\bar{y})+\frac{d^2(\bar{x},\bar{y})}{\epsilon}\right)+2\rho_1(\delta,n)+\rho_2(\delta,\epsilon;\beta,n). \end{aligned} \tag{3.25}$$

It remains to take $\lim_{\delta\to0}\limsup_{\epsilon\to0}\limsup_{\beta\to0}\limsup_{n\to+\infty}$ in (3.25) and use (3.18) to obtain a contradiction. ∎

Theorem 3.4 (Existence and uniqueness for bounded initial value functions)**.** *Let $(A1)-(A5)$ be satisfied, let g be bounded, and let for every $M>0$*

$$\sup\{|H(t,x,r,0)| : (t,x)\in(0,T)\times\mathbb{S}, |r|\le M\} = K_M < +\infty. \tag{3.26}$$

Then there exists a unique bounded metric viscosity solution u of (1.1)-(1.2) satisfying

$$\lim_{t\to0}|u(t,x)-g(x)|=0 \tag{3.27}$$

uniformly on bounded subsets of $\mathbb{S}$. The solution u is uniformly continuous on bounded subsets of $[0,T)\times\mathbb{S}$.

Proof. We need to produce a metric viscosity subsolution and a a metric viscosity supersolution of (1.1). The existence of a metric viscosity solution is then obtained by Perron's method which was established in [37] for a slightly less general equation. However the method and its proof remain the same for the current case and therefore will not reproduce it here.

We explain how to show the existence of a supersolution. Given $C>0$, using $(A2)$ we have

$$C+H(t,x,Ct+\|g\|_\infty,0)\ge C+H(t,x,0,0)\ge C-K_0.$$

In light of (3.26) and Lemma 2.8 we conclude that if $C>0$ is large enough, the function $w(t,x)=Ct+\|g\|_\infty$ is a metric viscosity supersolution of (1.1). Let $R>0$ and let σ_R be the modulus of continuity of g on $\{x : d(x,x_0)\le R+1\}$. For $\epsilon>0$ let $a_{\epsilon,R}>0$ be such that $\sigma_R(s)\le\epsilon+a_{\epsilon,R}s$, $s\ge0$ and $a_{\epsilon,R}>CT+2\|g\|_\infty$. Then for every $y\in\mathbb{S}$ such that $d(y,x_0)\le R$,

$$g(y)+\epsilon+a_{\epsilon,R}d(x,y)\ge g(x)\ x\in\mathbb{S},\quad g(y)+\epsilon+a_{\epsilon,R}d(x,y)>w(t,x)\ t\in(0,T), d(x,y)\ge1.$$

Therefore it follows from $(A2)$, $(A4)$, (3.26) and Lemma 2.8 that there are constants $C_{\epsilon,R}>0$ such that if $d(y,x_0)\le R$ then the function $w_{\epsilon,y}(t,x):=C_{\epsilon,R}t+g(y)+\epsilon+a_{\epsilon,R}d(x,y)$ satisfies

$$w(0,x)\ge g(x)\ x\in\mathbb{S},\quad w_{\epsilon,y}(t,x)>w(t,x)\ t\in(0,T), d(x,y)\ge1,$$

and the function $w_{\epsilon,y}$ is a strong metric (and hence viscosity) supersolution of (1.1) in $(0,T)\times\{x : d(x,y) < 1\}$. Then the function $\overline{w}_{\epsilon,y} := \min(w, w_{\epsilon,y})$ is a metric viscosity supersolution of (1.1)-(1.2) in $[0,T)\times\mathbb{S}$. Repeating the proof of Step 1 of Theorem 7.6 of [37] it is now easy to see that the function

$$\overline{u}(t,x) := \left(\inf\{\overline{w}_{\epsilon,y}(t,x) : \epsilon > 0, y \in \mathbb{S}\}\right)_*$$

is a bounded metric viscosity supersolution of (1.1) which satisfies

$$\lim_{t\to 0}[(\overline{u})^*(t,x) - g(x)]_+ = 0$$

uniformly on bounded subsets of $\mathbb{S}$. A bounded metric viscosity subsolution $\underline{u}$ of (1.1)-(1.2) satisfying

$$\lim_{t\to 0}[(\underline{u})_*(t,x) - g(x)]_- = 0$$

uniformly on bounded subsets of $\mathbb{S}$ is constructed by the same arguments applied to the subsolution case. We remind that by Proposition 3.3, comparison principle holds for bounded (in fact polynomially growing) sub- and supersolutions of (1.1)-(1.2) satisfying 3.9. It follows from construction and comparison that $\underline{u} \le (\overline{u})^*$ and $(\underline{u})_* \le \overline{u}$ and hence comparison ensures that $\underline{u} \le \overline{u}$. Therefore, by Perron's method, the function

$$u(t,x) = \sup\{v(t,x) : v \text{ is a metric viscosity subsolution of (1.1)}, \underline{u} \le v \le \overline{u}\}$$

is the unique bounded metric viscosity solution of (1.1)-(1.2). The uniform continuity of u on bounded subsets of $[0,T)\times\mathbb{S}$ follows from (3.10) applied with $v = u$. ■

The next theorem is a domain of dependence type result. Results of this type are known in Euclidean spaces, see Lemma VI.1 of [21], and in Hilbert spaces, see [22]. Our proof is an adaptation of the proof of Lemma VI.1 of [21] to the metric case. For $R > 0$ we denote

$$\Delta_R := \{(t,x) \in (0,T)\times\mathbb{S} : d(x,x_0) < Re^{-Lt} - 1\}.$$

Theorem 3.5 (Domain dependence and a comparison principle)**.** *Let $(A1)-(A4)$ be satisfied. Let u be a metric viscosity subsolution of (1.1)-(1.2) with initial condition g_1 and v be a metric viscosity supersolution of (1.1)-(1.2) with initial condition g_2. Let g_1, g_2 satisfy $(A5)$ and let*

$$\lim_{t\to 0}\left([u(t,x) - g_1(x)]_+ + [v(t,x) - g_2(x)]_-\right) = 0 \tag{3.28}$$

uniformly on bounded subsets of $\mathbb{S}$. Then for every $R > 0$

$$\sup_{\Delta_R}(u - v) \le \sup\{[g_1(x) - g_2(x)]_+ : d(x,x_0) < R - 1\}. \tag{3.29}$$

In particular, if $(A1)-(A5)$ are true then there is at most one viscosity solution of (1.1)-(1.2) among functions in $C([0,T)\times\mathbb{S})$ satisfying (3.27) uniformly on bounded subsets of $\mathbb{S}$.

Proof. Let $R > 0$. We define for $m \geq 1, h > 0$ the functions

$$w_m(t,x) := \exp\Big(m\Big[e^{Lt}\big(1 + d(x,x_0)\big) - R\Big]\Big)$$

and the sets

$$\Delta_R^h := \{(t,x) \in (0,T) \times \mathbb{S} : d(x,x_0) < (R+h)e^{-Lt} - 1\}.$$

We notice that

$$\lim_{m\to+\infty} w_m(t,x) = 0 \quad \text{for every } (t,x) \in \Delta_R. \tag{3.30}$$

We consider for $\mu > 0, \epsilon > 0, \beta > 0$ the function

$$\Psi(t,s,x,y) = u(t,x) - v(s,y) - \frac{\mu}{T-t} - \frac{\mu}{T-s} - w_m(t,x) - w_m(s,y) - \frac{d^2(x,y)}{2\epsilon} - \frac{(t-s)^2}{2\beta}$$

on $\overline{\Delta}_R^h \times \overline{\Delta}_R^h$, and we set $\Psi(t,s,x,y) = -\infty$ if either $t = T$ or $s = T$. Since u and v are locally bounded, the function ψ is bounded above on $\overline{\Delta}_R^h \times \overline{\Delta}_R^h$. Furthermore, it is upper semicontinuous there, and so by Lemma 2.9, for every $n \geq 1$ there are $a_n, b_n \in \mathbb{R}, |a_n| + |b_n| \leq \frac{1}{n}$, and $(\bar{t}, \bar{s}, \bar{x}, \bar{y}) \in \overline{\Delta}_R^h \times \overline{\Delta}_R^h$ such that

$$\tilde{\Psi}(t,s,x,y) := \Psi(t,s,x,y) + a_n t + b_n s - \frac{1}{n}(d(x,\bar{x}) + d(y,\bar{y}))$$

has a maximum over $\overline{\Delta}_R^h \times \overline{\Delta}_R^h$ at $(\bar{t}, \bar{s}, \bar{x}, \bar{y})$. Similarly as in the proof of Proposition 3.3 we obtain

$$\lim_{\beta\to 0} \limsup_{n\to+\infty} \frac{(\bar{t}-\bar{s})^2}{\beta} = 0 \quad \text{for every } m, \mu, \epsilon > 0, \tag{3.31}$$

$$\lim_{\epsilon\to 0} \limsup_{\beta\to 0} \limsup_{n\to\infty} \frac{d^2(\bar{x},\bar{y})}{\epsilon} = 0 \quad \text{for every } m, \mu. \tag{3.32}$$

Since

$$\lim_{m\to+\infty} \inf\{w_m(t,x) : (t,x) \in \partial\Delta_R^h \cap (0,T) \times \mathbb{S}\} = +\infty,$$

it follows that for large m we must have

$$(\bar{t}, \bar{x}, \bar{s}, \bar{y}) \notin \partial(\Delta_R^h \times \Delta_R^h) \cap \Big((0,T] \times \mathbb{S} \times (0,T] \times \mathbb{S}\Big).$$

