Quasi-static evolution and congested transport

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Hard congestion in crowd motion

The following crowd motion model is proposed by Maury, Roundneff-Chupin and Santambrogio (2010):

- $\rho(x,t)$: pedestrian population density, which cannot exceed a certain maximal value (which we assume to be 1).
- $-\nabla\Phi(x)$: The desired velocity field for an individual located at x. It may not be achieved due to the constraint $\|\rho(\cdot,t)\|_\infty \leq 1$.
- In the saturated zone $\{\rho(\cdot,t)=1\}$, the actual velocity field $\mathbf{v}(\cdot,t)$ must satisfy $\nabla\cdot\mathbf{v}(\cdot,t)\geq 0$, in order to not increase the density. (If so, we say $\mathbf{v}$ is *feasible*).

Hard congestion in crowd motion

- In [MRS], they consider the PDE system

$$(P) \qquad \begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0 \\ \mathbf{v}(\cdot, t) = -P_{C_\rho} \nabla \Phi, \end{cases}$$

  where P_{C_ρ} is the projection towards the space of feasible velocity fields in L^2 sense.

- They link this system with the following gradient flow: For $\rho \in \mathcal{P}_2(\mathbb{R}^d)$, let

$$E_\infty[\rho] = \begin{cases} \int_{\mathbb{R}^d} \rho(x)\Phi(x)dx & \|\rho\|_\infty \leq 1 \\ +\infty & \text{otherwise.} \end{cases}$$

Gradient flow approach for hard congestion

- Let $\rho_\infty(\cdot, t)$ be the gradient flow of E_∞ with respect to Wasserstein distance W_2 with initial data ρ_0. Existence and uniqueness of ρ_∞ can be shown following the book by Ambrosio, Gigli and Savaré (2005).
- [MRSV] showed that ρ_∞ is a weak solution to (P), however uniqueness of the weak solution to (P) is unknown since the velocity field $v = -P_{C_\rho}\nabla\Phi$ is only in L^2.
- Let $\Delta\Phi \geq 0$. Then formally speaking, the velocity field pulls the particles together. In this setting we will show a unique characterization of the velocity field v for ρ_∞.

Modification of the velocity

We expect the modified velocity to be of the form $v = \nabla p + \nabla \Phi$. The continuity equation for ρ then is expected to be of the form

$$\rho_t - \nabla \cdot (\rho(\nabla p + \nabla \Phi)) = 0,$$

where p is the pressure generated by the constraint, supported on the congested set $\{\rho = 1\}$.

A remark on the assumption $\Delta\Phi \geq 0$

When $\Delta\Phi \geq 0$, we expect the pressure to be nonzero in all of $\{\rho = 1\}$. Moreover the modified velocity should be incompressible in the congested region since the original velocity field only tries to compress the density. Thus p should solve

$$\nabla \cdot (\nabla p + \nabla\Phi) = 0 \text{ or } -\Delta p = \Delta\Phi \text{ in } \{\rho = 1\}.$$

Evolution of the congested region

Suppose ρ solves the modified discontinuity equation with ∇p discontinuously changing to zero across $\partial\Omega(t)$. Then denoting $\rho = \rho^I \chi_{\Omega(t)} + \rho^O$ we have

$$
\begin{aligned}
0 &= \partial_t[\textstyle\int_{\Omega(t)} (\rho^I)_t dx + \int_{\mathbb{R}^n - \Omega(t)} \rho^O dx] \\
&= \textstyle\int_{\Omega(t)} (\rho^I)_t + \int_{\mathbb{R}^n - \Omega(t)} (\rho^O)_t dx + \int_{\partial\Omega(t)} V(\rho^I - \rho^O) dS \\
&= \textstyle\int_{\Omega(t)} \nabla \cdot (\rho^I(\nabla p + \nabla\Phi) + \int_{\Omega(t)^c} \nabla \cdot (\rho^O \nabla\Phi) + \int_{\partial\Omega(t)} V(\rho^I - \rho^O) \\
&= \textstyle\int_{\partial\Omega(t)} \rho^I \partial_\nu p + (\rho^I - \rho^O)(\partial_\nu \Phi + V) dS,
\end{aligned}
$$

where $V = V_{x,t}$ denotes the (outward) normal velocity of $\Omega(t)$.

