Équation fractionnaire des films minces et fractures hydrauliques

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A fractional thin film equation

- Porous Media Equation

$$\partial_t u - \partial_x(u^n \partial_x u) = 0$$

- Thin Film Equation

$$\partial_t u - \partial_x(u^n \partial_x(-\partial_{xx} u)) = 0$$

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Outline of the talk

1. Derivation of the Equation: Hydraulic fractures.
   1. The lubrication approximation
   2. Pressure law
   3. The free boundary condition

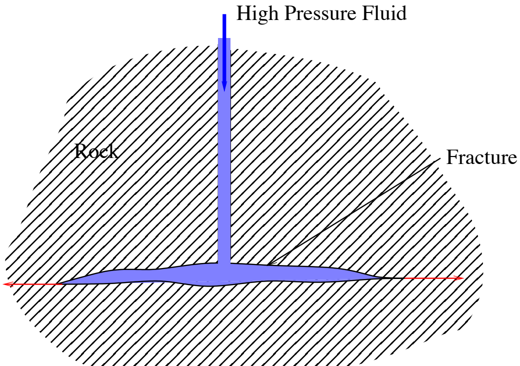
2. Self-Similar solutions
   1. When $K = 0$
   2. When $K \neq 0$

3. Existence of weak solutions when $K = 0$ for general initial data
   1. Integral inequalities
   2. Existence result

Hydraulic Fracture

Hydraulic Fracturing: Propagation of fractures in a rock layer caused by the presence of a pressurized fluid

- Occur naturally (volcanic dikes causes by magma pressure)
- Fracking: Artificial injection of a highly-pressurized fluid to create new channels in the rock, to increase the extraction rate of oil and natural gas (shale gas)

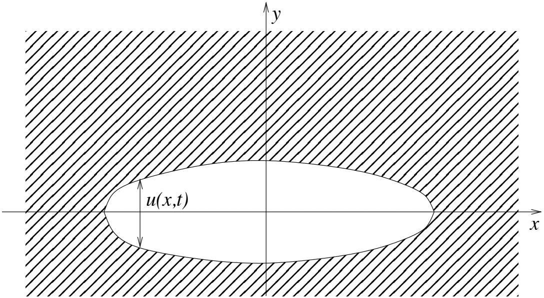


KGD model (Khristianovic, Geertsma and De Klerk)

Model developed by Khristianovic and Zheltov ('55) and Geertsma and De Klerk ('69).

- invariant with respect to z
- symmetric with respect to y

The fracture can then be entirely described by its opening $u(x, t)$ in the y direction:



Derivation: Lubrication approximation

- **Conservation of mass** for the fluid inside the fracture:

$$\partial_t(\rho u) + \partial_x(\rho u \overline{v}) = h(t)\delta \quad \text{in } \mathbb{R}$$

  where

  - ρ= density of the fluid = constant (= 1)
  - $\overline{v} = \frac{1}{u}\int_{-u/2}^{u/2} v_H(t,x,y)\,dy$ (averaged horizontal velocity of the fluid)
  - h is the injection rate.

- **Lubrication approximation**: Navier-Stokes equations reduce to

$$\mu \frac{\partial^2 v_H}{\partial y^2}(t,x,y) = \partial_x p(x,t)$$

  Assuming no-slip boundary condition ($v_H = 0$) at $y = \pm u/2$:

$$\overline{v}(x,t) = -\frac{u^2}{12\mu}\partial_x p(x,t).$$

Lubrication approximation

Putting everything together, we get:

$$\boxed{\partial_t u - \partial_x \left(\frac{u^3}{12\mu} \partial_x p \right) = h(t)\delta \qquad \text{in } \{u > 0\}}$$

Remark:

- The thin film equation corresponds to

$$\begin{aligned} p = & \text{ pressure due to surface tension at } z = u(x,t) \\ & \sim -\sigma \partial_{xx} u \end{aligned}$$

- If we use Navier slip condition instead of no-slip, we get:

$$\partial_t u - \partial_x \left((u^3 + \Lambda u^s) \partial_x p \right) = h(t)\delta \,, \qquad s = 1 \text{ or } 2$$

Pressure law

For hydraulic fractures:

Pressure $p(x, t)$ = Pressure exerted by the rock on the fluid.

