Méthode de convexité avec poids optimal pour la stabilisation des EDO et EDP

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Outline of the first lecture

1. Dissipative systems in finite and infinite dimensions
2. The optimal-weight convexity method in finite dimensions
3. The optimal-weight convexity method in infinite dimensions: direct method
4. Memory damping
5. The indirect method
6. Uniform discretization schemes
7. Optimality in infinite dimensions

Sommaire

Dissipative systems in finite dimensions

The scalar case :

$$u'' + \nu\, u + f(u) + \rho(u') = 0\,.$$

where

- $\nu > 0$
- u is a scalar unknown
- f is locally lipschitz continuous and $sf(s) \geq 0$ for all $s \in \mathbb{R}$
- $\rho(s)s \geqslant 0$ for all $s \in \mathbb{R}$, $\rho(0) = 0$.

Multiply the equation by u'. This gives the so-called dissipation relation:

$$\Big(\frac{1}{2}\Big(|u'(t)|^2 + \nu|u(t)|^2\Big) + F(u(t))\Big)' = -u'(t)\rho(u'(t))\,.,$$

where

$$F(u) = \int_0^u f(s)\, ds\,.$$

Define the energy

$$E(t) = \frac{1}{2}\Big(|u'(t)|^2| + \nu|u(t)|^2\Big) + F(u(t))\,,$$

Kinetic energy potential energy

Then the dissipation relation becomes

$$E'(t) = -u'(t)\rho(u'(t))\,.$$

Since $s\rho(s) \geq 0$ for all $s \in \mathbb{R}$, we have

$$E'(t) = -u'(t)\rho(u'(t)) \leq 0 \quad \forall t \geq 0\,.$$

$\Longrightarrow$ the energy is a Lyapunov function, however it is not a strict Lyapunov function, since the dissipation relation does not involve the potential part of the energy.

The vectorial case:

$$u'' + Au + f(u) + B\rho(u') = 0\,.$$

where the unknown $u \in \mathbb{R}^n$ and

- A is a symmetric, positive definite matrix
- $B = diag(b_i)_{1 \le i \le n}$, with $b_i \ge 0\ \forall\, i \in \{1, \dots, n\}$
- $(f(u))_i = f(u_i)\,, (\rho(u'))_i = \rho(u_i)$

Take the euclidian scalar product $\langle \cdot \rangle$ of the above equality with u', and denote by $|\cdot|$ the euclidian norm.

$$\langle u'', u' \rangle + \langle Au, u' \rangle + \sum_{i=1}^{n} (F(u_i))' = -\sum_{i=1}^{n} b_i \rho(u_i') u_i'\,.$$

where $F' = f$ as for the scalar case.

Thus, for this example the energy is given by

$$E(t) = \frac{1}{2}\left(|u'|^2 + \langle Au, u\rangle + \sum_{i=1}^{n}(F(u_i))\right),$$

and we have the dissipation relation (under the same hypothesis on ρ than in the scalar case)

$$E'(t) = -\sum_{i=1}^{n} b_i \rho(u_i')u_i' \leq 0 \quad \forall\, t \geq 0\,.$$

Thus, the energy is again a Lyapunov function, but not a strict one.

- Damping action $\rightsquigarrow$ Loss of energy
- Can we measure the effect of this damping, by measuring the decay rate of the energy?
- Is this measurement optimal in some way?

The motivation for vectorial system comes from the semi-discretization of wave-type equations.

Let us consider a frictional dissipative wave equation in the one-dimensional space domain $(0, 1)$.

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + f(u) + b(x)\rho(x, \partial_t u) = 0\,, 0 < t, 0 < x < 1\,, \\ u(t, x) = 0\,, \quad \text{for } x = 0, x = 1, 0 < t\,, \\ u(0, x) = u_0(x)\,,\, \partial_t u(0, x) = u_1(x)\,, \quad 0 < x < 1\,. \end{cases} \tag{1}$$

We assume that this system is dissipative, i.e.

- $b \geq 0$ is the damping coefficient (locally supported in general),
- ρ is monotone nondecreasing with respect to the second variable and $\rho(., 0) = 0$.

Additional hypotheses on f to guarantee existence for all $t \geq 0$.

A semi-discretization of the above equation in space, with for instance a uniform mesh $x_i = i\,h$ for $i = 0, \dots, n+1$ with a parameter of discretization $h = 1/(n+1)$
gives the finite dimensional system

$$\begin{cases} u_i'' - \dfrac{u_{i+1} + u_{i-1} - 2u_i}{h^2} + f(u_i) + b_i \rho_i(u_i') = 0\,, \quad 0 < t, i = 1, \dots, n\,, \\ u_0(t) = u_{n+1}(t) = 0\,, \quad 0 < t\,, \\ u_i(0) = u_{i,0}\,, \partial_t u_i(0) = u_{i,1}\,, \quad i = 1, \dots, n\,, \end{cases} \tag{2}$$

where

- u_i is a function of t which stands for an approximation of the solution u at point x_i
- $b_i = b(x_i)$,
- $\rho_i(s) = \rho(x_i, s)$ for all $s \in \mathbb{R}$.

$\rightsquigarrow$ similar analysis for the plate equation ...

Set

$$A = h^{-2} \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

which is symmetric and positive definite.

We can rewrite the semi-discretized equation as the vectorial system

$$u'' + Au + f(u) + \rho(u') = 0,$$

where the unknown $u \in \mathbb{R}^n$

- Here it is further important, not only to obtain optimal decay rates for fixed h, but also
- $\rightsquigarrow$ to trace the dependence of the estimates on the discretization parameters and
- to obtain numerical schemes which lead to uniform decay rates with respect to h.

Dissipative systems in infinite dimensions: examples

- Locally distributed damping

$$\begin{cases} u_{tt} - \Delta u + \rho(x, u_t) = 0 \\ u = 0 \text{ in } [0, \infty) \times \Gamma \\ (u, u_t)(0, .) = (u_0, u_1) \text{ in } \Omega\,, \end{cases}$$

where $\Gamma = \partial\Omega$ and $x \to \rho(x, \cdot)$ is non vanishing in a subset $\omega \subset \Omega$ which satisfies some suitable geometric conditions, $\rho(x, s)s \geqslant 0$ for all $(x, s) \in \Omega \times \mathbb{R}$.