If

$$\sup_{\Delta_R}(u - v) > \sup\{[g_1(x) - g_2(x)]_+ : d(x,x_0) < R - 1\}, \tag{3.33}$$

then there is $(t,x) \in \Delta_R$ and $\gamma > 0$ such that

$$u(t,x) - v(t,x) > \sup\{[g_1(x) - g_2(x)]_+ : d(x,x_0) < R - 1\} + 3\gamma.$$

It then follows from (3.30) that

$$\begin{aligned}\tilde{\Psi}(t,t,x,x) &> sup\Big\{[g_1(x)-g_2(x)]_+ : d(x,x_0) < R-1\Big\} + 3\gamma \\ &\quad - \frac{2\mu}{T-t} - 2w_m(t,x) + (a_n+b_n)t - \frac{1}{n}\big(d(x,\bar{x}) + d(x,\bar{y})\big) \\ &> \sup\Big\{[g_1(x)-g_2(x)]_+ : d(x,x_0) < R-1\Big\} + 2\gamma\end{aligned}$$

if $\mu < \mu_0$, $m > m_0$, $n > n_0$ for some μ_0, m_0, n_0 independent of ϵ and β. Therefore, since g_1, g_2 satisfy $(A5)$, we can assume that for $\mu < \mu_0$, $m > m_0$, $n > n_0$

$$\sup_{\Delta_R^h \times \Delta_R^h} \tilde{\psi} > \sup\{[g_1(x)-g_2(x)]_+ : d(x,x_0) < R+h-1\} + \gamma$$

if $0 < h < h_0$ for some $h_0 > 0$, and moreover

$$u(\bar{t},\bar{x}) > v(\bar{s},\bar{y}). \tag{3.34}$$

Now, $(A5)$, (3.28), (3.31) and (3.32) imply that $0 < \bar{t}, \bar{s} < T$ whenever $\mu < \mu_0$, $m > m_0$, $n > n_0$ and ϵ, β are sufficiently small. By the definition of metric viscosity subsolution we thus have

$$\frac{\mu}{(T-\bar{t})^2} + \frac{\bar{t}-\bar{s}}{\beta} + mLe^{L\bar{t}}(1+d(\bar{x},x_0))w_m(\bar{t},\bar{x}) - a_n + H_{|\nabla w_m(\bar{t},\bar{x})|^* + 1/n}\left(\bar{t},\bar{x},u(\bar{t},\bar{x}),\frac{d(\bar{x},\bar{y})}{\epsilon}\right) \leq 0,$$

which, together with $(A4)$ and $|\nabla w_m(\bar{t},\bar{x})|^* = me^{L\bar{t}}w_m(\bar{t},\bar{x})$, gives

$$\frac{\bar{t}-\bar{s}}{\beta} + H\left(\bar{t},\bar{x},u(\bar{t},\bar{x}),\frac{d(\bar{x},\bar{y})}{\epsilon}\right) \leq -\frac{\mu}{T^2} + \rho_1(\frac{1}{n}),$$

for some modulus ρ_1. Similarly we have

$$\frac{\bar{t}-\bar{s}}{\beta} + H\left(\bar{s},\bar{y},v(\bar{s},\bar{y}),\frac{d(\bar{x},\bar{y})}{\epsilon}\right) \geq \frac{\mu}{T^2} - \bar{\rho}_1(\frac{1}{n}).$$

Therefore, using $(A1)$, $(A2)$, (3.17) and (3.34) implies

$$\frac{2\mu}{T^2} \leq H\left(\bar{t},\bar{y},u(\bar{t},\bar{x}),\frac{d(\bar{x},\bar{y})}{\epsilon}\right) - H\left(\bar{t},\bar{x},u(\bar{t},\bar{x}),\frac{d(\bar{x},\bar{y})}{\epsilon}\right) + \rho_2(\epsilon;\beta,n)$$

for $\mu < \mu_0, m > m_0$, where $\lim_{\beta\to 0}\limsup_{n\to+\infty}\rho_2(\epsilon;\beta,n) = 0$. This yields a contradiction after we invoke $(A3)$, (3.18), and take $\lim_{\epsilon\to 0}\limsup_{\beta\to 0}\limsup_{n\to+\infty}$ as in the proof of Proposition 3.3. Therefore (3.29) must be true.

The uniqueness of viscosity solutions is a direct consequence of (3.29) when $g_1 = g_2$. ■

Theorem 3.6 (Existence and uniqueness for general initial value functions)**.** *Let* $(A1)-(A5)$ *be satisfied. Then there exists a metric viscosity solution* u *of (1.1)-(1.2). The solution* u *is uniformly continuous on bounded subsets of* $[0,T) \times \mathbb{S}$ *and is unique among functions in* $C([0,T) \times \mathbb{S})$ *satisfying (3.27).*

Proof. Let $h \in C_0^\infty(\mathbb{R})$ be such that $0 \le h \le 1$ and $h(\tau) = 1$ for $|\tau| \le 1$. Define for $n \ge 1$, $h_n(\tau) := h(\tau/n)$, and

$$H_n(t,x,r,s) := h_n(d(x,x_0))H(t,x,r,s), \quad g_n(x) := h_n(d(x,x_0))g(x).$$

It is easy to see that H_n, g_n satisfy $(A1)-(A5)$, with possibly different ω_R, σ_R, however with the same constant L in $(A4)$. Moreover H_n satisfies (3.26). Therefore, by Theorem 3.4, for every $n \ge 1$ the problem

$$\begin{cases} \partial_t u_n + H_n(t,x,u_n,|\nabla u_n|) = 0, & \text{in } (0,T)\times\mathbb{S}, \\ u_n(0,x) = g_n(x) & \text{on } \mathbb{S}, \end{cases} \tag{3.35}$$

has a unique bounded metric viscosity solution u_n which is uniformly continuous on bounded subsets of $[0,T)\times\mathbb{S}$. By Theorem 3.5 we have $u_n = u_m$ on Δ_n if $m > n$. Since for every $m > n$, $H_n(t,x,r,s) = H_m(t,x,r,s) = H(t,x,r,s)$ on $(0,T)\times\Delta_n\times\mathbb{R}\times\mathbb{S}$, it thus follows that the function

$$u(t,x) := \lim_{n\to+\infty} u_n(t,x)$$

is uniformly continuous on bounded subsets of $[0,T)\times\mathbb{S}$, $u = u_n$ on Δ_n for every $n \ge 1$, $u(0,x) = g(x)$ on $\mathbb{S}$, and u is a metric viscosity solution of

$$\begin{cases} \partial_t u + H(t,x,u,|\nabla u|) = 0, & \text{in } (0,T)\times\mathbb{S}, \\ u(0,x) = g(x) & \text{on } \mathbb{S}. \end{cases}$$

The uniqueness follows from Theorem 3.5. ∎

3.3. **Further results.** We show here how to relax some conditions in a comparison theorem if we know in advance that either a subsolution or a supersolution is Lipschitz continuous. To minimize technicalities we only consider Hamiltonians $H = H(x,s)$ however the result would also hold for more general H. Theorem 3.7 can also be regarded as an improvement of Theorem 4.2 if a subsolution or a supersolution is Lipschitz continuous, since no convexity of $H(x,\cdot)$ nor any restrictions on its growth are required. We make the following assumption which is weaker than $(A4)$:

- $(A4w)$ For every $R > 0$ there is a constant $L_R \ge 0$ such that for every $(x,r,s) \in \mathbb{S}\times\mathbb{R}\times\mathbb{R}$
$$|H(x,s) - H(x,r)| \le L_R\Big(1 + d(x,x_0)\Big)|s-r| \quad \text{if } \max(|r|,|s|) \le R.$$

Theorem 3.7 (A comparison principle)**.** *Assume $H \equiv H(x,s)$ satisfies $(A3), (A4w)$ and let g satisfy $(A5)$. Let u be a metric viscosity subsolution of (4.1) and v be a metric viscosity supersolution of (4.1) satisfying (3.9) uniformly on bounded subsets of $\mathbb{S}$. Suppose also that u, v satisfy (3.1) with $\bar{k} = 1$, and that either $u(t,\cdot)$ or $v(t,\cdot)$ is Lipschitz continuous, uniformly for $0 < t < T$. Then (3.10) holds.*

Proof. Without loss of generality we assume that there exists $L \geq 0$ such that

$$|u(t,x) - u(t,y)| \leq Ld(x,y) \quad \text{for } t \in (0,T), x, y \in \mathbb{S}. \tag{3.36}$$

Let $R := L + 18ec + 1$, where c is from (3.9), and we set $K := 4L_R$. We define $T_2 := \min(T, 1/K)/2, T_3 := 2T_2$. We will first prove (3.10) for $0 \leq t, s < T_2$. The proof initially proceeds like the proof of Proposition 3.3. If (3.10) does not hold for $0 \leq t, s < T_2$, we define the function

$$\Psi(t,s,x,y) = u(t,x) - v(s,y) - \frac{\mu}{T_3 - t} - \frac{\mu}{T_3 - s} - \frac{d^2(x,y)}{2\epsilon} - \frac{(t-s)^2}{2\beta}$$
$$- \delta e^{Kt}\big(1 + d^2(x,x_0)\big) - \delta e^{Ks}\big(1 + d^2(y,x_0)\big)$$

and for every $n \geq 1$ there are $a_n, b_n \in \mathbb{R}, |a_n| + |b_n| \leq \frac{1}{n}$, and $(\bar{t}, \bar{s}, \bar{x}, \bar{y}) \in [0,T_3] \times [0,T_3]) \times \mathbb{S} \times \mathbb{S}$ such that

$$\Psi(t,s,x,y) + a_n t + b_n s - \frac{1}{n}(d(x,\bar{x}) + d(y,\bar{y}))$$

has a global maximum over $[0,T_3] \times [0,T_3] \times \mathbb{S} \times \mathbb{S}$ at $(\bar{t}, \bar{s}, \bar{x}, \bar{y})$. We also have that (3.13), (3.17), (3.18) hold, and if $\mu, \delta, \epsilon, \beta$ are sufficiently small and n is sufficiently large then

$$\Psi(\bar{t}, \bar{s}, \bar{x}, \bar{y}) > 0 \tag{3.37}$$

and $0 < \bar{t}, \bar{s} < T_3$. By the definition of viscosity subsolution we thus get

$$\frac{\mu}{(T_3 - \bar{t})^2} + \frac{\bar{t} - \bar{s}}{\beta} + \delta K e^{K\bar{t}}(1 + d^2(\bar{x}, x_0)) - a_n + H_{2\delta e^{K\bar{t}} d(\bar{x},x_0) + 1/n}\left(\bar{x}, \frac{d(\bar{x},\bar{y})}{\epsilon}\right) \leq 0, \tag{3.38}$$

If follows from (3.9) and (3.37) that

$$c\big(2 + d(\bar{x},x_0) + d(\bar{y},x_0)\big) \geq u(\bar{t},\bar{x}) - v(\bar{s},\bar{y}) > \delta\Big(e^{K\bar{t}}\big(1 + d^2(\bar{x},x_0)\big) + e^{K\bar{s}}\big(1 + d^2(\bar{y},x_0)\big)\Big)$$
$$\geq \delta\big(2 + d^2(\bar{x},x_0) + d^2(\bar{y},x_0)\big) \geq \frac{\delta}{3}\big(1 + d(\bar{x},x_0) + d(\bar{y},x_0)\big)^2.$$

This gives us

$$\delta\big(1 + d(\bar{x},x_0) + d(\bar{y},x_0)\big) \leq 6c. \tag{3.39}$$

Since

$$\Psi(\bar{t},\bar{s},\bar{x},\bar{y}) + a_n\bar{t} + b_n\bar{s} \geq \Psi(\bar{t},\bar{s},\bar{y},\bar{y}) + a_n\bar{t} + b_n\bar{s} - \frac{1}{n}d(\bar{y},\bar{x}),$$

we obtain

$$Ld(\bar{x},\bar{y}) \geq u(\bar{t},\bar{x}) - u(\bar{t},\bar{y}) \geq \frac{d^2(\bar{x},\bar{y})}{\epsilon} + \delta e^{K\bar{t}}(d^2(\bar{x},x_0) - d^2(\bar{y},x_0)) - \frac{1}{n}d(\bar{y},\bar{x})$$
$$\geq \frac{d^2(\bar{x},\bar{y})}{\epsilon} - \delta e(d(\bar{x},x_0) + d(\bar{y},x_0))d(\bar{y},\bar{x}) - \frac{1}{n}d(\bar{y},\bar{x}).$$