Linear Elasticity:

- The **strain tensor** ϵ is related to the **displacement u** by:

$$\epsilon = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$$

- The **stress tensor**, $\boldsymbol{\sigma}$ satisfies $\operatorname{div} \boldsymbol{\sigma} = 0$ (no external forces)
- The **stress-strain** relations (Hooke's law):

$$\epsilon = \frac{1}{E}((1+\nu)\boldsymbol{\sigma} - \nu[\text{trace}(\boldsymbol{\sigma})I])$$

where E is Young's modulus and ν is Poisson's ratio.

Equations for plane-strain

We assume that the solid is in a **state of plane-strain**:

$$u_z = 0, \quad u_x \text{ and } u_y \text{ independent of the } z \text{ coordinate}$$

- **Equilibrium conditions**:

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} = 0 \\ \dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{xy}}{\partial x} = 0 \end{cases}$$

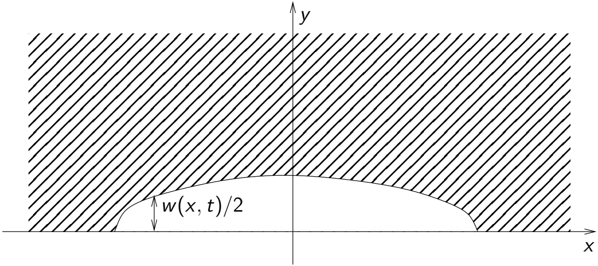
- **Stress-strain relations**

$$\begin{cases} \epsilon_{xx} = \frac{1}{2G}[\sigma_{xx} - \nu(\sigma_{xx} + \sigma_{yy})] \\ \epsilon_{yy} = \frac{1}{2G}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{yy})] \\ \epsilon_{xy} = \frac{1}{2G}\sigma_{xy} \end{cases}$$

where ν is Poisson's ratio and $G = \frac{1}{2}\frac{E}{1+\nu}$ is the shear modulus.

We get 5 equations with 5 unknowns: σ_{xx}, σ_{yy}, σ_{xy}, u_x, u_y.

Fracture in an infinite solid



$\sigma_{xy}(x,0) = 0, \qquad u_y(x,0) = w(x)/2$

- **Airy stress function**: There exists a bi-harmonic potential $U(x, y)$ such that

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 U}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y}.$$

  Equilibrium conditions are satisfied.
- Use **Fourier transform** with respect to x to solve the stress-strain relations.

Fracture in an infinite solid, cont.

Using the boundary conditions, we find

$$\widehat{U}(k,y) = A(k)(1 + y|k|)e^{-|k|y} \quad \text{for all } k \in \mathbb{R},\ y > 0.$$

with

$$A(k) = \frac{G}{2(1-\nu)}\frac{1}{|k|}\widehat{w}(k).$$

The **pressure** exerted by the rock in the y direction along $y = 0$ is given by

$$\begin{aligned} \widehat{p}(k) &:= -\widehat{\sigma_{yy}}(k,0) \\ &= k^2\widehat{U}(k,0) \\ &= \frac{G}{2(1-\nu)}|k|\widehat{w}(k) \quad \text{for all } k \in \mathbb{R}. \end{aligned}$$