Formally, multiply the equation by u' and integrate over Ω. Green's formula gives

$$\frac{d}{dt} E_u = - \int_\Omega u_t \rho(x, u_t) dx \leqslant 0,$$

where

$$E_u(t) = \frac{1}{2} \int_\Omega \left(|u_t|^2 + |\nabla u|^2 \right) dx.$$

- Boundary damping:

$$\begin{cases} u_{tt} - \Delta u = 0 \\ u = 0 \text{ in } [0,\infty) \times \Gamma_1 \\ \frac{\partial u}{\partial \nu} + \eta(.)u + \rho(.,u_t) = 0 \text{ in } [0,\infty) \times \Gamma_0 \\ (u,u_t)(0,.) = (u_0,u_1) \text{ in } \Omega\,, \end{cases}$$

where $\{\Gamma_0, \Gamma_1\}$ is a partition of Γ and where η is a nonnegative function. As before, multiplying by u_t and integrating over Ω, we get

$$\frac{d}{dt}E_u = -\int_{\Gamma_0} u_t \rho(x,u_t) d\sigma \leqslant 0,$$

where

$$E_u(t) = \frac{1}{2}\left(\int_\Omega |u_t(t)|^2 + |\nabla u(t)|^2\, dx + \int_{\Gamma_0} \eta |u|^2\, d\sigma\right)$$

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• Memory dampings

The damping depends on the memory of the "material" and is nonlocal with respect to time $\rightsquigarrow$ viscoelastic materials as in

$$\begin{cases} u_{tt} - \Delta u + k * \Delta u = 0 \\ u = 0 \text{ in } [0,\infty) \times \Gamma \\ (u, u_t)(0,.) = (u_0, u_1) \text{ in } \Omega\,, \end{cases}$$

where

$$(k * v)(t) = \int_0^t k(t-s)v(s)\,ds\,.$$

and the kernel k satisfies

- $k : [0,\infty) \mapsto (0,\infty)$
- k is continuously differentiable
- $k' \leqslant 0$
- $\int_0^\infty k(s)ds < 1$ (for well-posedness)

In this case, the natural energy is

$$E_u(t) = \frac{1}{2}\|u_t(t)\|^2_{L^2(\Omega)}\,dx + \frac{1}{2}\left(1 - \int_0^t k(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^t k(t-s)\|\nabla u(t) - \nabla u(s)\|^2_{L^2(\Omega)}\,ds$$

and the dissipation relation is:

$$E'_u(t) = -\frac{1}{2}k(t)\|\nabla u(t)\|^2 + \frac{1}{2}\int_0^t k'(t-s)\|\nabla u(s) - \nabla u(t)\|^2\,ds \leqslant 0$$

Model examples

But also : higher order systems: Petrowsky equation

$$\begin{cases} u_{tt} + \Delta^2 u + \rho(x, u_t) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}\,, \\ u = 0 = \Delta u \quad \text{on} \quad \Sigma = \Gamma \times \mathbb{R}\,, \\ (u, \partial_t u)(0) = (u^0, u^1) \quad \text{on} \quad \Omega\,. \end{cases}$$

We define the energy of a solution u by

$$E(t) = \frac{1}{2}\Big(\int_\Omega |u_t|^2 + |\Delta u|^2\Big)\, dx\,.$$

The dissipation relation is

$$E'(t) = -\int_\Omega u_t \rho(., u_t)\, dx \leq 0\,, \quad t \geq 0\,.$$

But can enter in this frame of analysis, coupled systems such as Timoshenko beams, Bresse system, examples of first order systems, Schrödinger equation, . . .

Infinite dimensional systems

These models can be recast in the following abstract

$$u'' + Au + \text{ damping operator}[u] = 0$$
$$(u, u')(0) = (u^0, u^1)$$

Here A stands for an unbounded linear operator in an Hilbert space H, which is

- closed,
- coercive
- self-adjoint
- with dense domain in H.

As for the finite dimensional case, one can associate an energy to the solutions and this energy decays through time.

The same questions than for the finite dimensional case arise, except now that there is a space dependence and the problem is set-up in an infinite dimensional framework (Sobolev spaces)

Moreover the damping can be locally or boundary supported

and in general the support of the damping coefficient has to satisfy geometric conditions.

We shall come back later on infinite dimensional systems.

Sommaire

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Let us start by the scalar case example:

$$u'' + \nu\, u + f(u) + \rho(u') = 0\,.$$

where

- $\nu > 0$
- u is a scalar unknown
- f is locally lipschitz continuous, $sf(s) \geq 0$ for all $s \in \mathbb{R}$

$$\exists\ \tilde{\mu} > 0 \text{ such that } 0 \leq F(s) \leq \tilde{\mu} s f(s)\,, \quad \forall\ s \in \mathbb{R}\,,$$

with

$$F(u) = \int_0^u f(s)\, ds$$

We assume that the damping ρ satisfies the assumption

$$(A1)\begin{cases} \rho \in \mathcal{C}(\mathbb{R})\,, \rho(s)s > 0 \forall\ s \in \mathbb{R}^*, \text{ is locally lipschitz} \\ \exists \text{ a strictly increasing odd function } g \text{ such that} \\ c\,|s| \leq |\rho(s)| \leq C\,|s|\,, \quad \forall |s|\ \geq 1 \\ c\,g(|s|) \leq |\rho(s)| \leq C\,g^{-1}(|s|)\,, \quad \forall\,|s| \leq 1 \\ \exists r_0 > 0 \text{ such that } g \in \mathcal{C}^1([0, r_0]), g(0) = g'(0) = 0 \end{cases}$$

where g^{-1} denotes the inverse function of g and where $c\,, C$ are positive constants.

Remark

The function g has an arbitrary sublinear growth (the interesting case) close to 0.
It captures the behavior of the damping close to 0.

Remark

If $g'(0) \neq 0$ then it is as for the linear damping case. So it will not be considered here.

The assumption of linear growth at infinity is made for the sake of simplicity.

It can be removed using a well-known strong stabilization result (based on Lasalle invariance principle) as follows

$E(t) \to 0$ *as* $t \to \infty$. *Thus for sufficiently large time* $|u'(t)| \leq r_0$.

The energy is defined as

$$E(t) = \frac{1}{2}\left(|u'(t)|^2 + \nu |u(t)|^2\right) + F(u(t))\,,$$

where

$$F(u) = \int_0^u f(s)\,ds\,.$$

Recall the dissipation relation:

$$E'(t) = -u'(t)\rho(u'(t))\,.$$

where E is the energy of the solution.

Assume that the function g that measures the behavior of the damping ρ close to 0 is an odd, strictly increasing smooth function on $\mathbb{R}$.

We assume that $g'(0) = 0$, otherwise it is easy to show that E decays exponentially to 0 at ∞ (linear damping case).

We set

$$H(s) = \sqrt{s}g(\sqrt{s}) \quad s \geq 0$$

We assume that H is strictly convex in a right neighbourhood of 0, let us say in $[0, r_0^2]$ to fix ideas.