Therefore, by (3.39), we obtain

$$\frac{d(\bar{x},\bar{y})}{\epsilon} \leq L + 6ec + \frac{1}{n}. \tag{3.40}$$

and thus, since $K\bar{t} \le 1$,

$$\frac{d(\bar{x},\bar{y})}{\epsilon} + 2\delta e^{K\bar{t}} d(\bar{x},x_0) \le L + 6ec + 1 + 2\delta e d(\bar{x},x_0) \le L + 18ec + 1 = R. \tag{3.41}$$

Using $(A4w)$ it thus follows from (3.38)

$$\begin{aligned} &\frac{\bar{t}-\bar{s}}{\beta} + H\left(\bar{x}, \frac{d(\bar{x},\bar{y})}{\epsilon}\right) \\ &\le -\frac{\mu}{T_3^2} - \delta K e^{K\bar{t}}(1 + d^2(\bar{x},x_0)) + 2\delta e^{K\bar{t}} L_R (1 + d(\bar{x},x_0)) d(\bar{x},x_0) + \rho_1(\mu,\delta,\epsilon,\beta;n) \\ &\le -\frac{\mu}{T_3^2} + \rho_1(\mu,\delta,\epsilon,\beta;n), \end{aligned} \tag{3.42}$$

where $\lim_{n\to+\infty} \rho_1(\mu,\delta,\epsilon,\beta;n) = 0$. Similarly we obtain

$$\frac{\bar{t}-\bar{s}}{\beta} + H\left(\bar{y}, \frac{d(\bar{x},\bar{y})}{\epsilon}\right) \ge \frac{\mu}{T_3^2} - \rho_1(\mu,\delta,\epsilon,\beta;n). \tag{3.43}$$

Set $R_\delta := \max(6c/\delta, L + 6ec + 1)$. Combining (3.42) and (3.43) and using $(A3)$ now gives

$$\frac{2\mu}{T_3^2} \le \omega_{R_\delta}\left(d(\bar{x},\bar{y})\left(1 + \frac{d(\bar{x},\bar{y})}{\epsilon}\right)\right) + 2\rho_1(\mu,\delta,\epsilon,\beta;n)$$

which produces a contradiction in light of (3.18) for μ,δ sufficiently small after we let $\lim_{\epsilon\to 0}\limsup_{\beta\to 0}\limsup_{n\to+\infty}$ above.

We can now reapply the procedure on intervals $[T_2/2, 3T_2/2], [T_2, 2T_2], ..., [kT_2/2, T)$, where $k \ge 1$ is such that the last interval has length less than or equal to T_2, to conclude the proof. We remark that we do not have an exact equivalent of condition (3.9) at time $T_2/2$, however (3.10) on $[0, T_2]$ gives a replacement for (3.9) and implies that the procedure can be restarted at $T_2/2$. ■

We remark that [21] contains several results on viscosity solutions of Hamilton-Jacobi equations with superlinear gradient terms in finite dimensional spaces that may be adaptable to the metric space case.

4. Hamiltonians Convex in the Momentum Variables

4.1. **A Comparison Principle.**

The next result is a comparison theorem for equations with superlinear growth in the local slope variable of the form

$$\begin{cases} \partial_t u + H(x, |\nabla u|) - f(x) = 0, & \text{in } (0,T)\times\mathbb{S}, \\ u(0,x) = g(x) & \text{on } \mathbb{S}. \end{cases} \tag{4.1}$$

We could include the dependence on t in the equation however we omit this easy extension since the presentation is already very technical. Our approach is different from that of [1]. Techniques of the type we use derive from the ideas of [48]. We remark that when H is independent of x, Theorem 7.5 of [37] gives comparison for (4.1) for metric viscosity sub- and supersolutions growing at most linearly, without any restrictions on H besides its continuity.

We make the following assumptions.

- $(B1)$ H is uniformly continuous on bounded subsets of $\mathbb{S} \times \mathbb{R}$, and for every $x \in \mathbb{S}$, $H(x, \cdot)$ is convex and nondecreasing.
- $(B2)$ There exist $C \geq 0, 0 \leq \kappa \leq 1, m > 1$ such that for every $(x, s) \in \mathbb{S} \times [0, +\infty)$
$$H(x, s) \leq C(1 + d(x, x_0))^\kappa s^m.$$
- $(B3)$ There exist functions $\gamma(t), \overline{\gamma}(t)$ such that $\gamma(t) > 1$ for $1 < t < t_0$, $\overline{\gamma}(t) \to 1$ as $t \to 1$, such that for every $t > 0$ and $(x, s) \in \mathbb{S} \times [0, +\infty)$
$$t\gamma(t)H(x, s) \leq H(x, ts) \leq \overline{\gamma}(t)H(x, s).$$
- $(B4)$ There exists $\theta > 0$ such that for every $(x, s) \in \mathbb{S} \times [0, +\infty)$
$$H(x, s) \geq \theta s^m.$$
- $(B5)$ For every $R > 0$ there is a modulus ω_R such that if $(x, y, s) \in \mathbb{S} \times \mathbb{S} \times [0, +\infty)$ and $\max(d(x, x_0), d(y, x_0)) \leq R$, then
$$|H(x, s) - H(y, s)| \leq \omega_R(d(x, y))(1 + s^m).$$
- $(B6)$ There exists a constant M such that
$$f \geq M \quad \text{on } \mathbb{S}.$$

We define α to be the solution of $\alpha = \kappa + m(\alpha - 1)$, i.e.
$$\alpha = 1 + \frac{1 - \kappa}{m - 1}.$$

Example 4.1. *It is easy to see that the Hamiltonian*
$$H(x, s) = \begin{cases} a(x)s^m & s \geq 0, \\ 0 & s < 0, \end{cases}$$
satisfies conditions $(B1) - (B5)$ if a is uniformly continuous on bounded subsets of $\mathbb{S}$, and
$$0 < \theta \leq a(x) \leq C(1 + d(x, x_0))^\kappa, \quad x \in \mathbb{S}.$$

We remark that since $H(x, \cdot)$ is nondecreasing, $H^\eta(x, s) = H(x, s + \eta)$ and $H_\eta(x, s) = H(x, s - \eta)$, $x \in \mathbb{S}, s \in \mathbb{R}, \eta > 0$.

Theorem 4.2. *Let $(B1) - (B6)$ be true and let f, g satisfy $(A5)$. Let u be a metric viscosity subsolution of (4.1) and v be a metric viscosity supersolution of (4.1) satisfying (3.9) uniformly on bounded subsets of $\mathbb{S}$. Suppose also that*

$$\limsup_{d(x,x_0)\to+\infty} \sup_{t\in[0,T)} \frac{u(t, x)}{1 + d^\alpha(x, x_0)} \leq 0, \quad \limsup_{d(x,x_0)\to+\infty} \sup_{t\in[0,T)} \frac{-v(t, x)}{1 + d^\alpha(x, x_0)} \leq 0. \tag{4.2}$$

Then $u \leq v$.

Proof. The proof initially proceeds like the proof of Proposition 3.3. Suppose that $\sup(u-v) \geq 2\nu_1 > 0$. Then if $\lambda_0 < 1$ is sufficiently close to 1 we also have $\sup(\lambda u - v) \geq \nu_1$ for $\lambda_0 < \lambda < 1$. We define for $\lambda_0 < \lambda < 1$, $\mu > 0$, $\delta > 0$, $\epsilon > 0$ and $\beta > 0$ the function

$$\Psi(t,s,x,y) = \lambda u(t,x) - v(s,y) - \frac{\mu}{T-t} - \frac{\mu}{T-s} - \frac{d^2(x,y)}{2\epsilon} - \frac{(t-s)^2}{2\beta} - \delta e^t(1+d^\alpha(x,x_0)) - \delta e^s(1+d^\alpha(y,x_0)),$$

$\Psi(t,s,x,y) = -\infty$ if either $t = T$ or $s = T$. We have by (4.2), $\Psi(t,s,x,y) \to -\infty$ as $\max(d(x,x_0), d(y,x_0)) \to +\infty$, uniformly for $t,s \in [0,T]$ and ϵ, β. Therefore, using Lemma 2.9, for every $n \geq 1$ there are $a_n, b_n \in \mathbb{R}, |a_n| + |b_n| \leq \frac{1}{n}$, and $(\bar{t},\bar{s},\bar{x},\bar{y}) \in [0,T] \times [0,T] \times B_{R_\delta}(x_0) \times B_{R_\delta}(x_0)$ for some $R_\delta > 0$ such that

$$\Psi(t,s,x,y) + a_n t + b_n s - \frac{1}{n}(d(x,\bar{x}) + d(y,\bar{y}))$$

has a maximum over $[0,T] \times [0,T] \times \mathbb{S} \times \mathbb{S}$ at $(\bar{t},\bar{s},\bar{x},\bar{y})$. Arguing as in the proof of Proposition 3.3 we have that (3.17) and (3.18) are satisfied and there is $\mu_0 > 0$ such that if $0 < \mu < \mu_0, \lambda_0 < \lambda < 1$, and $\delta, \epsilon, \beta, 1/n$ are sufficiently small, we must have $0 < \bar{t}$ and $\bar{s} < T$. It then follows from the definition of viscosity subsolution that

$$\frac{\mu}{(T-\bar{t})^2} + \frac{\bar{t}-\bar{s}}{\beta} + \delta e^{\bar{t}}(1+d^\alpha(\bar{x},x_0)) + \lambda H\left(\bar{x}, \frac{1}{\lambda}\Big(\frac{d(\bar{x},\bar{y})}{\epsilon} - \delta\alpha e^{\bar{t}} d^{\alpha-1}(\bar{x},x_0) - \frac{1}{n}\Big)\right) \leq a_n + \lambda f(\bar{x}).$$

Since by $(B6)$

$$-\lambda f(\bar{x}) \geq -f(\bar{x}) + (1-\lambda)M,$$

we thus obtain

$$\frac{\bar{t}-\bar{s}}{\beta} + \delta e^{\bar{t}}(1+d^\alpha(\bar{x},x_0)) + \lambda H\left(\bar{x}, \frac{1}{\lambda}\left(\frac{d(\bar{x},\bar{y})}{\epsilon} - \delta\alpha e^{\bar{t}} d^{\alpha-1}(\bar{x},x_0)\right)\right) \leq f(\bar{x}) - \frac{\mu}{T^2} + \rho_1(n) - (1-\lambda)M, \tag{4.3}$$

where $\lim_{n\to+\infty} \rho_1(n) = 0$ for fixed $\lambda, \mu, \delta, \epsilon, \beta$. Since for every $a, b \in \mathbb{R}, 0 < \eta < 1$,

$$\eta a = \eta(a-b) + (1-\eta)\left(\frac{\eta}{1-\eta} b\right),$$

the convexity of $H(\bar{x}, \cdot)$ implies

$$H(\bar{x}, \eta a) \leq \eta H(\bar{x}, a-b) + (1-\eta) H\left(\bar{x}, \frac{\eta}{1-\eta} b\right).$$