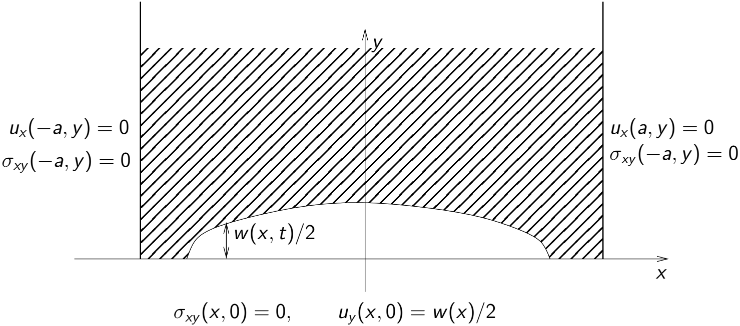
This corresponds to the relation

$$\boxed{p(x) = \frac{G}{2(1-\nu)}(-\Delta)^{1/2}w(x) \qquad \text{for } x \in \mathbb{R}.}$$

Fracture in a constrained solid

The solid is constrained by rigid lubricated walls on both sides:

- the horizontal displacement is zero
- the wall does not transmit a shear stress to the solid



Remark: On the lateral boundary, we have $\epsilon_{xy} = 0$ and so

$$\frac{\partial u_y}{\partial x} = 0 \quad \text{on } \partial\Omega \times \mathbb{R}.$$

Fracture in a contraint solid, cont.

We use Fourier sine series for u_x and Fourier cosine series for u_y:

If

$$w(x) = \sum_{k \in \mathbb{N}} \widehat{w}(k) \cos(kx),$$

then the pressure $p(x)$ is given by

$$\boxed{p(x) = \frac{G}{2(1-\nu)} \sum_{k \in \mathbb{N}} k \widehat{w}(k) \cos(kx).}$$

We denote

$$p(x) = I(w)$$

where I is a **nonlocal elliptic operator of order** 1 (half-Laplacian with Neumann boundary conditions).

What we have so far

The opening $u(x, t)$ solves

$$\partial_t u - \partial_x(u^n \partial_x(-\partial_{xx})^{1/2}(u)) = h(t)\delta \qquad \text{in } \{u > 0\}$$

with $n = 3$ ($n = 1$ or $n = 2$ also relevant).

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with $n = 3$ ($n = 1$ or $n = 2$ also relevant).

We need some **boundary conditions** on $\partial\{u > 0\}$:

- $u = 0$ (zero width)
- $u^n \partial_x(-\partial_{xx})^{1/2}(u) = 0$ (zero flux = zero fluid loss at the tip of the fracture)

Because $\{u > 0\}$ is not known a priori (free boundary problem), we need an additional condition.

In the case of the thin film equation, this is the contact angle condition: $|\partial_x u| = \gamma$.

The free boundary condition

Propagation condition:

$$K_I = K_{IC}$$

where

- K_{IC} =**rock toughness** (given coefficient)
- K_I = **stress intensity factor**. Assuming that $\{u > 0\} = (-1, 1)$, we have

$$\begin{aligned} K_I &:= \lim_{x\to 1^+} \sqrt{2\pi}\sqrt{x-1}\sigma_{yy}(x,0) \\ &= -\lim_{x\to 1^+} \sqrt{2\pi}\sqrt{x-1}p(x) \\ &= \frac{E}{4(1-\nu^2)} \lim_{x\to 1^-} \sqrt{2\pi}\sqrt{1-x}u'(x) \end{aligned}$$

We deduce

$$u(x) \sim K\sqrt{1-x} \qquad \text{as} \quad x \to 1^-$$

where $K := \sqrt{\frac{2}{\pi}} \frac{4(1-\nu^2)}{E} K_{IC}$

Fractional thin film equation

Summary

The width of the crack $u(x,t)$ solves the free boundary problem

$$\partial_t u - \partial_x(u^n \partial_x(-\partial_{xx})^{1/2} u) = h(t)\delta \qquad \text{in } \{u > 0\}$$

$$u = 0, \quad u^n \partial_x I(u) = 0 \qquad \text{on } \partial\{u > 0\}$$

$$u(t,x) = K\sqrt{|x - x_0|} + o\left(\sqrt{|x - x_0|}\right) \qquad \text{as } x \sim x_0 \in \partial\{u > 0\}$$

where $n \geq 1$.

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where $n \geq 1$.