Evolution of ρ when $\Delta\Phi \geq 0$

Above calculation suggests the following evolution for $\rho(\cdot, t) = \chi_{\Omega_t} + \rho^O$, where the congested set $\Omega(t) := \{p(\cdot, t) > 0\}$ is determined by the following free boundary problem for $p \geq 0$:

$$\begin{cases} -\Delta p(\cdot, t) = \Delta\Phi & \text{in} \quad \{p(\cdot, t) > 0\}; \\ V = \frac{1}{(1-\rho^O)}(-\partial_\nu p) - \partial_\nu \Phi & \text{on} \quad \partial\{p(\cdot, t) > 0\}. \end{cases}$$

- Note that the velocity law

$$V = \frac{1}{1-\rho^O}|Dp| - \partial_\nu \Phi$$

indicates that there is a generic discontinuity of ρ across $\partial\Omega(t)$.

- The well-posedness of the free boundary problem can be shown by viscosity solutions theory.
- We are interested in connecting this problem with the gradient flow solution ρ_∞. In addition to the assumption $\Delta\Phi \geq 0$, we will also assume that the initial data is patch.

Quasi-static evolution: patch case

Suppose $\rho_0 = \chi_{\Omega_0}$, and consider the following free boundary problem that p solves with the initial data $\{p(\cdot, 0) > 0\} = \Omega_0$:

$$(FB) \qquad \begin{cases} -\Delta p(\cdot, t) = \Delta\Phi & \text{in} \quad \{p(\cdot, t) > 0\} =: \Omega_t; \\ V = |Dp| - \partial_\nu \Phi & \text{on} \quad \partial\Omega_t. \end{cases}$$

Our goal is to prove that the gradient flow solution ρ_∞ with initial data ρ_0 satisfies $\rho_\infty(\cdot, t) = \chi_{\Omega_t}$.

Approximation by Porous Medium Equation

- Let ρ_m solve the following porous medium equation with drift:

$$\rho_t = \Delta \rho^m + \nabla \cdot (\rho \nabla \Phi),$$

  with initial data $\rho(\cdot, 0) = \chi_{\Omega_0}$.

- It is well known (Otto, 2001) that ρ_m is the gradient flow for

$$E_m[\rho] = \frac{1}{m} \int \rho^m dx + \int \rho \Phi dx.$$

- We will show that ρ_m converging to ρ_∞ as $m \to \infty$, with a rate.
- We will also show that ρ_m converges to χ_{Ω_t}, yielding the desired statement, $\rho_\infty = \chi_{\Omega_t}$.

Convergence as $m \to \infty$

Theorem (Alexander-K-Yao., 2013)

Let Ω_0 be a compact set in $\mathbb{R}^d$ with locally Lipschitz boundary, then

(a) *Assuming $\Delta\Phi \geq 0$. Then there is a unique family of compact sets Ω_t evolving with (FB). As $m \to \infty$, $\rho_m \to \chi_{\Omega_t}$ locally uniformly away from $\partial\Omega(t)$.*

(b) *Assume $\|D^2\Phi\|_\infty \leq \infty$ and $\inf \Phi$ is finite. Then $\rho_m(\cdot, t)$ converges to $\rho_\infty(\cdot, t)$ in W_2 distance uniformly in $t \in [0, T]$, with convergence rate*

$$\sup_{t\in[0,T]} W_2(\rho_m(t), \rho_\infty(t)) \lesssim \frac{1}{m^{1/24}}.$$

Convergence as $m \to \infty$

Corollary

Since $\rho_m(\cdot, t)$ converges to both χ_{Ω_t} and $\rho_\infty(\cdot, t)$ as $m \to \infty$, χ_{Ω_t} and $\rho_\infty(\cdot, t)$ must be equal almost everywhere.

- While the gradient flow approach cannot directly deal with general nonconvex bounded domains (with e.g. Neumann boundary conditions), the viscosity solution approach still applies and we have $\rho_m \to \chi_{\Omega_t}$.
- On the other hand, without the condition $\Delta\Phi \geq 0$, the gradient flow approach still works and we still have $\rho_m(\cdot, t) \to \rho_\infty(\cdot, t)$ but the characterization of the modified velocity remains open.

$\rho_m \to \chi_{\Omega_t}$: Heuristics

First let us discuss the convergence of ρ_m to χ_{Ω_t}. The following heuristics suggest that $\rho_m(\cdot, t)$ should converge to χ_{Ω_t} as $m \to \infty$. The main trick is to consider the limit of the corresponding pressure variable $p_m = \frac{m}{m-1}\rho_m^{m-1}$ rather than ρ_m, which remains continuous as $m \to \infty$.