Applying this with

$$\eta = \frac{1+\lambda}{2}, \ a = \frac{d(\bar{x},\bar{y})}{\lambda\epsilon}, \ b = \frac{\delta\alpha e^{\bar{t}} d^{\alpha-1}(\bar{x},x_0)}{\lambda}$$

we obtain

$$\begin{aligned}
&\lambda H\left(\bar{x}, \frac{1}{\lambda}\Big(\frac{d(\bar{x},\bar{y})}{\epsilon} - \delta\alpha e^{\bar{t}} d^{\alpha-1}(\bar{x},x_0)\Big)\right)\\
&\geq \frac{\lambda}{\eta} H\left(\bar{x}, \frac{\eta}{\lambda}\frac{d(\bar{x},\bar{y})}{\epsilon}\right) - \frac{(1-\eta)\lambda}{\eta} H\left(\bar{x}, \frac{\eta}{1-\eta}\frac{\delta\alpha e^{\bar{t}} d^{\alpha-1}(\bar{x},x_0)}{\lambda}\right)\\
&\geq \gamma\Big(\frac{1+\lambda}{2\lambda}\Big) H\left(\bar{x}, \frac{d(\bar{x},\bar{y})}{\epsilon}\right) - C_1(\alpha,\lambda,T)\delta^m\big(1+d(\bar{x},x_0)\big)^\kappa d^{m(\alpha-1)}(\bar{x},x_0) \qquad (4.4)
\end{aligned}$$

where we used $(B2)$ and $(B3)$ to get the last inequality. Since $m > 1$ it thus follows by the definition of α that if δ is sufficiently small, say $\delta < \delta_0 = \delta_0(\lambda, T, \alpha, \kappa, m)$, then

$$+\,\delta e^{\bar{t}}(1 + d^\alpha(\bar{x}, x_0)) - C_1(\alpha,\lambda,T)\delta^m(1 + d(\bar{x},x_0))^\kappa d^{m(\alpha-1)}(\bar{x},x_0) \geq 0. \qquad (4.5)$$

Therefore, for $\delta < \delta_0$, we finally have using (4.3), (4.4) and (4.5)

$$\frac{\bar{t}-\bar{s}}{\beta} + \gamma\left(\frac{1+\lambda}{2\lambda}\right) H\left(\bar{x}, \frac{d(\bar{x},\bar{y})}{\epsilon}\right) - f(\bar{x}) \leq -\frac{\mu}{T^2} + \rho_1(n) - (1-\lambda)M. \qquad (4.6)$$

Similarly, using the definition of viscosity supersolution, we have

$$\frac{\bar{t}-\bar{s}}{\beta} - \delta e^{\bar{s}}(1 + d^\alpha(\bar{y},x_0)) + H\left(\bar{y}, \frac{d(\bar{x},\bar{y})}{\epsilon} + \delta\alpha e^{\bar{s}} d^{\alpha-1}(\bar{y},x_0)\right) - f(\bar{y}) \geq \frac{\mu}{T^2} - \rho_1(n) \qquad (4.7)$$

for some function, still denoted by ρ_1, such that $\lim_{n\to+\infty}\rho_1(n) = 0$ for fixed $\lambda, \mu, \delta, \epsilon, \beta$. We choose τ such that

$$r := \gamma\left(\frac{1+\lambda}{2\lambda}\right) - \tau\overline{\gamma}\left(\frac{1}{\tau}\right) > 0. \qquad (4.8)$$

The convexity of H, together with $(B2)$ and $(B3)$, implies

$$\begin{aligned}
&H\left(\bar{y}, \frac{d(\bar{x},\bar{y})}{\epsilon} + \delta\alpha e^{\bar{s}} d^{\alpha-1}(\bar{y},x_0)\right)\\
&\leq \tau H\left(\bar{y}, \frac{1}{\tau}\frac{d(\bar{x},\bar{y})}{\epsilon}\right) + (1-\tau) H\left(\bar{y}, \frac{1}{1-\tau}\delta\alpha e^{\bar{s}} d^{\alpha-1}(\bar{y},x_0)\right)\\
&\leq \tau\overline{\gamma}\left(\frac{1}{\tau}\right) H\left(\bar{y}, \frac{d(\bar{x},\bar{y})}{\epsilon}\right) + C_2(\alpha,\lambda,T)\delta^m(1+d(\bar{y},x_0))^\kappa d^{m(\alpha-1)}(\bar{y},x_0) \qquad (4.9)
\end{aligned}$$

Combining (4.7) with (4.9) we thus again have that for $\delta < \delta_1 = \delta_1(\lambda, T, \alpha, \kappa, m)$

$$\frac{\bar{t}-\bar{s}}{\beta} + \tau\overline{\gamma}\left(\frac{1}{\tau}\right) H\left(\bar{y}, \frac{d(\bar{x},\bar{y})}{\epsilon}\right) - f(\bar{y}) \geq \frac{\mu}{T^2} - \rho_1(n). \qquad (4.10)$$

Subtracting (4.10) from (4.6) and using (4.8) we obtain

$$\begin{aligned}
&rH\left(\bar{x}, \frac{d(\bar{x},\bar{y})}{\epsilon}\right) + \tau\overline{\gamma}\left(\frac{1}{\tau}\right)\left(H\left(\bar{x}, \frac{d(\bar{x},\bar{y})}{\epsilon}\right) - H\left(\bar{y}, \frac{d(\bar{x},\bar{y})}{\epsilon}\right)\right) + f(\bar{y}) - f(\bar{x})\\
&\leq -\frac{2\mu}{T^2} + 2\rho_1(n) - (1-\lambda)M
\end{aligned}$$

which, by $(B4), (B5)$ and $(A5)$, implies

$$r\theta\left(\frac{d(\bar{x},\bar{y})}{\epsilon}\right)^m - \tau\overline{\gamma}\left(\frac{1}{\tau}\right)\omega_{R_\delta}(d(\bar{x},\bar{y}))\left(1+\left(\frac{d(\bar{x},\bar{y})}{\epsilon}\right)^m\right)$$
$$\leq \sigma_{R_\delta}(d(\bar{x},\bar{y})) - \frac{2\mu}{T^2} + 2\rho_1(n) - (1-\lambda)M.$$

We now take $\lambda_0 < \lambda < 1$ such that

$$-\frac{2\mu}{T^2} - (1-\lambda)M \leq -\frac{\mu}{T^2}.$$

Then, for such λ, $0 < \mu < \mu_0$, $0 < \delta < \min(\delta_0, \delta_1)$, and $\epsilon, \beta, 1/n$ sufficiently small, we have

$$\left(r\theta - \tau\overline{\gamma}\left(\frac{1}{\tau}\right)\omega_{R_\delta}(d(\bar{x},\bar{y}))\right)\left(\frac{d(\bar{x},\bar{y})}{\epsilon}\right)^m$$
$$\leq -\frac{\mu}{T^2} + \tau\overline{\gamma}\left(\frac{1}{\tau}\right)\omega_{R_\delta}(d(\bar{x},\bar{y})) + \sigma_{R_\delta}(d(\bar{x},\bar{y})) + 2\rho_1(n). \tag{4.11}$$

Taking $\lim_{\epsilon\to 0}\limsup_{\beta\to 0}\limsup_{n\to+\infty}$ in (4.11) and using (3.18) gives a contradiction since the left hand side will become nonnegative and the right hand side will become negative. ∎

4.2. **Existence of a solution; The value function.** Throughout this subsection we assume that $L \in C\big(\mathbb{S}\times[0,\infty)\big)$, for each $x \in \mathbb{S}$, $L(x,\cdot)$ is monotone nondecreasing, and there exist a monotone nondecreasing function $0 \leq \varrho \in C([0,\infty)$ and for each $R > 0$ there exists a modulus $\bar{\omega}_R$ such that if $d(x,x_0), d(y,x_0) \leq R$ then for $r \geq 0$ we have

$$|L(x,r) - L(y,r)| \leq \varrho(r)\bar{\omega}_R\big(d(x,y)\big). \tag{4.12}$$

We also assume there are functions $\mathcal{L} : [0,\infty) \to \mathbb{R}$ and $\mathcal{W} : \mathbb{S} \to \mathbb{R}$ such that if $r \geq 0$ and $x \in \mathbb{S}$ then

$$L(x,r) \geq \mathcal{L}(r) - \mathcal{W}(x), \tag{4.13}$$

where there are real numbers $C_0, C_1 \geq 0$, $\theta \geq 1$ and a continuous function $\alpha : [0,+\infty) \to [0,+\infty)$ such that $\lim_{r\to+\infty}\alpha(r) = +\infty$ and

$$\mathcal{L}(r) \geq \alpha(r)r^\theta - C_0, \quad r \geq 0, \tag{4.14}$$

$$\mathcal{W}(x) \leq C_1\big(d^\theta(x_0,x) + 1\big), \quad x \in \mathbb{S}. \tag{4.15}$$

Moreover we assume that g satisfies (A5) and

$$-g(x) \leq C_1\big(d^\theta(x_0,x) + 1\big). \tag{4.16}$$

We define the Hamiltonian

$$H(x,s) = \sup_{r>0}\{sr - L(x,r)\}, \quad s \geq 0. \tag{4.17}$$

The proof of the following lemma is elementary and will be skipped.

Lemma 4.3. *We have:*

(i) *For each* $x \in \mathbb{S}$, $H(x,\cdot)$ *is monotone nondecreasing.*

(ii) *If* $0 \le s < \eta$, *then*

$$H_\eta(x,s) = H(x,0) = \sup_{r>0} \{(s-\eta)r - L(x,r)\}.$$

We define the value function u:

$$u(t,x) = \inf_\sigma \{ \int_0^t L\big(\sigma(s), |\sigma'|(s)\big) ds + g(\sigma(0)) : \ \sigma(t) = x\},$$

where the infimum is performed over the set of absolutely continuous paths $\sigma : [0,t] \to \mathbb{S}$, and $|\sigma'|(s)$ is the metric derivative of σ, see [2], pages 23-24.

The path $\sigma(s) \equiv x$ is used to obtain

$$u(t,x) \le tL(x,0) + g(x). \tag{4.18}$$

Remark 4.4. *Let* $t \in (0,T]$, *let* $x \in \mathbb{S}$ *and let* $\sigma : [0,t] \to \mathbb{S}$ *be an absolutely continuous curve such that* $\sigma(t) = x$.