When $K = 0$ (**zero toughness or pre-cracked**), we can solve instead

$$\partial_t u - \partial_x(u^n \partial_x(-\partial_{xx})^{1/2} u) = h(t)\delta \qquad \text{in } \mathbb{R} \times (0, \infty)$$

and hope to recover

$$u(t,x) = o\left(\sqrt{|x - x_0|}\right) \qquad \text{as } x \sim x_0 \in \partial\{u > 0\}$$

(this corresponds to the complete wetting regime for thin film).

The exponent $n = 3$

In 1971, Huh and Scriven noted that for $n = 3$, the motion of the contact line leads to infinite dissipation of energy for the **thin film equation**:

The velocity of the fluid is given by

$$v = -u^2 \partial_x p$$

and the dissipation of energy is given by

$$D(u) = \int u^3 (\partial_{xxx} u)^2 \, dx = \int \frac{v^2}{u}$$

If $u \sim x_+$ and $v \neq 0$ at the free boundary this is infinite.

For the **Hydraulic fractures**, we have a similar computation but $u \sim x_+^{1/2}$, so the dissipation is finite even when $v \neq 0$.

In fact, a similar argument shows that $n = 4$ is the critical exponent for our equation.

References

- Derivation of the equation:
  - Khristianovic-Zheltov ('55)
  - Geertsma-De Klerk ('69)
- Formal asymptotic at the tip of the fracture (self-similar solutions):
  - Spence and Sharp ('85)
  - Adachi-Detournay ('94)
  - Mitchell-Kuske-Peirce ('06).
- Numerical computations:
  - Peirce-Detournay ('08,'09)
  - Peirce-Siebrits ('05).

Particular solutions

Stationary solution

When $h = 0$ (no injection of fluid) and $K \neq 0$, then the free boundary problem has a stationary solution supported in $(-1, 1)$:

$$\boxed{V(x) = \frac{K}{\sqrt{2}}\sqrt{(1 - x^2)_+}.}$$

When $K = 0$, there is no stationary solution in $\mathbb{R}$ (solutions are always expanding).

Self-similar solutions

Self-similar solutions:

$$u(t,x) = t^{-\alpha} U(t^{-\beta} x)$$

With U even and supported in $(-a, a)$ satisfying

$$-\alpha U(t^{-\beta}x) - \beta t^{-\beta} x U'(t^{-\beta}x) + t^{-n\alpha+1-3\beta}(U^n I(U)')'(t^{-\beta}x) = t^{1+\alpha} h(t) t^{-\beta} \delta(t^{-\beta}x)$$

So we must have

$$1 - 3\beta = n\alpha$$

$$h(t) = \lambda t^{-\alpha-1+\beta} \qquad \lambda \in \mathbb{R}$$

and

$$\boxed{-\alpha U - \beta y U' + (U^n I(U)')' = \lambda\delta \qquad \text{in } (-a,a).}$$

+ boundary conditions

Boundary conditions

- The null flux condition gives:

$$(\beta - \alpha) \int U(y)\, dy = \lambda$$

- If U satisfies $U(x) = K\sqrt{|x-a|} + o\left(\sqrt{|x-a|}\right)$, then $u(t,x)$ satisfies

$$u(t,x) = Kt^{-\alpha-\beta/2}|x-a(t)|^{1/2} + o(|x-a(t)|^{1/2})$$

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Either $K = 0$ (zero toughness) or if $\alpha = -\frac{\beta}{2}$

- When $h(t) = 0$ there exist a self-similar solution (with constant mass) only if $\alpha = \beta = \frac{1}{n+3}$ and $K = 0$ (zero toughness case)
- For given toughness coefficient $K > 0$, a self similar solution with $\beta = -2\alpha = \frac{2}{n-6}$ and the injection rate is given by

$$h(t) = \lambda t^{\frac{n-3}{6-n}}$$

Existence of self similar solution $K = 0$

Theorem (Imbert, M. 2015 part 1)

Let $n \in [1, 4)$. Assume that $K = 0$ and $h(t) = 0$.
Then, for any $m > 0$ there exists a self-similar solution

$$u(t,x) = t^{-\frac{1}{n+3}} U(t^{-\frac{1}{n+3}} x)$$

satisfying $\int_{\mathbb{R}} u(t,x)\,dx = m$ for all $t > 0$.