We set

$$\widehat{H}(x) = \begin{cases} H(x)\text{ , if } x \in [0, r_0^2]\,, \\ +\infty\text{ , if } x \in \mathbb{R} - [0, r_0^2]\,, \end{cases}$$

We define the following two functions

$$\widehat{H}^*(y) = \sup(xy - \widehat{H}(x)) \quad \text{convex conjugate function of } \widehat{H}$$

$$\Lambda_H(x) = \frac{H(x)}{xH'(x)} \quad x \in [0, r_0^2].$$

Theorem (Optimality Theorem, A.-B. JDE 2010)

Assume that f *is a continuous and locally Lipschitz function on* $\mathbb{R}$ *as above,* ρ *satisfies* $(A1)$*, and* H *is strictly convex on* $[0, r_0^2]$*. Let* $0 < |u^1| + |u^0|$ *be given,* u *be the corresponding solution and* E *be its energy. Assume that* $\limsup_{x\to 0} \Lambda_H(x) < 1$*. Then* E *satisfies*

$$E(t) \leqslant \max(\eta_1, \eta_2 E(0))(H')^{-1}\left(\frac{\eta_3}{t}\right)$$

for sufficiently large t*. Moreover if* $\rho = g$ *in* $(A1)$ *and if either*

$$0 < \liminf_{x\to 0} \Lambda_H(x) \leq \limsup_{x\to 0} \Lambda_H(x) < 1\,,$$

or that there exists $\mu > 0$ *such that*

$$0 < \liminf_{x\to 0}\left(\frac{H(\mu\, x)}{\mu\, x}\int_x^{z_1}\frac{1}{H(y)}\,dy\right),\ \text{and}\ \limsup_{x\to 0}\Lambda_H(x) < 1\,,$$

for a certain $z_1 \in (0, z_0]$ *(arbitrary).*

Theorem (continued)

Then the energy of solution satisfies the estimate

$$E(t) = O(v^2(t)) = O\Big((H')^{-1}(\frac{1}{t})\Big), \quad \textit{uniformly for large time}$$

where where v *is the solution the ODE* $v' + g(v) = 0$*,* $v(0) = \sqrt{2E(0)}$*.*

Here η_i, $i = 1, 2, 3$ are explicit constants which do not depend on $E(0)$.

The proof relies on a key comparison lemma that relies on convexity properties and allows up to compare

the lower estimate $v^2(t)$ which is an "energy-type" estimate, since v^2 is the energy associated to the ode $v' + g(v) = 0$

to the

upper time-pointwise estimate $(H')^{-1}(Cste/t)$

Remarks

The function Λ_H introduces a "classification" of the nonlinearity of the dampings

- $\lim_{x\to 0} \Lambda_H(x) = 0$ for $g(x) = e^{-1/x}$, $x > 0$ and more generally for very degenerate dampings (converging to 0 exponentially for instance)
- $\lim_{x\to 0} \Lambda_H = 2/(p+1)$ for $g(x) = |x|^{p-1}x$ and more generally this $\liminf \in (0,1)$ for polynomial-logartithmic behavior close to 0
- $\lim_{x\to 0} \Lambda_H(x) = 1$ for $g(x) = x|\ln(x)|^{-1}, x > 0$ and more generally for dampings which "are close" to a linear behavior at the origin.
- for dampings for which this $\limsup < 1$, all the solutions have the same asymptotic behavior at infinity
- we have also an upper estimate when $\limsup \Lambda_{H_{x\to 0}}(x) = 1$

The condition *limsup*$_{x\to 0^+}\Lambda_H(x) < 1$ for optimality excludes the dampings which are "close" to a linear behavior at the origin (see the 3rd example above).

Is this sharp?

Consider a linear case, that is

$$u'' + \nu\, u + \upsilon u' = 0\,.$$

where $\nu > 0$, $\upsilon > 0$ are given constants and u is a scalar unknown. Then if $\upsilon^2 - 4\nu > 0$ there are two linearly independent solutions, namely

$$u_1(t) = e^{\frac{-\upsilon-\sqrt{\upsilon^2-4\nu}}{2}}\,, \quad \forall\, t \geq 0\,,$$

and

$$u_2(t) = e^{\frac{-\upsilon+\sqrt{\upsilon^2-4\nu}}{2}}\,, \quad \forall\, t \geq 0\,,$$

The energies

$$E_i(t) = \frac{1}{2}\left(|u_i'(t)|^2 + \nu|u_i(t)|^2\right), \quad i = 1, 2\,, t \geq 0\,,$$

of these two solutions decay exponentially as t goes to ∞, but at different rates and their ratio satisfies

$$\lim_{t\to\infty} \frac{E_2(t)}{E_1 t)} = \infty\,.$$

This implies different lower and upper bounds and different decay rates depending on the initial data

Same behavior for $\upsilon^2 - 4\nu = 0$,

If $\upsilon^2 - 4\nu < 0$, all the energies of all the solutions decay at the same speed $e^{-\upsilon t}$ as t goes to ∞.

A similar situation (with more cases) arises in the linear vectorial case.

We conjecture that a similar situation, that is

two branches of solutions with different asymptotic behavior

may arise for nonlinear dampings close to linear growth at the origin

that is for dampings for which

$$limsup_{x \to 0^+} \Lambda_H(x) = 1$$

.

Sketch of the optimal-weight convexity method for sharp upper energy estimates

1. Use multiplier and a weight function $w(E)$ to be determined later on to prove a weighted dominant energy estimate
2. Use convexity arguments to determine an optimal weight function $w(E)$ to prove that E satisfies a nonlinear weighted integral inequality
3. Derive an optimal semi-explicit upper estimate for nonnegative, nonincreasing function E satisfying general nonlinear weighted integral inequalities
4. Use the feedback classification induced by Λ_H to simplify the upper energy estimate for dampings which are away from a linear behavior around 0

all this leads to the sharp simplified one-step upper estimate

$$\max(\eta_1, \eta_2 E(0))(H')^{-1}\left(\frac{\eta_3}{t}\right)$$

Then

1. Prove an energy comparison principle to derive a lower energy estimate
2. Use comparison arguments to prove equivalence (up to constant multiplicative factors) between the lower and upper estimates

Step 1: Dominant energy estimates

Let for the moment w be a nonnegative $\mathcal{C}^1$ and strictly increasing function defined from $[0, r_0^2)$ onto $[0, +\infty)$.

w is going to be an optimal-weight function to be determined later on

We multiply the left hand side of the equation

$$u'' + \nu u + f(u) + \rho(u') = 0$$

by $w(E(t))u(t)$ and integrate the resulting equation on $[S, T]$.

Since E is nonincreasing, w is nondecreasing and thanks to our assumption on f, this gives after some computations

continued.