(i) *There is a constant* C_2 *independent of* x *and* $z \in \mathbb{S}$ *such that*

$$-g(z),\ W(z) \le C_2\Big(d^\theta(z,x) + d^\theta(x,x_0) + 1\Big), \quad z \in \mathbb{S}. \tag{4.19}$$

(ii) *There is a constant* C_T *independent of* t, x, σ, *such that*

$$\int_0^t L(\sigma(s), |\sigma'|(s))ds + g(\sigma(0)) \ge \frac{1}{2}\int_0^t \alpha(|\sigma'|(\tau))|\sigma'|^\theta(\tau)d\tau - C_T(d^\theta(x,x_0)+1). \tag{4.20}$$

(iii) *For every* $R > 0$ *there exists a constant* $C(R)$ *depending only on* R *such that if* $d(x_0,x) \le R$ *and*

$$u(t,x) \ge -1 + \int_0^t L\big(\sigma(s), |\sigma'|(s)\big)ds + g(\sigma(0)) \tag{4.21}$$

then

$$\int_0^t \alpha(|\sigma'|(\tau))|\sigma'|^\theta d\tau \le C(R) \quad \textit{and} \quad \mathcal{W}(\sigma(s)) \le C(R). \tag{4.22}$$

Proof. (i) Using the fact that $a \to a^\theta$ is a convex function, (4.15) and (4.16) yield (4.19).

(ii) We recall Jensen's inequality:

$$\Big(\int_0^t |\sigma'| d\tau\Big)^\theta \le t^{\theta-1}\int_0^t |\sigma'|^\theta d\tau.$$

We set $z = \sigma(s)$ in (4.19), where $s \in [0,t]$ and first use the fact that $d(\sigma(s),\sigma(t)) \le \int_0^t |\sigma'|d\tau$ and then Jensen's inequality to obtain

$$\begin{aligned} -g(\sigma(0)),\ W(\sigma(s)) &\le C_2\bigg(\Big(\int_0^t |\sigma'|d\tau\Big)^\theta + d^\theta(x,x_0) + 1\bigg) \\ &\le C\bigg(t^{\theta-1}\int_0^t |\sigma'|^\theta d\tau + d^\theta(x,x_0) + 1\bigg) \end{aligned} \tag{4.23}$$

and so,

$$\int_0^t -W(\sigma(s))ds + g(\sigma(0)) \geq -\tilde{C}_T\Big(\int_0^t |\sigma'|^\theta d\tau + d^\theta(x,x_0) + 1\Big).$$

This, together with (4.13) and (4.14) yields

$$\int_0^t L(\sigma(s), |\sigma'|(s))ds + g(\sigma(0)) \geq \int_0^t \Big(\mathcal{L}(|\sigma'|) - \tilde{C}_T|\sigma'|^\theta\Big) d\tau - \tilde{C}_T(d^\theta(x,x_0)+1)$$
$$\geq \frac{1}{2}\int_0^t \alpha(|\sigma'|(\tau))|\sigma'|^\theta(\tau)d\tau - C_T(d^\theta(x,x_0)+1)$$

for some C_T.

(iii) We combine (4.18), (4.20) and (4.21) to obtain

$$\int_0^t \alpha(|\sigma'|)|\sigma'|^\theta d\tau \leq 2(1 + C_T(d^\theta(x,x_0)+1) + u(t,x))$$
$$\leq 2(1 + C_T(d^\theta(x,x_0)+1) + g(x) + tL(x,0)) \leq C(R)$$

for some constant $C(R)$ by (A5) and (4.12). This, together with (4.23) completes the proof of (iii) after we readjust $C(R)$. ∎

Lemma 4.5. *If σ is a path satisfying (4.21), $\sigma(t) = x$, and $d(x,x_0) \leq R$, then there is a modulus of continuity ρ_R, independent of t, x, σ, such that*

$$d\big(\sigma(s_1), \sigma(s_2)\big) \leq \bar{\rho}_R(|s_2 - s_1|), \quad 0 \leq s_1, s_2 \leq t. \tag{4.24}$$

Moreover there is a constant $C_1(R)$ such that

$$d(\sigma(s), x_0) \leq C_1(R), \quad 0 \leq s \leq t. \tag{4.25}$$

Proof. We will only deal with the case $\theta = 1$ since the case $\theta > 1$ follows easily from Hölder's inequality. If (4.24) is not true then there is $\delta > 0$, points $0 \leq t_n \leq T, x_n \in \mathbb{S}, d(x_n, x_0) \leq R$, paths $\sigma_n, \sigma_n(t_n) = x_n$, and $0 \leq s_n^1 \leq s_n^2 \leq t_n$, such that $s_n^2 - s_n^1 =: \epsilon_n \to 0$ as $n \to +\infty$ and

$$d\big(\sigma(s_n^1), \sigma(s_n^2)\big) \geq \delta.$$

Denote

$$A_n := \left\{ s : s_n^1 \leq s \leq s_n^2, |\sigma_n'|(s) \geq \frac{\delta}{2\epsilon_n} \right\}.$$

Since

$$\int_{s_n^1}^{s_n^2} |\sigma_n'| ds \geq \delta,$$

we must have

$$\int_{A_n} |\sigma_n'| ds \geq \frac{\delta}{2}.$$

Thus, by (4.22),

$$C(R) \geq \int_{A_n} \alpha(|\sigma_n'|)|\sigma_n'| ds \geq \alpha\left(\frac{\delta}{2\epsilon_n}\right) \int_{A_n} |\sigma_n'| ds \geq \alpha\left(\frac{\delta}{2\epsilon_n}\right)\frac{\delta}{2}.$$

This gives a contradiction if n is large enough. Inequality (4.25) now follows from

$$d(\sigma(s), x_0) \leq d(\sigma(s), \sigma(t)) + d(x, x_0) \leq \rho_R(t) + d(x, x_0) \leq \rho_R(T) + R =: C_1(R).$$

∎

Lemma 4.6. *Assume $0 \leq t < t+h \leq T$ and $x \in \mathbb{S}$ is such that $d(x_0, x) \leq R, R > 0$. Then there exists a modulus of continuity e_R depending only on R such that*

$$|u(t+h, x) - u(t, x)| \leq e_R(h).$$

Proof. For each $\epsilon \in (0,1)$, let $\sigma_\epsilon : [0,t] \to \mathbb{S}$ be a path such that $\sigma_\epsilon(t) = x$ and

$$u(t,x) \geq -\epsilon + \int_0^t L\big(\sigma_\epsilon(s), |\sigma_\epsilon'|(s)\big)ds + g(\sigma_\epsilon(0)). \tag{4.26}$$

We extend σ_ϵ to $(t, t+h]$ by setting its value to be x there, and we continue to denote the extension σ_ϵ. We have

$$u(t+h, x) \leq \int_0^{t+h} L\big(\sigma_\epsilon(s), |\sigma_\epsilon'|(s)\big)ds + g(\sigma_\epsilon(0)) \leq \epsilon + u(t,x) + \int_t^{t+h} L\big(x, 0\big)ds.$$

We set $r = 0$ and $y = x_0$ in (4.12) and use the fact that ϵ is arbitrary to obtain a constant $c_0(R) > 0$ such that

$$u(t+h, x) - u(t, x) \leq c_0(R)h. \tag{4.27}$$

For each $\epsilon \in (0,1)$, let $\sigma_\epsilon : [0, t+h] \to \mathbb{S}$ be a path such that $\sigma_\epsilon(t+h) = x$ and

$$u(t+h, x) \geq -\epsilon + \int_0^{t+h} L\big(\sigma_\epsilon(s), |\sigma_\epsilon'|(s)\big)ds + g(\sigma_\epsilon(0)). \tag{4.28}$$

By (4.24) and (4.25) we have

$$d\big(\sigma_\epsilon(s_1), \sigma_\epsilon(s_2)\big) \leq \bar{\rho}_R(|s_2 - s_1|), \quad d(\sigma_\epsilon(s), x_0) \leq C_1(R), \quad 0 \leq s_1, s_2 \leq t+h. \tag{4.29}$$

Therefore (4.13) and (4.15) yield for $s \in [0, t+h]$

$$L(\sigma_\epsilon(s), |\sigma_\epsilon'|(s)) \geq -C_0 - C_1\Big(1 + (C_1(R))^\theta\Big). \tag{4.30}$$

Define

$$\bar{\sigma}_\epsilon(s) := \sigma_\epsilon(s+h) \quad s \in [0,t].$$

We have $\bar{\sigma}_\epsilon(t) = x$ and so,

$$\begin{aligned} u(t+h,x) - u(t,x) &\geq -\epsilon + \int_0^{t+h} L\big(\sigma_\epsilon(s), |\sigma_\epsilon'|(s)\big)ds + g(\sigma_\epsilon(0)) \\ &\quad - \int_0^t L\big(\bar{\sigma}_\epsilon(s), |\bar{\sigma}_\epsilon'|(s)\big)ds - g(\bar{\sigma}_\epsilon(0)) \\ &= -\epsilon + g(\sigma_\epsilon(0)) - g(\sigma_\epsilon(h)) + \int_0^h L\big(\sigma_\epsilon(s), |\sigma_\epsilon'|(s)\big)ds \end{aligned} \tag{4.31}$$

We use (4.29) and the fact that g satsifies (A5) to obtain

$$|g(\sigma_\epsilon(0)) - g(\sigma_\epsilon(h))| \leq \sigma_{C_1(R)}\Big(d(\sigma_\epsilon(0), \sigma_\epsilon(h))\Big) \leq \sigma_{C_1(R)}\Big(\bar{\rho}(h)\Big).$$

This, together with (4.30), gives

$$u(t+h,x) - u(t,x) \geq -\epsilon - \sigma_{C_1(R)}\Big(\bar{\rho}(h)\Big) - \Big(C_0 + C_1\Big(1 + (C_1(R))^\theta\Big)\Big)\, h.$$

We can now send $\epsilon \to 0$ in this inequality and combine it with (4.27) to conclude the proof of the lemma. ∎

Proposition 4.7. *Under the assumptions of this section, for every $R > 0$, u is uniformly continuous on $[0,T] \times \{x : d(x,x_0) \leq R\}$.*

Proof. Fix $R > 0$ and let $t \in [0,T]$. Let $x, y \in \mathbb{S}$ be such that $d(x,x_0), d(y,x_0) \leq R$. We set $s := d(x,y)$. Substituting x by y if necessary, we assume without loss of generality that $u(t,x) \leq u(t,y)$.