For all $t > 0$, there exists a constant $C(t) > 0$ such that

$$u(t,x) = \begin{cases} C(t)|x-x_0|^{\frac{3}{2}} + \mathcal{O}(|x-x_0|^{\frac{2}{n}}) & \text{if } n \in [1, \frac{4}{3}) \\ C(t)|x-x_0|^{\frac{3}{2}} \left| \ln|x-x_0| \right|^{\frac{3}{4}} + \mathcal{O}(|x-x_0|^{\frac{3}{2}}) & \text{if } n = \frac{4}{3} \\ C(t)|x-x_0|^{\frac{2}{n}} + o(|x-x_0|^{\frac{2}{n}}) & \text{if } n \in (\frac{4}{3}, 4) \end{cases}$$

when $x \to x_0$, for any $x_0 \in \partial\{u(t,\cdot) > 0\}$.

Existence of self similar solution $K \neq 0$

Theorem (Imbert, M. 2015, part 2)

Let $n \in [1,4)$. Given $K > 0$ and $\lambda > 0$.
There exists a self-similar solution

$$u(t,x) = t^{\frac{1}{6-n}} U(t^{-\frac{2}{6-n}} x)$$

with injection rate $h(t) = \lambda \, t^{\frac{n-3}{6-n}}$

Furthermore, u satisfies

$$u(t,x) = K\sqrt{|x-x_0|} + \begin{cases} \mathcal{O}(|x-x_0|^{\frac{3}{2}}) & \text{if } n \in [1,2) \\ \mathcal{O}\left(|x-x_0|^{\frac{3}{2}} \ln\left(\frac{1}{|x-x_0|}\right)\right) & \text{if } n = 2 \\ \mathcal{O}(|x-x_0|^{\frac{5-n}{2}}) & \text{if } n \in (2,4) \end{cases}$$

when $x \to x_0$, for any $x_0 \in \partial\{u(t,\cdot) > 0\}$.

Remarks

1. Thin film equation [*Bernis-Peletier-Williams ('92), Ferreira and Bernis ('97)*]
   Self similar solutions for the zero contact angle case ($K = 0$).
2. In the physical case ($n = 3$), we find $h(t) = \lambda$ (constant injection rate).
   We recover several known (formal) results concerning the rate of growth of hydraulic fractures [*Adachi-Detournay ('94), Garagash ('06), Mitchell-Kuske-Peirce ('06)*]
3. When $K = 0$ we have

   $$\lim_{t \to 0^+} u(t, x) = m\delta$$

   (Source-type solution).
   When $K \neq 0$, we have

   $$\lim_{t \to 0^+} ||u(t, x)||_{L^\infty} = 0.$$

4. Uniqueness is a completely open problem.

Idea of the proof (Ferreira-Bernis ('97))

In the case $K = 0$, $h = 0$, the equation for U reads:

$$[-\alpha x U - U^n (-\Delta)^{1/2} (U)']' = 0 \quad \text{in } (-a, a)$$

which using the boundary condition gives:

$$\boxed{(-\Delta)^{1/2} (U)' = -\alpha x U^{1-n} \quad \text{in } (-a, a)}$$

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Up to some rescaling we can take $\alpha = 1$ and $a = 1$

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$$[-\alpha x U - U^n(-\Delta)^{1/2}(U)']' = 0 \quad \text{in } (-a, a)$$

which using the boundary condition gives:

$$\boxed{(-\Delta)^{1/2}(U)' = -xU^{1-n} \quad \text{in } (-1, 1)}$$

We prove the existence by a fixed point argument on the integral formulation

$$U(x) = \int_{-1}^{1} g(x,z)\, z\, (U(z))^{1-n}\, dz.$$

where $g(x,z)$ is the green function for

$$-(-\Delta)^{1/2}(U)' = f \quad \text{in } (-1,1), \quad U = 0 \quad \text{in } \mathbb{R} \setminus (-1,1)$$