Thus, we have

$$\int_S^T Ew(E)\,dt \leqslant \frac{2}{\sqrt{\nu\theta}}E(S)w(E(S))+$$

$$\frac{2}{\theta}\int_S^T w(E)|u'|^2\,dt + \frac{1}{2\nu\theta^2}\int_S^T w(E)|\rho(u')|^2\ , \quad \forall\, 0 \leq S \leq T.$$

weighted linear kinetic energy   weighted nonlinear kinetic energy □

The next steps will be devoted to *control* the weighted integrals of the

linear kinetic

and

nonlinear kinetic energies

in the right hand side of the above inequality.

Step 2: Optimal weight function w and weighted integral inequalities

Theorem (A.-B. 2005, 2010)

We make the above hypotheses and further assume that H is strictly convex on a right neighborhood of 0*, denoted by* $[0, r_0^2]$*,* $r_0 > 0$*. Then there is a nonnegative smooth strictly increasing weight function w such that E satisfies the following integral inequality*

$$\int_S^T w(E)E\,dt \leq C_0 E(S) \quad \forall\, 0 \leq S \leq T\,,$$

Proof

Set

$$\theta = \min\left(1, \frac{1}{2\tilde{\mu}}\right).$$

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We first remark that thanks to our hypotheses on ρ, we have (up to the positive constants c and C, which may change)

$$\begin{cases} c\,|s| \le |\rho(s)| \le C\,|s|\,, & \forall |s| \ge r_0\,, \\ c\,g(|s|) \le |\rho(s)| \le C\,g^{-1}(|s|)\,, & \forall\, |s| \le r_0\,, \end{cases}$$

First step: estimate of the linear kinetic energy

For $|u'| \le r_0$, we have

$$H(|u'|^2) = |u'|g(|u'|) \le \frac{1}{c}u'\rho(u')\,.$$

This, together with Young's inequality imply for $|u'| \le r_0$

$$w(E)|u'|^2 \le \widehat{H}^*\left(w(E)\right) + H(|u'|^2) \le \widehat{H}^*\left(w(E)\right) + \frac{1}{c}u'\rho(u')\,.$$

On the other hand, we have

$$w(E)|u'|^2 \le \frac{1}{c}w(E)u'\rho(u')\,, \quad \text{for } |u'| \ge r_0\,.$$

continued.

Combining the above two inequalities and the dissipation relation, we obtain

$$\int_S^T w(E)|u'|^2\,dt \leq \int_S^T \widehat{H}^*\big(w(E)\big)\,dt + \frac{1}{c}\big[1 + w(E(S))\big]E(S)\,, \quad \forall\, 0 \leq S \leq T\,.$$

Second step: estimate of the nonlinear kinetic energy

For $|u'| \leq r_0$, we have, thanks to Young's inequality

$$w(E)\frac{|\rho(u')|^2}{C^2} \leq \widehat{H}^*\big(w(E)\big) + H\Big(\Big|\frac{\rho(u')}{C}\Big|^2\Big) \leq \widehat{H}^*\big(w(E)\big) + \frac{1}{C}u'\rho(u')\,, \quad \text{for } |u'| \leq r_0\,.$$

continued.

On the other hand, we have

$$w(E)|\rho(u')|^2 \le Cw(E)u'\rho(u')\,, \qquad \text{for } |u'| \ge r_0\,.$$

Combining the above two inequalities and the dissipation relations, as above, we have

$$\int_S^T w(E)|\rho(u')|^2\,dt \le$$
$$C^2 \int_S^T \widehat{H}^*(w(E))\,dt + C\big[1 + w(E(S))\big]E(S)\,, \quad \forall\, 0 \le S \le T\,.$$

Using the above estimates, we obtain the estimate □

continued.

$$\int_S^T Ew(E)\,dt \leq \beta \int_S^T \widehat{H}^*\big(w(E)\big)\,dt + \\ \left(\frac{2}{\sqrt{\nu\theta}} + \frac{2}{\theta c} + \frac{C}{2\nu\theta}\right)\big[1 + w(E(S))\big]E(S)\,, \quad \forall\, 0 \leq S \leq T\,,$$

where $\beta = \max(\eta_1, \eta_2 E(0))$ can be easily computed and does not depend on w.

• β is not only chosen as the constant appearing in the above right hand side, it also should be chosen in relation with $E(0)$ such that the weight function $s \mapsto w(s)$ is defined in the range $[0, E(0))$ (indeed we choose it even with a stronger criterium below for technical reasons).

continued.

We can now choose the weight by requesting

$$\beta\widehat{H}^*(w(E)) = \frac{1}{2}Ew(E)$$

We define a function L by

$$L(y) = \frac{\widehat{H}^*(y)}{y}$$

Then the weight w should satisfy

$$L(w(E)) = E/(2\beta)$$

that is, if L is invertible

$$w(.) = L^{-1}(./2\beta)\,.$$

continued.

Assume for the moment that w is well-defined and satisfies the desired properties (non negativity, strictly increasing...).

We saw before that the energy E satisfies

$$\int_S^T Ew(E)\,dt \leq \beta \int_S^T \widehat{H}^*\big(w(E)\big)\,dt + \Big(\frac{2}{\sqrt{\nu\theta}} + \frac{2}{\theta c} + \frac{C}{2\nu\theta}\Big)\big[1 + w(E(S))\big]E(S)\,, \quad \forall\, 0 \leq S \leq T\,,$$

With this choice of weight function, we deduce that

$$\int_S^T Ew(E)\,dt \leq C_0 E(S)\,,\ \forall\, 0 \leq S \leq T$$

where $C_0 > 0$, so that we prove that E satisfies a generalized weighted inequality.

We now check that the optimal-weight w is indeed well-defined by this way

Proposition (A.-B. 2005)

Let g be a given odd, strictly increasing $\mathcal{C}^1$ function from $\mathbb{R}$ to $\mathbb{R}$ such that $g'(0) = 0$. We assume that there exists $r_0 > 0$ such that the function H is strictly convex on $[0, r_0^2]$. Then the function L defined by

$$L(y) = \begin{cases} \dfrac{\widehat{H}^*(y)}{y} & , \text{if } y \in (0, +\infty)\,, \\ 0 & , \text{if } y = 0\,, \end{cases}$$

is the strictly increasing continuous onto function from $[0, +\infty)$ on $[0, r_0^2)$ given by:

Proposition (continued)

$$L(y) = \begin{cases} (H')^{-1}(y) - \dfrac{H((H')^{-1}(y))}{y} & , \textit{if } y \in [0, H'(r_0^2)]\,, \\ r_0^2 - \dfrac{H(r_0^2)}{y} & , \textit{if } y \in [H'(r_0^2), +\infty)\,. \end{cases}$$

The weight function w is thus uniquely determined as

$$w(s) = L^{-1}\Big(\frac{s}{2\beta}\Big) \quad \forall\, s \in [0, 2\beta r_0^2)\,.$$

where $\beta > 0$ is a suitable constant (independent on w) which satisfies $\beta > E(0)/(2r_0^2)$.