For $\epsilon > 0$ arbitrary, we choose σ_ϵ as in (4.26). We extend σ_ϵ to $(t, t+s]$ to be a geodesic connecting x to y so that the extension, which we continue to denote σ_ϵ, satisfies

$$|\sigma_\epsilon'|(\tau) = \frac{d(x,y)}{s} = 1, \quad \tau \in (t, t+s).$$

Using the fact that $\sigma_\epsilon(t+s) = y$ and that (4.26) holds, we have

$$\begin{aligned} u(t,y) &\leq \int_s^{t+s} L\big(\sigma_\epsilon(\tau), |\sigma_\epsilon'|(\tau)\big)d\tau + g(\sigma_\epsilon(s)) \\ &\leq \epsilon + u(t,x) - \int_0^s L\big(\sigma_\epsilon(\tau), |\sigma_\epsilon'|(\tau)\big)d\tau + \int_t^{t+s} L\big(\sigma_\epsilon(\tau), \frac{d(x,y)}{s}\big)d\tau + g(\sigma_\epsilon(s)) - g(\sigma_\epsilon(0)). \end{aligned}$$

We have

$$d(\sigma_\epsilon(\tau), x) \leq d(x,y) \implies d(\sigma_\epsilon(\tau), x_0) \leq 3R.$$

Moreover, denoting $t_1 = \min(s,t)$, we have by (4.24)

$$\begin{aligned} |g(\sigma_\epsilon(s)) - g(\sigma_\epsilon(0))| &\leq |g(\sigma_\epsilon(s)) - g(\sigma_\epsilon(t_1))| + |g(\sigma_\epsilon(t_1)) - g(\sigma_\epsilon(0))| \\ &\leq \sigma_{3R}(s - t_1) + \sigma_{3R}(\bar{\rho}(t_1)) \leq \sigma_{3R}(d(x,y)) + \sigma_{3R}(\bar{\rho}(d(x,y))). \end{aligned}$$

These, together with (4.12), (4.14) (4.15), yield for some constant $C_3(R)$

$$\begin{aligned} |u(t,y) - u(t,x)| &\leq \epsilon + C_3(R)d(x,y) + d(x,y) \sup_{d(z,x_0)\leq 3R} |L\big(z,1\big)| \\ &+ \sigma_{3R}(d(x,y)) + \sigma_{3R}(\bar{\rho}(d(x,y))). \end{aligned}$$

The lemma follows by sending $\epsilon \to 0$ above and invoking Lemma 4.6. ∎

We will use the principle of optimality called the Dynamic Programming Principle. It states that for every $0 \leq t - \epsilon \leq t \leq T$ and $x \in \mathbb{S}$,

$$u(t,x) = \inf_\sigma\Big\{\int_{t-\epsilon}^t L\big(\sigma(s), |\sigma'|(s)\big)ds + u(t-\epsilon, \sigma(t-\epsilon)) : \ \sigma(t) = x\Big\}, \tag{4.32}$$

where the infimum is taken over the set of absolutely continuous paths $\sigma : [0,t] \to \mathbb{S}$. Its proof is the same as for finite dimensional spaces (see e.g. [29]) and will be omitted.

Theorem 4.8. *Let the assumptions of this section be satisfied. Then:*

(i)

$$\lim_{t\to 0^+} |u(t,x) - g(x)| = 0$$

uniformly on bounded subsets of $\mathbb{S}$*.*

(ii) *u is a metric viscosity subsolution of*

$$\partial_t u + H(x, |\nabla u|) = 0, \quad u(0,\cdot) = u_0. \tag{4.33}$$

(iii) *u is a metric viscosity supersolution of (4.33).*

Proof. For all $x \in \mathbb{S}$ we have $u(0,x) = g(x)$ and (i) follows from Proposition 4.7.

Viscosity sub-solution. Let $\psi \in \underline{\mathcal{C}}$ be such that $u - \psi$ achieves its local maximum at $(t,x) \in (0,T) \times \mathbb{S}$. Fix $r > 0$ arbitrary. Since $|\nabla^- \psi_1| = |\nabla \psi_1|$ there exists a sequence $\{x_n\}_n \subset \mathbb{S}$ such that

$$|\nabla^- \psi_1(t,x)| = \lim_{n\to+\infty} \frac{\psi_1(t,x) - \psi_1(t,x_n)}{\mathrm{dist}(x_n,x)}. \tag{4.34}$$

Set

$$\epsilon_n = \frac{\mathrm{dist}(x_n,x)}{r}.$$

Note that if σ_n is a geodesic of constant speed connecting x_n and x between times $t - \epsilon_n$ and t then, whenever $t_1, t_2 \in [0,t]$,

$$|\sigma_n'| = \frac{\mathrm{dist}(x_n,x)}{\epsilon_n} = r, \quad \mathrm{dist}(\sigma_n(t_2), \sigma_n(t_1)) = |t_2 - t_1| \frac{\mathrm{dist}(x_n,x)}{\epsilon_n} = |t_2 - t_1| r. \tag{4.35}$$

By the Dynamic Programming Principle (4.32)

$$u(t,x) \le u(t - \epsilon_n, x_n) + \int_{t-\epsilon_n}^{t} L(\sigma_n, |\sigma_n'|) ds,$$

or equivalently, thanks to (4.35) and using that $\sigma_n(t) = x$,

$$u(t,x) \le u(t - \epsilon_n, x_n) + \epsilon_n L(x,r) + \int_{t-\epsilon_n}^{t} \Big(L(\sigma_n(s), r) - L(\sigma_n(t), r) \Big) ds. \tag{4.36}$$

Setting $t_2 = t$ in (4.35) we conclude that

$$\mathrm{dist}(x_0, \sigma_n(t_1)) \le rt + d(x, x_0) =: R. \tag{4.37}$$

We use the modulus of continuity provided by (4.12) to obtain

$$\int_{t-\epsilon_n}^{t} \Big| L(\sigma_n(s), r) - L(\sigma_n(t), r) \Big| ds \le \int_{t-\epsilon_n}^{t} \varrho(r) \bar{\omega}_R \Big(\mathrm{dist}(\sigma_n(s), \sigma_n(t)) \Big) ds.$$

This, together with the second identity in (4.35), yields

$$\int_{t-\epsilon_n}^{t} \Big| L(\sigma_n(s), r) - L(\sigma_n(t), r) \Big| ds \leq \int_{t-\epsilon_n}^{t} \varrho(r)\bar{\omega}_R\Big(|t-s|r\Big) ds \leq \epsilon_n \varrho(r)\bar{\omega}(\epsilon_n r)$$

and so, by (4.36),

$$u(t,x) \leq u(t-\epsilon_n, x_n) + \epsilon_n L(x,r) + \epsilon_n \bar{\omega}_R(\epsilon_n r)\varrho(r). \tag{4.38}$$

Observe that, by the local maximality of $u - \psi$ at (t,x) and (4.34),

$$\begin{aligned}
\frac{u(t,x) - u(t-\epsilon_n, x_n)}{\epsilon_n} &\geq \frac{\psi(t,x) - \psi(t-\epsilon_n, x_n)}{\epsilon_n} \\
&= \frac{\psi_1(t,x) - \psi_1(t,x_n)}{\epsilon_n} + \frac{\psi_2(t,x) - \psi_2(t,x_n)}{\epsilon_n} + \frac{\psi(t,x_n) - \psi(t-\epsilon_n, x_n)}{\epsilon_n} \\
&\geq (|\nabla\psi_1(t,x)| - |\nabla\psi_2(t,x)| + \gamma_1(n))r + \int_{t-\epsilon_n}^{t} \frac{\partial_t \psi(s,x_n)}{\epsilon_n} ds \\
&= (|\nabla\psi_1(t,x)| - |\nabla\psi_2(t,x)|)r + \partial_t\psi(t,x) + \gamma_2(n),
\end{aligned} \tag{4.39}$$

where $\lim_{n\to+\infty} \gamma_1(n) = \lim_{n\to+\infty} \gamma_2(n) = 0$. Therefore (4.39), together with (4.38), implies

$$\Big(|\nabla\psi_1(t,x)| - |\nabla\psi_2(t,x)|\Big)r + \partial_t\psi(t,x) + \gamma_2(n) \leq L(x,r) + \varrho(r)\bar{\omega}_R(\epsilon_n r).$$

Letting n tend to $+\infty$ we get

$$\Big(|\nabla\psi_1(t,x)| - |\nabla\psi_2(t,x)|\Big)r - L(x,r) + \partial_t\psi(t,x) \leq 0. \tag{4.40}$$

Maximizing over $r > 0$ in (4.40), Lemma 4.3 thus yields

$$H_{|\nabla\psi_2(t,x)|}\Big(x, |\nabla\psi_1(t,x)|\Big) + \partial_t\psi(t,x) \leq 0. \tag{4.41}$$

Viscosity super-solution. Let $\psi \in \bar{\mathcal{C}}$ be such that $u - \psi$ achieves its local minimum at $(t,x) \in (0,T) \times \mathbb{S}$. For each $\epsilon > 0$ there exists $\sigma_\epsilon : [0,t] \to \mathbb{S}$ such that $\sigma_\epsilon(t) = x$ and

$$u(t,x) \geq -\epsilon^2 + \int_0^t L(\sigma_\epsilon, |\sigma'_\epsilon|) ds + g(\sigma_\epsilon(0)). \tag{4.42}$$

Thus, setting $\sigma_\epsilon(t-\epsilon) = x_\epsilon$, we must also have

$$u(t,x) \geq -\epsilon^2 + u(t-\epsilon, x_\epsilon) + \int_{t-\epsilon}^{t} L(\sigma_\epsilon, |\sigma'_\epsilon|) ds. \tag{4.43}$$

By Lemma 4.5, (4.42) implies $\lim_{\epsilon\to 0} d(x, x_\epsilon) = 0$. By the minimality of $u - \psi$ at (t,x)

$$\frac{\psi(t,x) - \psi(t-\epsilon, x^\epsilon)}{\epsilon} \geq \frac{u(t,x) - u(t-\epsilon, x^\epsilon)}{\epsilon}. \tag{4.44}$$

Similarly as before one checks that

$$\frac{\psi(t,x) - \psi(t-\epsilon, x^\epsilon)}{\epsilon} \leq \Big(|\nabla\psi_1(t,x)| + |\nabla\psi_2(t,x)| + \gamma_3(\epsilon)\Big)\frac{\operatorname{dist}(x_\epsilon, x)}{\epsilon} + \partial_t\psi(t,x) + \gamma_3(\epsilon)$$

for some modulus γ_3. Therefore we conclude that

$$\frac{\psi(t,x) - \psi(t-\epsilon, x^\epsilon)}{\epsilon} \leq \Big(|\nabla\psi_1(t,x)| + |\nabla\psi_2(t,x)| + \gamma_3(\epsilon)\Big)\frac{\int_{t-\epsilon}^{t} |\sigma'_\epsilon| ds}{\epsilon} + \partial_t\psi(t,x) + \gamma_3(\epsilon). \tag{4.45}$$