Idea of the proof

- Regularize: For all $k > 0$ there exists U_k such that

$$\begin{cases} U_k(x) = \displaystyle\int_{-1}^{1} zg(x,z)\left(\frac{1}{k} + U_k(z)\right)^{1-n} dz & \text{for } x \in (-1,1) \\ U_k(x) = 0 & \text{for } x \notin (-1,1). \end{cases}$$

- Pass to the limit $k \to \infty$.

The second step requires delicate asymptotic results on the solution of

$$-(-\Delta)^{1/2}(U)' = f \quad \text{in } (-1,1), \quad U = 0 \quad \text{in } \mathbb{R} \setminus (-1,1)$$

Lemma

If $f : (-1,1) \to \mathbb{R}$ satisfies $|f(z)| \leq M(1-z^2)^a$ for some $a > -\frac{3}{2}$ then

$$|U'(x)| \leq CM\, F(1-x^2), \quad \text{with } F(y) = \begin{cases} y^{a+1} & \text{if } -\frac{3}{2} < a < -\frac{1}{2} \\ y^{\frac{1}{2}} \ln\left(\frac{1}{y}\right) & \text{if } a = -\frac{1}{2} \\ y^{\frac{1}{2}} & \text{if } a > -\frac{1}{2}. \end{cases}$$

Idea of the proof

Lemma

If

$$f(z) = z(1-z^2)^a h(z)$$

where $h(z) \geq 0$ *is a bounded even function on* $(-1,1)$ *and* $a > -\frac{3}{2}$, *then*

$$U'(x) = \begin{cases} -C_0(1-x^2)^{a+1} + o((1-x^2)^{a+1}) & \text{if } -\frac{3}{2} < a < -\frac{1}{2} \\ -C_0(1-x^2)^{\frac{1}{2}} \ln\left(\frac{1}{(1-x^2)}\right) + \mathcal{O}((1-x^2)^{\frac{1}{2}}) & \text{if } a = -\frac{1}{2} \\ -C_0(1-x^2)^{\frac{1}{2}} + \mathcal{O}((1-x^2)^{a+1}) & \text{if } a > -\frac{1}{2}. \end{cases}$$

where

$$C_0 = \begin{cases} c_a h(1) & \text{when } -\frac{3}{2} < a \leq -\frac{1}{2} \\ \frac{1}{2\pi}\int_0^1 2f(\sqrt{1-v})v^{-1/2}dv & \text{when } a > -\frac{1}{2} \end{cases}$$

for some constant c_a *depending only on* a.

Solution of the Cauchy problem when $K = 0$

The Cauchy value problem

We now want to solve the Cauchy problem

$$\begin{aligned} &\partial_t u - \partial_x(u^n \partial_x(-\partial_{xx})^{1/2} u) = 0 && \text{in } \mathbb{R} \times (0, \infty) \\ &u(x, 0) = u_0(x) && \text{in } \mathbb{R} \end{aligned}$$

and hope to recover

$$u(t, x) = o\left(\sqrt{|x - x_0|}\right) \quad \text{as } x \sim x_0 \in \partial\{u > 0\} \qquad (K = 0)$$

The Cauchy value problem

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$$\begin{aligned} &\partial_t u - \partial_x(u^n \partial_x(-\partial_{xx})^{1/2} u) = 0 && \text{in } \Omega \times (0,\infty) \\ &u(x,0) = u_0(x) && \text{in } \Omega \end{aligned}$$

and hope to recover

$$u(t,x) = o\left(\sqrt{|x - x_0|}\right) \quad \text{as } x \sim x_0 \in \partial\{u > 0\} \qquad (K = 0)$$

We work in a (fixed) bounded domain $\Omega \subset \mathbb{R}$.