Remark

Note that for general dampings the inverse of L is not defined on all $\mathbb{R}$, and the further requested computations are not explicit, contrarily to the linear or polynomial case.

Step 3: Semi-explicit upper energy estimates

Hence E is a nonnegative, nonincreasing continuous function satisfying a weighted integral inequality.

If $w \equiv 1$ this leads to exponential decay If w is polynomial, it leads to a power-like decay rate $\rightsquigarrow$ Haraux 1978, ..., Komornik 1994.

The situation is more tricky in the general case. Define

- a function K_r from $(0, r]$ on $[0, +\infty)$ by:

$$K_r(\tau) = \int_\tau^r \frac{dy}{yL^{-1}(y)}, \quad \text{where } r \text{ is chosen as } r = L(H'(r_0^2))$$

- a function ψ_r as the strictly increasing onto function defined from $[\frac{1}{L^{-1}(r)}, +\infty)$ on $[\frac{1}{L^{-1}(r)}, +\infty)$ by:

$$\psi_r(z) = z + K_r(L(\frac{1}{z})) \geq z\ , \quad \forall\, z \geq \frac{1}{L^{-1}(r)}.$$

Hence $\psi_r^{-1}(\frac{t}{T_0})$ goes to ∞ as t goes to ∞.

Theorem (A.-B. AMO 2005)

Let H be a given strictly convex $\mathcal{C}^1$ function from $[0, r_0^2]$ to $\mathbb{R}$ such that $H(0) = H'(0) = 0$, where $r_0 > 0$ is sufficiently small.
Assume E be a given nonincreasing , absolutely continuous, nonnegative real function defined on $[0, +\infty)$, $C_0 > 0$ be a fixed real number and $\beta > 0$ a given real number such that E satisfies the nonlinear Gronwall inequality

$$\int_S^T E(t) L^{-1}\left(\frac{E(t)}{2\beta}\right) dt \leq C_0 E(S) , \quad \forall\, 0 \leq S \leq T \, .$$

under the condition

$$0 < \frac{E(0)}{2L(H'(r_0^2))} \leq \beta \, ,$$

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Theorem (continued)

Then E satisfies the estimate

$$E(t) \leqslant \beta L\Big(\frac{1}{\psi_r^{-1}(\frac{t}{C_0})}\Big)\,, \quad \forall\, t \geq \frac{C_0}{H'(r_0^2)}\,.$$

Step 4: Simplification of the upper estimate via classification

Introduce the function Λ_H. Then one can rewrite ψ_r as follows for $x \geqslant 1/H'(r_0^2)$

$$\psi_r(x) = \frac{1}{H'(r_0^2)} + \int_{1/x}^{H'(r_0^2)} \frac{1}{y^2(1 - \Lambda_H((H')^{-1}(y)))}\, dy.$$

Theorem (A.-B. JDE 2010)

If $\limsup_{x\to 0^+} \Lambda_H(x) < 1$, *then for sufficiently large t*

$$E(t) \leq \beta (H')^{-1}\Big(\frac{\eta_3}{t}\Big)\,, \text{ where } \eta_3 > 0 \text{ is independent of } E(0).$$

If $g(s) = |s|^{p-1}s$, then $H(s) = s^{(p+1)/2}$ so that

$$H'(s) = \frac{1}{2}(p+1)s^{(p-1)/2}$$

Thus we recover the well-known upper estimate

$$E(t) \leq C(E(0))t^{-2/(p-1)}.$$

Moreover one can show that the weight function is of the form $w(s) = Cs^{(p-1)/2}$, that is we recover as a particular example the weight introduced by Haraux and Komornik.

$\rightsquigarrow$ **Important to note that the introduction of Λ_H is a key point if one wants general one-step, easily "computable" formulas for an "optimal" decay rate.**

$\rightsquigarrow$ Requires new ideas. Is more technical since except for the linear and polynomial damping cases, the functions to be manipulated are never explicit.

Prove the optimality of our upper estimate

Next Step 5: lower energy estimates: an energy comparison principle

Lemma (A.-B. JDE 2010)

Assume that f is a continuous and locally Lipschitz function on $\mathbb{R}$ which satisfies the above assumptions, and that $\rho = g$ satisfies $(A1)$. Moreover assume that H is increasing and $H(0) = 0$. Let u be a solution of the scalar ode and E be its energy. Then the following lower estimate holds

$$\frac{1}{2}v^2(t) \leq E(t)\,, \quad \forall\, t \geq 0\,,$$

where v is the solution the ODE $v' + g(v) = 0$, $v(0) = \sqrt{2E(0)}$.

Proof.

Thanks to the dissipation relation, and to our assumptions on g, we have

$$-\quad E'(t) \quad \leqslant \quad u'(t)g(u'(t)) \quad = \quad H\big((u')^2\big)\,, \quad \forall \quad t \quad \geqslant \quad 0\,.$$

Hence, thanks to the ode satisfied by v, we have

$$\Big(\frac{v^2}{2} - E\Big)'(t) \leqslant H\big((u')^2\big) - H\big((v(t))^2\big) \leqslant H(2E(t)) - H\big((v(t))^2\big)\,, \quad \forall\, t \geqslant 0\,,$$

and

$$v^2(0) = 2E(0)\,.$$

□

Proof.

Since H is strictly increasing on $\mathbb{R}$, we deduce easily by comparison principles for ODE's that the stated lower energy estimate holds. □

This is an energy comparison principle:

we compare the energy of the nonlinear harmonic oscillator

$$u'' + \nu\, u + f(u) + g(u') = 0\,.$$

which is a second order ODE, to the energy of the first order ODE

$$v' + g(v) = 0 \rightsquigarrow \text{ corresponds to } w'' + g(w') = 0$$

$\rightsquigarrow$ i.e. : drop the potential terms in the original equation □

Lemma (A.-B. JDE 2010)

Let H be a given strictly convex $\mathcal{C}^1$ function from $[0, r_0^2]$ to $\mathbb{R}$ such that $H(0) = H'(0) = 0$, where $r_0 > 0$ is sufficiently small. Let z be the solution of the ordinary differential equation:

$$z'(t) + \kappa\, H(z(t)) = 0\,,\ z(0) = z_0 \quad t \geq 0\,, \tag{3}$$

where $z_0 > 0$ and $\kappa > 0$ are given. Then $z(t)$ is defined for every $t \geq 0$ and decays to 0 at infinity. Moreover assume that

$$0 < \liminf_{x \to 0} \Lambda_H(x) \leq \limsup_{x \to 0} \Lambda_H(x) < 1\,,$$

or that there exists $\mu > 0$ such that

$$0 < \liminf_{x \to 0} \left(\frac{H(\mu\, x)}{\mu\, x} \int_x^{z_1} \frac{1}{H(y)}\, dy \right),\ \text{and} \ \limsup_{x \to 0} \Lambda_H(x) < 1\,,$$

for a certain $z_1 \in (0, z_0]$ (arbitrary).