We combine (4.43), (4.44) and (4.45) to obtain

$$-\epsilon-\gamma_3(\epsilon)\leq\frac{1}{\epsilon}\int_{t-\epsilon}^{t}\Big(\Big(|\nabla\psi_1(t,x)|+|\nabla\psi_2(t,x)|+\gamma_3(\epsilon)\Big)|\sigma_\epsilon'|-L(\sigma_\epsilon,|\sigma_\epsilon'|)\Big)ds+\partial_t\psi(t,x).$$

Thus,

$$-\epsilon-\gamma_3(\epsilon)\leq\frac{1}{\epsilon}\int_{t-\epsilon}^{t}H\Big(\sigma_\epsilon,|\nabla\psi_1(t,x)|+|\nabla\psi_2(t,x)|+\gamma_3(\epsilon)\Big)ds+\partial_t\psi(t,x).$$

Using the equicontinuity of σ_ϵ and the fact that H is continuous, and letting ϵ tend to 0, we conclude that

$$\begin{aligned}0&\leq H\Big(x,|\nabla\psi_1(t,x)|+|\nabla\psi_2(t,x)|\Big)+\partial_t\psi(t,x)\\&=H^{|\nabla\psi_2(t,x)|}\Big(x,|\nabla\psi_1(t,x)|\Big)+\partial_t\psi(t,x).\end{aligned}$$

∎

Remark 4.9. *The proof of Theorem 4.8 would also work in the case when $L(x,\cdot)$ is not nondecreasing. The only difference would be that in the proof of the subsolution part, if $|\nabla\psi_1(t,x)|-|\nabla\psi_2(t,x)|<0$ we would get instead of (4.41)*

$$\tilde{H}_{|\nabla\psi_2(t,x)|}\Big(x,|\nabla\psi_1(t,x)|\Big)+\partial_t\psi(t,x)\leq 0,$$

where $\tilde{H}(x,s)=\sup_{r>0}\{sr-L(x,r)\}$, $s\in\mathbb{R}$. We have $\tilde{H}(x,s)=H(x,s)$ if $s\geq 0$ however we would need to use this extension of H to define a metric viscosity subsolution. Thus we would obtain that u is a metric viscosity solution of (4.33) with H replaced by $\tilde{H}$. This kind of extension was also used in [1]. *However using negative values for the local slope variable seems rather artificial. We suggest an idea how one can get around this even though we do not pursue it here. If we take $\bar{L}(x,s)=H^*(x,s):=\sup_{r>0}\{sr-H(x,r)\}$, then $(\bar{L})^*=H$ but $\bar{L}(x,\cdot)$ is nondecreasing. Assuming that $\bar{L}$ has properties similar to these of L, one can then show that the value function $\bar{u}$ for the problem associated with $\bar{L}$ is continuous and $\bar{u}$ is a metric viscosity solution of (4.33) (with the original H). Since $\tilde{H}_r\leq H_r$, $\bar{u}$ is also a metric viscosity solution of (4.33) with H replaced by $\tilde{H}$. So u and $\bar{u}$ are both viscosity solutions of the same equation. If $\tilde{H}$ satisfies the properties needed for comparison theorem we then obtain $\bar{u}=u$, i.e. u is a metric viscosity solution of the original equation (4.33). This is why we stated the assumptions in this paper for Hamiltonians which are also defined for negative values of the local slope variable.*

5. Stationary Equations

We present three comparison theorems. The first is a typical result for time independent equations with Hamiltonians that have at most linear growth in the local slope variable. The second is a stationary version of Theorem 4.2, and the third is a result for equations of eikonal type.

5.1. **Hamiltonians with sublinear growth.**

Theorem 5.1 (A comparison principle for bounded solutions). *Let $\Omega = \mathbb{S}$. Let $(A1), (A3), (A4)$, and $(A2)$ with $\nu > 0$ be satisfied. Let u be a metric viscosity subsolution of (1.3) and v be a metric viscosity supersolution of (1.3) such that u and $-v$ are bounded from above. Then*

$$m := \lim_{R\to+\infty} \lim_{\gamma\to 0} \sup \Big\{ u(x) - v(y) : \ d(x,y) < \gamma, \ d(x,x_0) + d(y,x_0) \le R \Big\} \le 0. \tag{5.1}$$

In particular $u \le v$ in $\mathbb{S}$.

Proof. The proof follows the lines of the proof of Proposition 3.3. Suppose $m > 0$. Define for $\epsilon, \delta > 0$

$$\Psi(x,y) = u(x) - v(y) - \delta d^2(x,x_0) - \delta d^2(y,x_0) - \frac{d^2(x,y)}{2\epsilon},$$

and let, for $n \ge 1$, $\bar{x}, \bar{y} \in \mathbb{S}$ be such that

$$\Psi(x,y) - \frac{1}{n}(d(x,\bar{x}) + d(y,\bar{y}))$$

has a maximum over $\mathbb{S}\times\mathbb{S}$ at $(\bar{x},\bar{y})$. Obviously $d(\bar{x},x_0) + d(\bar{y},x_0) \le R_\delta$ for some $R_\delta > 0$. Similarly to the proof in the time dependent case we show that

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{d^2(\bar{x},\bar{y})}{\epsilon} = 0 \quad \text{for every } \delta > 0, \tag{5.2}$$

$$\lim_{\delta\to 0} \limsup_{\epsilon\to 0} \limsup_{n\to\infty} \delta(d^2(\bar{x},x_0) + d^2(\bar{y},x_0)) = 0, \tag{5.3}$$

and that for sufficiently small δ, ϵ and sufficiently large n

$$\lim_{\delta\to 0} \limsup_{\epsilon\to 0} \limsup_{n\to\infty} (u(\bar{x}) - v(\bar{y})) > \frac{m}{2}. \tag{5.4}$$

We now have

$$H_{2\delta d(\bar{x},x_0)+1/n}\left(\bar{x}, u(\bar{x}), \frac{d(\bar{x},\bar{y})}{\epsilon}\right) \le 0,$$

which by Remark 3.1 and (5.3) implies

$$H\left(\bar{x}, u(\bar{x}), \frac{d(\bar{x},\bar{y})}{\epsilon}\right) \le L(1 + d(\bar{x},x_0))(2\delta d(\bar{x},x_0) + 1/n) \le \sigma(\delta,\epsilon,n),$$

where $\lim_{\delta\to 0}\limsup_{\epsilon\to 0}\limsup_{n\to\infty} \sigma(\delta,\epsilon,n) = 0$. In the same way we obtain

$$H\left(\bar{y}, v(\bar{y}), \frac{d(\bar{x},\bar{y})}{\epsilon}\right) \ge -\sigma(\delta,\epsilon,n).$$

Therefore, by $(A2)$ and $(A3)$,

$$\begin{aligned}\nu(u(\bar{x}) - v(\bar{y})) &\le H\left(\bar{y}, u(\bar{x}), \frac{d(\bar{x},\bar{y})}{\epsilon}\right) - H\left(\bar{x}, u(\bar{x}), \frac{d(\bar{x},\bar{y})}{\epsilon}\right) + 2\sigma(\delta,\epsilon,n) \\ &\le \omega_{\tilde{R}_\delta}\left(d(\bar{x},\bar{y})\Big(1 + \frac{d(\bar{x},\bar{y})}{\epsilon}\Big)\right) + 2\sigma(\delta,\epsilon,n)\end{aligned}$$

for some $\tilde{R}_\delta > 0$. This, together with (5.2) and (5.4), produces a contradiction. ∎

5.2. **Hamiltonians with superlinear growth.** We consider the equation

$$u + H(x, |\nabla u|) - f(x) = 0 \quad \text{in } \mathbb{S}. \tag{5.5}$$

Theorem 5.2 (A comparison principle for solutions which may be unbounded)**.** *Let $(B1)-(B6)$ be true and let f satisfy $(A5)$. Let u be a metric viscosity subsolution of (5.5) and v be a metric viscosity supersolution of (5.5) satisfying*

$$\limsup_{d(x,x_0)\to+\infty} \frac{u(x)}{1+d^\alpha(x,x_0)} \le 0, \qquad \limsup_{d(x,x_0)\to+\infty} \frac{-v(x)}{1+d^\alpha(x,x_0)} \le 0. \tag{5.6}$$

Then $u \le v$.

Proof. The proof mostly repeats the arguments of the proof of Theorem 4.2. If $\sup(u-v) \ge 2\nu_1 > 0$ then there is $\lambda_0 < 1$ such that $\sup(\lambda u - v) \ge \nu_1$ for $\lambda_0 < \lambda < 1$. We define for $\lambda_0 < \lambda < 1, \delta > 0, \epsilon > 0$ the function

$$\Psi(x,y) = \lambda u(x) - v(y) - \delta(1+d^\alpha(x,x_0)) - \delta(1+d^\alpha(y,x_0)) - \frac{d^2(x,y)}{2\epsilon}.$$

Since by (5.6), $\Psi(x,y) \to -\infty$ as $\min(d(x,x_0), d(y,x_0)) \to +\infty$, uniformly for ϵ, using Lemma 2.9, for every $n \ge 1$ there are $(\bar{x},\bar{y}) \in B_{R_\delta}(x_0) \times B_{R_\delta}(x_0)$ for some $R_\delta > 0$ such that

$$\Psi(x,y) - \frac{1}{n}(d(x,\bar{x}) + d(y,\bar{y}))$$

has a maximum over $\mathbb{S}\times\mathbb{S}$ at $(\bar{x},\bar{y})$. Moreover by an argument as in the proof of Proposition 3.3, (5.2) is satisfied and, for λ sufficiently close to 1,

$$\lim_{\delta\to 0}\limsup_{\epsilon\to 0}\limsup_{n\to\infty}(\lambda u(\bar{x}) - v(\bar{y}) - \delta(1+d^\alpha(\bar{x},x_0)) - \delta(1+d^\alpha(\bar{y},x_0))) > \frac{\nu_1}{2}. \tag{5.7}$$