The operator $(-\partial_{xx})^{1/2}$ is replaced with

$$I : \sum_{k=0}^{\infty} c_k \varphi_k \longmapsto \sum_{k=0}^{\infty} c_k \lambda_k^{\frac{1}{2}} \varphi_k$$

where (λ_k, φ_k) are the e-value and e-function of $-\Delta$ in Ω with **Neuman** boundary conditions

The Cauchy value problem

We consider

$$
\begin{aligned}
&\partial_t u - \partial_x(u^n \partial_x I(u)) = 0 && \text{in } \Omega \times (0,\infty) \\
&u^n \partial_x I(u) = 0 && \text{in } \partial\Omega \times (0,\infty) \\
&u(x,0) = u_0(x) && \text{in } \Omega
\end{aligned}
$$

where I = half-Laplacian with Neumann boundary condition.

The Cauchy value problem

We consider

$$\begin{aligned} &\partial_t u - \partial_x(u^n \partial_x I(u)) = 0 && \text{in } \Omega \times (0,\infty) \\ &u^n \partial_x I(u) = 0 && \text{in } \partial\Omega \times (0,\infty) \\ &u(x,0) = u_0(x) && \text{in } \Omega \end{aligned}$$

where I = half-Laplacian with Neumann boundary condition.

- Reminiscent of the Porous media equation, but no maximum principle. Even the fact that $u_0(x) \geq 0 \Longrightarrow u(x,t) \geq 0$ is non trivial.
- The analysis relies mostly on integral inequalities
- Similar to the thin film equation but
  - the equation is non local
  - the energy is not as good.

Integral inequalities (Lyapunov functionals)

Strong solutions satisfies three important inequalities

- Conservation of mass

$$\int_\Omega u(t,x)\,dx = \int_\Omega u_0(x)\,dx$$

- Energy inequality

$$\frac{d}{dt}\int_\Omega u\,I(u)\,dx + \int_\Omega u^n(\partial_x I(u))^2\,dx = 0$$

- Entropy inequality

$$\frac{d}{dt}\int_\Omega G(u)\,dx + \int_\Omega u_x\,I(u)_x\,dx = 0$$

where G is a non-negative convex function satisfying $G''(s) = s^{-n}$.

Functional spaces

Recall that $I : \sum_{k=0}^{\infty} c_k \varphi_k \longmapsto \sum_{k=0}^{\infty} c_k \lambda_k^{\frac{1}{2}} \varphi_k$

We define

$$H_N^s(\Omega) = \left\{ u = \sum_{k=0}^{\infty} c_k \varphi_k \; ; \; \sum (1 + \lambda_k^s) c_k^2 < \infty \right\}$$

with the norm

$$\|u\|_{H_N^s}^2 \;=\; \sum (1 + \lambda_k^s) c_k^2$$

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with the norm

$$\|u\|_{H_N^s}^2 \; = \; \sum (1 + \lambda_k^s) c_k^2$$

- If $0 \leq s < 3/2$, then $H_N^s(\Omega) = H^s(\Omega)$
- If $3/2 < s < 7/2$, then $H_N^s(\Omega) = \{u \in H^s(\Omega) \, ; \, u_\nu = 0 \text{ on } \partial\Omega\}$.
- if $s = 3/2$, then

$$H_N^{3/2}(\Omega) = \left\{ u \in H^{3/2}(\Omega) \, ; \, \int_\Omega \frac{u_x^2}{d(x)} \, dx < \infty \right\}$$

Existence theorem

Theorem (Imbert-M. (2011))

Consider $u_0 \in H^{\frac{1}{2}}(\Omega)$ *such that* $\int_\Omega G(u_0)dx < +\infty$. *There exists a non-negative weak solution u.*