The equation for p_m is

$$(\rho_m)_t - \nabla \cdot (\rho_m(\nabla p_m + \nabla\Phi) = 0$$

One then considers the corresponding PDE for p_m, and show the convergence of p_m to p by barrier (viscosity solutions) argument.

Convergence of ρ_m, p_m as $m \to \infty$: Relevant work

- When $\Phi = 0$ but with source term, in the patch case: Gil and Quirós (1998), K(2003).
- Weak solutions theory is developed for general initial data in Perthame-Quiros-Vazquez (2013), where they study the $m \to \infty$ limit of

$$\rho_t - \Delta(\rho^m) = \rho G(p).$$

- We first define viscosity solution for (FB), and prove the comparison principle.
- Then we construct the "lower limit" of $\{p_m\}_m$ as $m \to \infty$:

$$u_2(x,t) := \lim_{n\to\infty} \inf_{\substack{m\geq n \\ |(x,t)-(y,s)|<1/n}} p_m(y,s),$$

and use comparison arguments to show that u_2 is a supersolution of the Hele-Shaw problem (P) with the initial pressure satisfying $\{p_0 > 0\} = \Omega_0$.

Roughly speaking, the strategy is then to show that the corresponding "upper limit" u_1 of $\{p_m\}_m$ is a subsolution of (P) with the initial data p_0 . Then due to the comparison principle we can conclude that $u_1 \leq (u_2)^*$.

Since $u_1 \geq u_2$ by definition, it follows that $u_1 = (u_2)^*$ and $(u_1)_* = u_2$. Let $\Omega_t := \{u_1(\cdot, t) > 0\}$. Then $(\overline{\Omega_t})^0 = (\Omega_t)^0$ and ρ_m uniformly converges to χ_{Ω_t} away from $\partial\Omega_t$ with initial data $u_0 = \chi_{\Omega_0}$.

$\rho_m \to \rho_\infty$: based on the JKO scheme

Next we proceed to show $\rho_m \to \rho_\infty$. The goal is to show

$$W_2(\rho_m, \rho_\infty) \leq C m^{-1/24},$$

where W_2 is the 2-Wasserstein distance.

To this end, let us first compare discrete-time solutions over one time step. Let μ_m and μ_∞ be the respective minimizer of the following JKO scheme for one time step:

$$\mu_m = \operatorname*{argmin}_{\rho \in \mathcal{P}_2(\mathbb{R}^d)} E_m[\rho] + \frac{1}{2\Delta t} W_2^2(\rho, \rho_0)$$

$$\mu_\infty = \operatorname*{argmin}_{\rho \in \mathcal{P}_2(\mathbb{R}^d)} E_\infty[\rho] + \frac{1}{2\Delta t} W_2^2(\rho, \rho_0)$$

We want to estimate $W_2(\mu_m, \mu_\infty)$: the main difficulty is that μ_m may not be in L^∞.

Towards a contradiction, suppose $W_2(\mu_m, \mu_\infty)$ is large; in this case we want to find a better competitor $\tilde{\mu}$, such that

$$\begin{aligned} &E_m[\tilde{\mu}] \quad + \frac{1}{2\Delta t} W_2^2(\tilde{\mu}, \rho_0) \quad + E_\infty[\tilde{\mu}] \quad + \frac{1}{2\Delta t} W_2^2(\tilde{\mu}, \rho_0) \\ <&E_m[\mu_m] + \frac{1}{2\Delta t} W_2^2(\mu_m, \rho_0) + E_\infty[\mu_\infty] + \frac{1}{2\Delta t} W_2^2(\mu_\infty, \rho_0) \end{aligned}$$

This means $\tilde{\mu}$ would at least beat one of μ_m and μ_∞!