By the definition of viscosity subsolution we have

$$u(\bar{x}) + H\left(\bar{x}, \frac{1}{\lambda}\Big(\frac{d(\bar{x},\bar{y})}{\epsilon} - \delta\alpha d^{\alpha-1}(\bar{x},x_0)\Big)\right) - f(\bar{x}) \le \rho_1(n), \tag{5.8}$$

where $\lim_{n\to+\infty}\rho_1(n) = 0$ for fixed $\lambda, \delta, \epsilon$. Repeating the arguments that led to (4.4) we obtain

$$\begin{aligned} H\left(\bar{x}, \frac{1}{\lambda}\Big(\frac{d(\bar{x},\bar{y})}{\epsilon} - \delta\alpha d^{\alpha-1}(\bar{x},x_0)\Big)\right) &\ge \frac{1}{\eta}H\left(\bar{x}, \frac{\eta}{\lambda}\frac{d(\bar{x},\bar{y})}{\epsilon}\right) - \frac{(1-\eta)}{\eta}H\left(\bar{x}, \frac{\eta}{1-\eta}\frac{\delta\alpha d^{\alpha-1}(\bar{x},x_0)}{\lambda}\right) \\ &\ge \frac{1}{\lambda}\gamma\left(\frac{1+\lambda}{2\lambda}\right)H\left(\bar{x}, \frac{d(\bar{x},\bar{y})}{\epsilon}\right) \\ &\quad - C_1(\alpha,\lambda)\delta^m(1+d(\bar{x},x_0))^\kappa d^{m(\alpha-1)}(\bar{x},x_0), \end{aligned} \tag{5.9}$$

where $\eta = (1+\lambda)/2$ and we used $(B2)$ and $(B3)$. Since for $\delta < \delta_0 = \delta_0(\lambda,\alpha,\kappa,m)$

$$C_1(\alpha,\lambda)\delta^m(1+d(\bar{x},x_0))^\kappa d^{m(\alpha-1)}(\bar{x},x_0) \le \frac{\delta}{\lambda}(1+d^\alpha(\bar{x},x_0)),$$

it thus follows from (5.8) and (5.9) that for $\delta < \delta_0$

$$u(\bar{x}) - \frac{\delta}{\lambda}(1 + d^\alpha(\bar{x}, x_0)) + \frac{1}{\lambda}\gamma\left(\frac{1+\lambda}{2\lambda}\right) H\left(\bar{x}, \frac{d(\bar{x},\bar{y})}{\epsilon}\right) - f(\bar{x}) \le \rho_1(n),$$

which, by $(B6)$ implies

$$\lambda u(\bar{x}) - \delta(1 + d^\alpha(\bar{x}, x_0)) + \gamma\left(\frac{1+\lambda}{2\lambda}\right) H\left(\bar{x}, \frac{d(\bar{x},\bar{y})}{\epsilon}\right) - f(\bar{x}) \le \rho_1(n) - (1-\lambda)M. \tag{5.10}$$

Defining τ and r as in (4.8) and arguing like in (4.9) we also obtain for $\delta < \delta_1 = \delta_1(\lambda, \alpha, \kappa, m)$

$$v(\bar{y}) + \delta(1 + d^\alpha(\bar{y}, x_0)) + \tau\bar{\gamma}\left(\frac{1}{\tau}\right) H\left(\bar{y}, \frac{d(\bar{x},\bar{y})}{\epsilon}\right) - f(\bar{y}) \ge -\rho_1(n). \tag{5.11}$$

Subtracting (5.11) from (5.10) yields

$$\begin{aligned}
&\lambda u(\bar{x}) - v(\bar{y}) - \delta(1 + d^\alpha(\bar{x}, x_0)) - \delta(1 + d^\alpha(\bar{y}, x_0)) \\
&\le -rH\left(\bar{x}, \frac{d(\bar{x},\bar{y})}{\epsilon}\right) - \tau\bar{\gamma}\left(\frac{1}{\tau}\right)\left(H\Big(\bar{x}, \frac{d(\bar{x},\bar{y})}{\epsilon}\Big) - H\Big(\bar{y}, \frac{d(\bar{x},\bar{y})}{\epsilon}\Big)\right) \\
&+ f(\bar{x}) - f(\bar{y}) + 2\rho_1(n) - (1-\lambda)M \\
&\le -r\theta\left(\frac{d(\bar{x},\bar{y})}{\epsilon}\right)^m + \tau\overline{\gamma}\left(\frac{1}{\tau}\right)\omega_{R_\delta}(d(\bar{x},\bar{y}))\left(1 + \Big(\frac{d(\bar{x},\bar{y})}{\epsilon}\Big)^m\right) \\
&+ \sigma_{R_\delta}(d(\bar{x},\bar{y})) + 2\rho_1(n) - (1-\lambda)M
\end{aligned} \tag{5.12}$$

We now obtain a contradiction if we choose $\lambda_0 < \lambda < 1$ such that $-(1-\lambda)M < \nu_1/4$, take $\lim_{\delta\to 0}\limsup_{\epsilon\to 0}\limsup_{n\to\infty}$ in (5.12), and use (5.2) and (5.7). ■

5.3. **Eikonal type equations.** In this section we consider equations which are not "proper", i.e. they are not strictly monotone in the zero order variable. In such cases a typical technique is to perturb a subsolution/supersolution so that a perturbed function is a subsolution/supersolution of the equation with strict inequality. There are various ways to do it. Here we present a standard technique which applies to equations of eikonal type. We refer to [48] for more on such techniques in domains of $\mathbb{R}^n$.

Let Ω be an open and bounded subset of $\mathbb{S}$. We consider the equation

$$\begin{cases} a(x)|\nabla u| - f(x) = 0, & \text{in } \Omega, \\ u(x) = g(x) & \text{on } \partial\Omega. \end{cases}$$

Defining

$$H(x,s) = \begin{cases} a(x)s & s \ge 0, \\ 0 & s < 0, \end{cases}$$

we rewrite the above equation as

$$\begin{cases} H(x, |\nabla u|) - f(x) = 0, & \text{in } \Omega, \\ u(x) = g(x) & \text{on } \partial\Omega. \end{cases} \tag{5.13}$$

We assume that

- $(D1)$

$$|a(x) - a(y)| \le C d(x, y), \quad \text{for all } x, y \in \Omega.$$

Theorem 5.3. *Let Ω be an open and bounded subset of $\mathbb{S}$, let $(D1)$ be true, and let f satisfy $(A5)$ on Ω and $(B6)$ on Ω with $M > 0$. Let g be uniformly continuous on $\partial\Omega$. Let u be a metric viscosity subsolution of (5.13) and v be a metric viscosity supersolution of (5.13). Suppose that*

$$u(y) \le g(x) + \sigma_0(d(x, y)), \quad v(y) \ge g(x) - \sigma_0(d(x, y)) \quad \text{for all } x \in \partial\Omega, y \in \Omega, \tag{5.14}$$

for some modulus σ_0. Then $u \le v$ in $\overline{\Omega}$.

Proof. Recall that since Ω is bounded and u is locally bounded, u is bounded on Ω. If $u \not\le v$ then for $0 < \lambda < 1$ sufficiently close to 1, we have $\sup_\Omega(\lambda u - v) \ge 2\nu_1 > 0$. By Lemma 2.9, for $0 < \lambda < 1, \epsilon > 0, n \ge 1$ there are points $(\bar{x}, \bar{y}) \in \overline{\Omega} \times \overline{\Omega}$ such that the function

$$\Psi(x, y) := \lambda u(x) - v(y) - \frac{d^2(x, y)}{2\epsilon} - \frac{1}{n}(d(x, \bar{x}) + d(y, \bar{y}))$$

has a maximum over $\overline{\Omega} \times \overline{\Omega}$ at $(\bar{x}, \bar{y})$. Moreover as before we have that (5.2) is satisfied. Since Ω is a bounded set and $(\bar{x}, \bar{y})$ maximizes Ψ then for n large enough

$$\lambda u(\bar{x}) - v(\bar{y}) = \Psi(\bar{x}, \bar{y}) + \frac{d^2(\bar{x}, \bar{y})}{2\epsilon} > \nu_1. \tag{5.15}$$

If we assume for instance that $\bar{x} \in \partial\Omega$ then, since $u(\bar{x}) \le g(\bar{x})$ and (5.14) holds,

$$\lambda u(\bar{x}) - v(\bar{y}) \le \lambda g(\bar{x}) - v(\bar{y}) \le -(1-\lambda) g(\bar{x}) + \sigma_0\big(d(\bar{x}, \bar{y})\big) \le -(1-\lambda) u(\bar{x}) + \sigma_0\big(d(\bar{x}, \bar{y})\big). \tag{5.16}$$

Thus, since $|\lambda - 1| << 1$ and $d(\bar{x}, \bar{y}) << 1$ if ϵ and $1/n$ are small, (5.15) contradicts (5.16). Consequently, $(\bar{x}, \bar{y}) \in \Omega \times \Omega$ if λ is sufficiently close to 1 and $\epsilon, 1/n$ are sufficiently small. Therefore by the definition of viscosity subsolution

$$H_{1/(n\lambda)}\left(\bar{x}, \frac{1}{\lambda}\frac{d(\bar{x}, \bar{y})}{\epsilon}\right) - f(\bar{x}) \le 0,$$

which implies

$$H\left(\bar{x}, \frac{d(\bar{x}, \bar{y})}{\epsilon}\right) - \lambda f(\bar{x}) \le \rho_1(\lambda, \epsilon; n), \tag{5.17}$$

where $\lim_{n \to +\infty} \rho_1(\lambda, \epsilon; n) = 0$ for fixed λ, ϵ. Since $f \ge M > 0$, it follows

$$-\lambda f(\bar{x}) \ge -f(\bar{x}) + M(1 - \lambda),$$

and thus we obtain in (5.17)

$$H\left(\bar{x}, \frac{d(\bar{x}, \bar{y})}{\epsilon}\right) - f(\bar{x}) \le -M(1 - \lambda) + \rho_1(\lambda, \epsilon; n). \tag{5.18}$$

(We remark that the same argument shows that in fact λu is a metric viscosity subsolution of $H(x,|\nabla u|) - f + M(1-\lambda) = 0$.) Using the definition of viscosity supersolution we obtain

$$H\left(\bar{y}, \frac{d(\bar{x},\bar{y})}{\epsilon}\right) - f(\bar{y}) \geq -\rho_1(\lambda,\epsilon;n). \tag{5.19}$$

Subtracting (5.18) from (5.19) we thus have

$$\begin{aligned} M(1-\lambda) &\leq H\left(\bar{y}, \frac{d(\bar{x},\bar{y})}{\epsilon}\right) - H\left(\bar{x}, \frac{d(\bar{x},\bar{y})}{\epsilon}\right) + f(\bar{x}) - f(\bar{y}) + 2\rho_1(\lambda,\epsilon;n) \\ &\leq \frac{d(\bar{x},\bar{y})}{\epsilon}|a(\bar{y}) - a(\bar{x})| + \sigma(d(\bar{x},\bar{y})) + 2\rho_1(\lambda,\epsilon;n) \\ &\leq C\frac{d^2(\bar{x},\bar{y})}{\epsilon} + \sigma(d(\bar{x},\bar{y})) + 2\rho_1(\lambda,\epsilon;n) \end{aligned} \tag{5.20}$$

which gives a contradiction if we let $\lim_{\epsilon\to 0}\limsup_{n\to\infty}$ above and use (5.2). ■

We remark that if we know in advance that either u or v is more regular then condition $(D1)$ can be relaxed. In particular (see the proof of Theorem 3.7) if either u or v is Lipschitz continuous, then $(D1)$ can be replaced by the uniform continuity of a.

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