Moreover, for a.e. $t \in (0, T)$,

- *(Mass)* $\int_\Omega u(t)\,dx = \int_\Omega u_0\,dx$;
- *(Energy)* $\|u(t)\|^2_{\dot{H}^{\frac{1}{2}}(\Omega)} + 2\int_0^t \int_\Omega g^2(s,x)\,dx\,ds \leq \|u_0\|^2_{\dot{H}^{\frac{1}{2}}(\Omega)}$

  where $g = \partial_x(u^{n/2} I(u)) - \frac{n}{2} u^{\frac{n}{2}-1} \partial_x u\ I(u) \in L^2$
- *(Entropy)* $\int_\Omega G(u(t))\,dx + \int_0^t \|u(s)\|^2_{\dot{H}_N^{\frac{3}{2}}(\Omega)} ds \leq \int_\Omega G(u_0)\,dx$

Idea of the proof

The general strategy follows [Bernis-Friedman ('90)]:

- Regularize the diffusion coefficient $u^n \to f_\varepsilon(u) = \varepsilon + |u|^n$, and prove the existence of a solution u^ε.
- Energy inequality + mass conservation give a bound on u^ε in

$$H^{1/2}(\Omega) \subset L^p(\Omega) \quad \text{for all } p < \infty$$

  and shows that the flux $h^\varepsilon = f_\varepsilon(u^\varepsilon) I(u^\varepsilon)_x$ is bounded in $L^2(0, T, L^{2-}(\Omega))$.
- Since $G'' = u^{-n}$, we have $G(s) = +\infty$ for $s < 0$ whenever $n \geq 1$, so the entropy inequality implies $\lim u^\varepsilon \geq 0$.
- However, for $n \geq 2$, $G(0) = +\infty$, so the entropy inequality requires positive initial data, which is a major limitation of the result.

Idea of the proof

We have a solution of

$$\partial_t u^\varepsilon - \partial_x h^\varepsilon = 0.$$

We have to show that

$$h^\varepsilon = f_\varepsilon(u^\varepsilon) I(u^\varepsilon)_x \longrightarrow u^n I(u)_x.$$

Idea:

- $h^\varepsilon \to 0$ wherever $\lim u^\varepsilon = 0$
- $I(u^\varepsilon)_x$ bounded in L^2 wherever $\lim u^\varepsilon \geq \delta > 0$.

Difficulty: to identify $\lim I(u^\varepsilon)_x$ in $\mathcal{D}'$, we need u continuous, which we do not get from the energy ($H^{1/2}$ bound):

Use the entropy dissipation ($H_N^{3/2}$ bound).

Drawback: we cannot lift the assumption $\int_\Omega G(u_0)dx < +\infty$.

Remarks

- Rana Tarhini ('15) extends this analysis to the case $I = (-\partial_{xx})^{\alpha}$ with $\alpha \in (0,2)$.
  The case $\alpha = \frac{1}{2}$ is critical in the sense that if $\alpha > 1/2$ (e.g. for the thin film equation), the energy inequality gives a bound in

$$H^{\alpha}(\Omega) \subset C^{0,\beta}(\Omega).$$

- Free boundary condition: If $\{u > 0\} \neq \Omega$, then the condition $u \in L^2(0,T;\dot{H}_N^{\frac{3}{2}}(\Omega))$ gives $u \in L^2(0,T;C^{\alpha})$ for all $\alpha < 1$ and so

$$u(x) = o(|x - x_0|^{\beta}) \text{ as } x \to x_0 \in \partial\{u > 0\} \qquad \text{for all } \beta < 1$$

  that is $K = 0$ (and this is consistent with the behavior of self-similar solutions when $n < 2$).

Final Remarks

- Existence of solutions in the case $n = 3$ with compactly supported initial data is still unknown in either the case $K = 0$ or $K \neq 0$ (free boundary problem):
  - We need to get more regularity from the dissipation term

$$\int_0^T \int_\Omega u^n |\partial_x I(u)|^2 \, dx$$

  - or obtain other entropy type estimates (α-entropy inequality for the thin film equation).
- Other properties that are verified for the porous media equation and the thin film equation:
  - Finite speed of propagation of the support
  - initial waiting time phenomenon