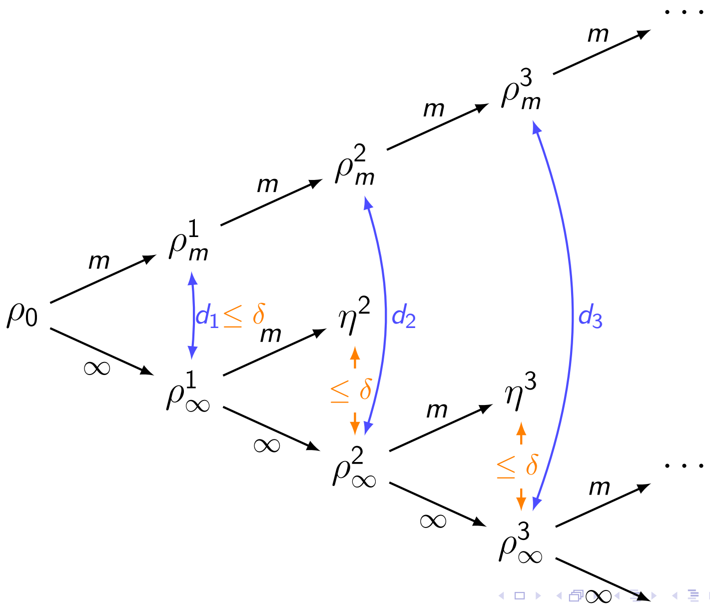
How do we find such $\tilde{\mu}$?

- First guess: Choose $\tilde{\mu}$ as the midpoint (along the generalized geodesics) of μ_m and μ_∞. This choice saves the distance.
- But $E_\infty[\tilde{\mu}]$ may be infinite.

A suitable competitor $\tilde{\mu}$ can be found as follows:

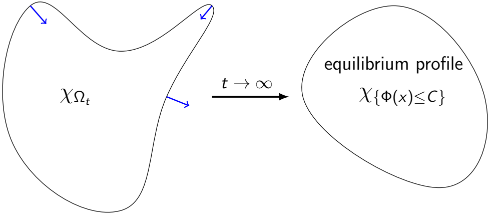
1. Even though the maximum density of μ_m may exceed 1, $\int(\mu_m - 1)_+ dx \lesssim m^{-1/2}$ for $m > 2$.
2. Thus we can find some $\|\eta_m\|_\infty \leq 1$, such that $W_2(\eta_m, \mu_m) \lesssim m^{-1/4}$, and $E_m[\eta_m] \approx E_m[\mu_m]$.
3. One can then choose $\tilde{\mu}$ as the midpoint (along the generalized geodesics) of η_m and μ_∞. (Note that $\|\tilde{\mu}\|_\infty \leq 1$.)
4. $\tilde{\mu}$ would be better than either μ_m or μ_∞ if $W_2(\mu_m, \mu_\infty) \gg m^{-1/8}$.

$\rho_m \to \rho_\infty$:Controlling the distance for multiple time steps



Confinement and long time behavior

- For $1 < m \leq \infty$, if $\Phi(x) \to \infty$ as $|x| \to \infty$, using the comparison principle, we know that if the initial data is compactly supported, the discrete solution to JKO scheme will be uniformly confined for all time steps.
- If Φ is strictly convex, ρ_∞ converges to the global minimizer (which is $\chi_{\{\Phi(x)\leq C\}}$ for some C) exponentially fast in W_2 distance.



Many open questions remain. The ultimate goal would be to try to generalize the continuity equation with L_∞ constraint, possibly characterized as the singular limit of the porous medium equation with drift

$$\rho_t - \Delta(\rho^m) + \nabla \cdot (\vec{v}\rho) = 0.$$

as $m \to \infty$.

Part II: Aggregation with Height constraint

Next we discuss a different but relevant problem, which can be formally viewed as the singular limit of Patlak-Keller-Segel (PKS) equation:

$$\rho_t - \Delta(\rho^m) - \nabla \cdot (\rho \nabla (\rho * \mathcal{N})) = 0.$$

This is joint work with
Katy Craig (UCLA) and Yao Yao (UW Madison).

Interaction energy

- Let $\rho_\infty(\cdot, t)$ be the gradient flow of

$$\tilde{E}_\infty[\rho] = \begin{cases} \int_{\mathbb{R}^2} \rho(\rho * \mathcal{N})dx & \text{for } \|\rho\|_\infty \leq 1 \\ +\infty & \text{otherwise} \end{cases}$$

  where $\mathcal{N}$ is the Newtonian potential.
- Uniqueness of the minimizer follows from certain convexity properties of $\tilde{E}_\infty$ given by Carrillo, Lisini and Mainini (2012). The stability of the JKO scheme is much weaker due to the weak convexity properties of the energy $\tilde{E}_\infty$.