Lemma (continued)

Then there exists $T_1 > 0$ such that for all $R > 0$ there exists a constant $C > 0$ such that

$$(H')^{-1}\left(\frac{R}{t}\right) \leqslant C\, z(t)\,, \quad \forall\, t \geq T_1\,,$$

where T_1 is a positive constant.

Recall that $v' + g(v) = 0$. This implies that

$$(v^2)' + 2H(v^2) = 0, v^2(0) = E(0).$$

We apply our Lemma to $z = v^2$, $\kappa = 2$ and $R = \eta_3$ and get

$$\max(\eta_1, \eta_2 E(0))(H')^{-1}\left(\frac{\eta_3}{t}\right) \leqslant \max(\eta_1, \eta_2 E(0)) C v^2(t),$$

for sufficiently large t. This concludes the proof of optimality.

Examples of decay rates and optimality

Examples of decay rates

Example 1 **(polynomial case):**

let g be given by $g(x) = x^p$ where $p > 1$ on $(0, r_0]$.
Then

$$E(t) \leq C\beta_{E(0)}\, t^{\frac{-2}{p-1}} \,,$$

for t sufficiently large and for all (u_0, u_1) in $\mathbb{R}^2$.

Moreover this estimate is optimal.

Example 2 **(exponential case):**

let g be given by $g(x) = e^{-\frac{1}{x^2}}$ on $(0, r_0]$.
Then

$$E(t) \leq C\beta_{E(0)}(\ln(t))^{-1} \,,$$

for large t. Moreover, this estimate is optimal.

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Example 3 **(polynomial-logarithmic, close to linear)**:

let g be given by $g(x) = x(ln(\frac{1}{x}))^{-p}$ where $p > 0$.
Then

$$E(t) \leq C\,\beta_{E(0)}e^{-2(\frac{p\,t}{DT_0})^{1/(p+1)}}t^{-1/(p+1)} \tag{4}$$

for t sufficiently large.

Optimality cannot be asserted.

Example 4 **(faster than any polynomial less than exponential)** :
vskip 2mm let g be given by $g(x) = e^{-(ln(\frac{1}{x}))^p}$, $p > 2$, $x \in [0, r_0]$.
Then

$$E(t) \leq C\,\beta_{E(0)}e^{-2(\ln(t))^{1/p}}$$

This estimate is optimal.

Similar results with sharp energy decay rates, and optimality results for the vectorial case.

These optimal estimates depends on the dimension of the system.

Moreover these results applied to semi-discretized PDE's such as the wave or plate equations and give optimal rates of decay.

Therefore when applied to semi-discretized PDE's, they are not uniform with respect to the discretization parameter.

Sommaire

The direct method

We will sketch the optimal-weight convexity on a model example. It works in a similar way for boundary dampings, higher order equations . . .

We consider the locally damped wave equation

$$\begin{cases} u_{tt} - \Delta u + \rho(x, u_t) = 0 \\ u = 0 \text{ in } [0, \infty) \times \Gamma \\ (u, u_t)(0, .) = (u_0, u_1) \text{ in } \Omega\,, \end{cases}$$

$$(HF_g) \quad \begin{cases} \rho \in C(\overline{\Omega} \times \mathbb{R})\,, \rho(x, 0) = 0 \quad \forall x \in \Omega\,, \\ s \mapsto \rho(x, s) \text{ is nondecreasing } \forall\, x \in \Omega, \\ \exists\, c > 0, \exists\, a \in C(\overline{\Omega}) \text{ and } \exists\, g \in \mathcal{C}^1(\mathbb{R}) \text{ such that} \\ a(x)|s| \leqslant |\rho(x, s)| \leqslant ca(x)|s|, \quad \forall x \in \Omega\,, |s| \geqslant 1, \\ a(x)g(|s|) \leqslant |\rho(x, s)| \leqslant ca(x)g^{-1}(|s|), \quad \forall x \in \Omega\,, |s| \leqslant 1, \text{ with} \\ a \geqslant 0 \text{ on } \Omega\,, \exists\, a_- > 0 \text{ such that } a(x) \geqslant a_-, \quad \forall x \in \omega_d \subset \Omega, \\ g \text{ is a strictly increasing and odd function.} \end{cases}$$

Stabilization results for wave-like systems require geometric conditions in the region where the feedback is active. We consider multiplier type condition, that is the Piecewise Multiplier Geometric Condition (denoted by (PMGC), in short), that is:

Definition (PMGC, K. Liu 1997)

We say that a subset $\omega \subset \Omega$ satisfies the (PMGC), if there exist subsets $\Omega_j \subset \Omega$ having Lipschitz boundaries and points $x_j \in \mathbb{R}^N$, $j = 1, \cdots, J$, such that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$ and ω contains an ε-neighborhood in Ω of the set $\cup_{j=1}^J \gamma_j(x_j) \cup (\Omega \setminus \cup_{j=1}^J \Omega_j)$, where $\gamma_j(x_j) = \{x \in \partial\Omega_j : (x - x_j) \cdot \nu_j(x) \geqslant 0\}$ and ν_j is the outward unit normal to $\partial\Omega_j$.

For $J = 1$ and a single observation point $x_0 \rightsquigarrow$ Zuazua's condition.

We assume that ω_d satisfies $(PMGC)$ and the same hypotheses on H than in the finite dimensional case.

Theorem (A.-B. AMO 2005, JDE2010)

Assume the above hypotheses and that $\limsup_{x\to 0} \Lambda_H(x) < 1$. *Then* E *satisfies*

$$E(t) \leqslant \max(\eta_1, \eta_2 E(0))(H')^{-1}\left(\frac{\eta_3}{t}\right)$$

for sufficiently large t.

The steps follow the same ideas than in the finite dimensional case, except that one has to deal with the space dependency.

Step 1: dominant energy estimates

The proof is different than in the finite dimensional case but the spirit of the result is the same.

Let w be a weight function to be determined later on. We only assume that w is non decreasing, nonnegative and differentiable. We prove that under the above hypotheses, we have for any such function w

The dominant energy estimate

$$\int_S^T E(t)w(E(t))\,dt \leq \delta_1 E(S)w(E(S)) + \delta_2 \int_S^T w(E(t)) \int_\Omega |\rho(., u_t)|^2\,dxdt + \delta_3 \int_S^T w(E(t)) \int_\omega |u_t|^2\,dxdt\,, \quad \forall\, 0 \leq S \leq T\,.$$

where the $\delta_i > 0$ are constants for $i = 1, \ldots 3$.