Open questions:

- Does the discrete-time solutions converge to a gradient flow solution in a stable way?
- Can one characterize the corresponding gradient flow solution?
- As $t \to \infty$, does ρ_∞ eventually converge to the characteristic function of a ball? **yes if n=2**
- Does the solutions of (PKS) converge to ρ_∞ as $m \to \infty$? **Open**

Convexity property of E_∞

To show the convergence of discrete-time scheme with E_∞, we recall the notion of ω-convexity:

Definition (Carrillo-Lisini-Mainini)

E is called ω- *convex* if for $\rho_1, \rho_2 \in \mathcal{P}_{2,ac}(\mathbb{R}^d)$ we have

$$\begin{aligned} E(\rho_t) &\leq (1-t)E(\rho_0) + tE(\rho_1) \\ &+C[(1-t)\omega(t^2 W_2^2(\rho_0,\rho_1)) + t\omega((1-t)^2 W_2^2(\rho_0,\rho_1))], \end{aligned}$$

where $\omega(x) = x|\ln x|$ for small x.

We then have the following contraction inequality.

Theorem (K. Craig)

- *$\tilde{E}_\infty$ is ω-convex.*
- *Suppose E is ω-convex, then the corresponding solutions of the JKO scheme satisfies*

$$f_\tau(W_2(\mu_\tau, \nu_\tau)) \leq W_2^2(\mu_0, \nu_0) + C\tau^2 \ln \tau,$$

where $f_\tau(x) = x - c\tau\omega(x)$.

Based on above inequality, one can follow the argument of Crandall-Liggett to obtain a recursive inequality to estimate $W_2(\mu_\tau, \nu_h)$, and thus to show that JKO scheme converges to a unique limit ρ_∞ in Wasserstein distance.

Next we consider characterizing ρ_∞ with a free boundary problem. The corresponding approximating energy is

$$\tilde{E}_m(\rho) := \int \frac{1}{m}\rho^m + \rho\mathcal{N} * \rho,$$

But then we realize that the corresponding gradient flow of above energy is hard to analyze due to the lack of convexity properties of $\tilde{E}_m$.

The main difficulty lies in the lack of L^∞-bound for the discrete-time gradient flow solutions. In fact it is open whether the discrete-time solutions converges to the continuum solutions of (PKS) in spite of their formal connection.

Characterization of the gradient flow solution

Hence we will use instead the following energy

$$E_m(\rho) := \int \frac{1}{m}\rho^m + \rho\mathcal{N} * \rho_\infty,$$

which is ω-convex, and show that the corresponding gradient flow solution ρ_m converges.

As before, we only consider the patch case, i.e. when $\rho_0 = \chi_{\Omega_0}$. Here the corresponding free boundary problem is:

$$(FB) \qquad \begin{cases} -\Delta p = \Delta \Phi = 1 & \text{in} \quad \{p(\cdot, t) > 0\} =: \Omega(t); \\ V = |Dp| - \partial_\nu \Phi & \text{on} \quad \partial\Omega(t), \end{cases}$$

where $\Phi = \chi_{\Omega(t)} * \mathcal{N}$.

Theorem (Craig-K-Yao)

ρ_m converges to $\rho_\infty = \chi_{\Omega(t)}$ where $\Omega(t)$ solves (FB).

Long time behavior of ρ_∞

Using the free boundary formulation for ρ_∞, we can show the following:

Theorem (Craig-K-Yao)

When $n = 2$, Ω_t converges to a ball as $t \to \infty$.

We prove this by showing that the second moment $\int \rho_\infty(x,t)|x|^2 dx$ decreases in time unless $\rho_\infty = \chi_B$.

Computing the second moment

The evolution of $M_2[\rho(t)]$ is given by

$$\begin{aligned}
\frac{d}{dt}M_2[\rho(t)] &= -\int_{\mathbb{R}^2} \rho\nabla(\rho * \mathcal{N})\cdot\vec{x}dx - \int_{\mathbb{R}^2}\rho\nabla p\cdot\vec{x}dx \\
&= -\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\rho(x)\rho(y)\frac{(\vec{x}-\vec{y})\cdot\vec{x}}{|x-y|^2}dydx - \int_{\Omega(t)}\nabla p\cdot\vec{x}dx \\
&= -\frac{1}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\rho(x)\rho(y)dydx + 2\int_{\Omega(t)}p(x)dx \\
&= -\frac{1}{4\pi}|\Omega(t)|^2 + 2\int_{\Omega(t)}p(x)dx.
\end{aligned}$$

The quantity above is negative unless $\Omega(t)$ is a ball due to [Talenti, 1976].

References

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Thank you for your attention!