The terms in purple are the nonlinear and localized linear kinetic energies.
The proofs relies on the use of multipliers of the form $K(u)w(E)$ in several steps. This works as for the linear case. It involves geometrical aspects (see A.-B CIME Lectures 2012, Springer).
Estimate the part of the energy in the undamped region by the energy localized in a subset Q_1 of the damped region ω_d and the linear and nonlinear kinetic energies

- $\rightsquigarrow$ This first result is proved as follows

Let $0 < \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \varepsilon$ and define $Q_i := N_{\varepsilon_i}[\cup_{j=1}^{J}\gamma_j(x_j) \cup (\Omega \setminus \cup_{j=1}^{J}\Omega_j)](i = 0, 1, 2)$. Since $(\overline{\Omega_j} \backslash Q_1) \cap \overline{Q_0} = \emptyset$, we introduce a cut-off function $\psi_j \in C_0^\infty(R^N)$ satisfying

$$0 \leqslant \psi_j \leqslant 1, \qquad \psi_j = 1 \quad \text{on } \overline{\Omega_j} \backslash Q_1; \qquad \psi_j = 0 \quad \text{on } Q_0.$$

For $m_j(x) = x - x_j$, we define the C^1 vector field on Ω:

$$\theta(x) = \begin{cases} \psi_j(x) m_j(x) & \text{if } x \in \Omega_j, j = 1, \cdots, J \\ 0 & \text{if } x \in \Omega \backslash \cup_{j=1}^{J} \Omega_j \end{cases}$$

Use the multiplier $w(E(t))\left[\theta \cdot \nabla u + \frac{n-1}{2} u\right] \rightsquigarrow$ desired first estimate

- $\rightsquigarrow$ Then estimate the part of the potential energy

$$\int_S^T w(E(t)) \int_{Q_1} |\nabla u|^2 dxdt$$

by the weighted nonlinear kinetic energy $+$ the weighted localized linear kinetic energy $+$ weighted localized zero order term

$\rightsquigarrow$ performed via multipliers of the form $w(E(t))$cut-off function $\times\ u$

- $\rightsquigarrow$ get the desired estimate by estimating the weighted localized zero order term by using a suitable multiplier

- $\rightsquigarrow$ The dominant energy estimate is proved

We now turn to step 2 as in the finite dimensional version of the optimal-weight convexity method.

Introduce a suitable partition of Ω , separating the domain in which the velocity is *sufficiently small* and its complementary.

This argument goes back to an idea of Zuazua

That is, we introduce

$$\Omega_t = \{x \in \Omega\,, |u_t(t,x)| \leq \varepsilon_0\}$$

where $\varepsilon_0 > 0$ is chosen sufficiently small in relation with g and the convexity arguments, roughly speaking, to guarantee that we stay in the domain of convexity of H.

Thanks to Jensen's inequality

$$\begin{cases} H\Big(\frac{1}{|\Omega_t|}\int_{\Omega_t}|\rho(.,u_t)|^2\,dt\Big) \leq \\ \frac{1}{|\Omega_t|}\int_{\Omega_t}|\rho(.,u_t)|\,g(|\rho(.,u_t)|) \end{cases}$$

Thanks to the assumption on ρ close to 0 and our choice of ε_0

$$g(|\rho(.,u_t)|) \leq c|u_t|\,,\ \text{on}\ \Omega_t\,,$$

Hence if w is nonnegative function (to be determined), we deduce from the above estimate that

$$\int_S^T w(E(t))\,dt\int_{\Omega_t}|\rho(.,u_t)|^2\,dx \leq \int_S^T|\Omega_t|w(E(t))\,dt \times H^{-1}\Big(\frac{1}{|\Omega_t|}\int_{\Omega_t}u_t\rho(.,u_t)\,dx\Big)\,.$$

Now, thanks to Young's inequality, we have

$$w(E)H^{-1}\Big(\frac{1}{|\Omega_t|}\int_{\Omega_t} u_t\rho(.,u_t)\,dx\Big) \leq$$
$$H^*(w(E(t))) + \frac{1}{|\Omega_t|}\int_{\Omega_t} u_t\rho(.,u_t)\,dx\,,$$

But thanks to the dissipation relation

$$E'(t) = -\int_{\Omega} u_t\rho(.,u_t)\,dx$$

and the fact that $s\rho(.,s) \geq 0$, we deduce that

$$\int_S^T w(E(t))\,dt \int_{\Omega_t} |\rho(.,u_t)|^2\,dx \leq |\Omega| \int_S^T H^*(w(E(t))) + cE(S)\,.$$

Optimal energy-weight

In the set $\Omega \backslash \Omega_t$ the damping has a linear growth $\leadsto$ easy to treat.

$$\int_S^T w(E(t))\,dt \int_\Omega |\rho(.,u_t)|^2\,dx \leq |\Omega| \int_S^T H^*(w(E(t))) + cE(S)(1 + w(E(S))$$

We can treat the linear kinetic energy term in a similar way using the other side of the inequality between ρ and g, g^{-1}, a suitable subset $\omega_d^t = \{x \in \omega_d, |u_t(t,x)| \leqslant \varepsilon_1\}$
Finally, we show that for a suitable $\beta = \max(\eta_1, \eta_2 E(0))$, we have

$$\int_S^T E(t)w(E(t))\,dt \leq \beta \int_S^T H^*(w(E(t))) + \frac{C_0}{2}E(S)$$

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Let us choose the weight function w such that

$$\frac{s}{2\beta} = \frac{\widehat{H}^*(w(s))}{w(s)} \, ,$$

This optimal weight function w is uniquely determined in the following way :

$$w(s) = L^{-1}(s/2\beta) \quad s \in [0, 2\beta r_0^2) \, ,$$

where we recall that L is defined as follows :

$$L(y) = \begin{cases} \dfrac{\hat{H}^*(y)}{y} \text{ if } y \in (0, +\infty) \, , \\ 0 \text{ if } y = 0 \, , \end{cases}$$

Hence we have $\beta \int_S^T H^*(w(E(t))) = \frac{1}{2} \int_S^T E(t) w(E(t)) \, dt$ by this construction of w.

Hence we prove

$$\int_S^T w(E(t))E(t)dt \leqslant C_0 E(S) \quad \forall\, 0 \leqslant S \leqslant T.$$

Then we can proceed exactly as in the finite dimensional case, and prove the general semi-explicit energy decay formula and simplify it using the damping classification with Λ_H, i.e.

If $\limsup_{x\to 0} \Lambda_H(x) < 1$. Then E satisfies

$$E(t) \leqslant \max(\eta_1, \eta_2 E(0))(H')^{-1}\left(\frac{\eta_3}{t}\right)$$

for sufficiently large t and a general semi-explicit decay estimate if $\limsup_{x\to 0} \Lambda_H(x) = 1$.
We can treat in a similar way PDE's with boundary damping and give an abstract framework for this (A.-B. 2005), so the method works as well for unbounded damping operators.

Literature (non exhaustive):

• Exponential and polynomial growing dampings : Haraux 1978, Zuazua, Nakao, Komornik . . .

• Lasiecka Tataru 1993 arbitrary growth, non explicit formula, only two explicit decays: for linear and polynomially growing dampings

• Martinez 1999 arbitrary growth, convexity and integral inequalities first explicit decay formulas, does not recover the optimal polynomial decay for polynomial dampings, W. Liu and Zuazua 1999, Eller et al 2002 . . .

• Explicit and sharp upper estimates are requested if one wants to deal with optimality questions.

$\rightsquigarrow$ The optimal-weight convexity method gives such type of estimates. It is also related to physical properties: dominant energies, energy comparison principle . . .

We have seen that we can proceed in the infinite dimensional case as for the finite dimensional with more complex arguments

- $\leadsto$: in step 1 the geometry comes in and one has to use multiplier arguments to prove the dominant energy estimate, under geometric conditions on the damping region

- $\leadsto$: in step 2 the geometry also comes in and one has to also use Jensen's in addition to Young's inequalities to build the optimal-weight function w

- $\leadsto$: the steps 3 and 4 are not linked to the finite or infinite character they are specific to weighted nonlinear integral inequalities, thus these two steps are similar in both finite and infinite dimensions

The major difference between the finite dimensional case and the infinite dimensional one is in

1. Obtention of lower estimates? $\longrightarrow$ tough, and exists only for specific cases: Haraux 1995, A.-B. 2010, 2011
2. Lower estimates $\approx$ upper estimates, optimality? $\longrightarrow$ very tough, exists only for specific cases Martinez and Vancostenoble 2001, A.-B. 2005

The energy comparison principle of the finite dimension case is no longer valid, as for instance for globally distributed dampings

$$-E'(t) = \int_\Omega H(u_t^2)dx$$

whereas

$$2E(t) = \int_\Omega \left(|u_t|^2 + |\nabla u|^2\right) dx$$ in finite D easy to compare $u'^2(t)$ and $E(t)$.

- One can only show lower estimates using interpolation arguments for certains cases (1-D waves with local or boundary dampings, plates in 2-D. some systems such as Timoshenko) A.-B. 2010, 2011
- Better lower estimates if the solutions are smoother, Haraux 1995, A.-B. 2010, 2011
- The lower estimate is not of the same order than the upper one $\longrightarrow$ no optimality proof except wave 1-D, boundary dampings, very specific initial data Martinez Vancostenoble 2001

I shall come back to this question at the end

Sommaire

Consider

$$\begin{cases} u_{tt} - \Delta u + k * \Delta u = 0 \\ u = 0 \text{ in } [0, \infty) \times \Gamma \\ (u, u_t)(0, .) = (u_0, u_1) \text{ in } \Omega \,, \end{cases}$$

$$(k * v)(t) = \int_0^t k(t-s)v(s)\,ds\,.$$

and the kernel k is positive, nonincreasing, and decaying at infinity. Moreover, for well-posedness, one considers the condition

$$\int_0^\infty k(t)\,dt < 1$$

In this case, the natural energy is

$$E_u(t) = \frac{1}{2}\|u_t(t)\|^2_{L^2(\Omega)}\,dx + \frac{1}{2}\left(1 - \int_0^t k(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^t k(t-s)\|\nabla u(t) - \nabla u(s)\|^2_{L^2(\Omega)}\,ds$$

and the dissipation relation is:

$$E_u'(t) = -\frac{1}{2}k(t)\|\nabla u(t)\|^2 + \frac{1}{2}\int_0^t k'(t-s)\|\nabla u(s) - \nabla u(t)\|^2\,ds \leq 0$$

The optimal-weight convexity method can be extended, in an involved way, to treat nonlocal dissipation.

- The dissipation terms are very different from the frictional case
- the problem is not invariant by translation in time.

The case of memory damping

Assume

$$k'(t) \leq -\chi(k(t)) \quad \text{for a.e.} \quad t \geq 0$$

and

- χ is a nonnegative measurable function on $[0, k_0]$, for some $k_0 > 0$, strictly increasing and of class $\mathcal{C}^1$ on $[0, k_1]$, for some $k_1 \in (0, k_0]$, such that

$$\chi(0) = \chi'(0) = 0$$

- $$\exists\ \chi_0 > 0 \text{ such that } \chi \geq \chi_0 \text{ on } [k_1, k_0]$$

$$\int_0^{k_0} \frac{dx}{\chi(x)} = \infty\,, \int_0^{k_0} \frac{x}{\chi(x)}\,dx < 1$$

Theorem (A.-B.-Cannarsa 2009)

Assume that the convolution kernel $k : [0,\infty) \to [0,\infty)$ *is a locally absolutely continuous function as above and that* χ *is strictly convex on an interval of the form* $(0,\delta]$ *with* $\delta > 0$ *and*

- $$\liminf_{x\to 0^+} \Lambda(x) > \frac{1}{2},$$

  where

  $$\Lambda(x) = \frac{\chi(x)/x}{\chi'(x)}, \quad x \in (0,\delta].$$

- $$\limsup_{x\to 0^+} \Lambda(x) < 1 \qquad \text{and} \qquad k'(t) = -\chi(k(t)) \quad \text{for a.e.} \quad t \geq 0.$$

Then

$$E_u(t) \leq \kappa(E_u(0))k(t), \quad \forall\, t \geq T_1$$

where $T_1 > 0$ *and* $\kappa(E_u(0))$ *are explicit positive constants.*

We can also give an explicit estimate if the $\limsup = 1$ but it has a different expression.

The approach works as well for abstract PDE's and applies to elasticity, higher order PDE's. for semi linear terms (with a growth condition ...

It requires a sharp analysis of the damping mechanisms and of the intrinsic properties of such dissipative systems.

Remark

The exponential or power-like case was known (see e.g. Munoz-Rivera and Salvaterria 2001 , A.-B.-Cannarsa-Sforza 2008, it is much more difficult to get a sharp estimate for generally decaying kernels k (also easily computable for all examples) satisfying the condition (to guarantee well-posedness):

$$\int_0^\infty k(s)\,ds < 1$$

Sommaire

Let H be a Hilbert space with norm $|\cdot|$, A be bounded linear self-ajoint coercive operator on H and B be a linear bounded self-adjoint nonnegative operator on H.

We consider the following two problems

- The conservative problem

$$\phi'' + A\phi = 0\,,$$

- The linearly damped problem

$$y'' + Ay + By' = 0\,.$$

Then one can prove that the linearly damped system is exponentially stable, that is there exists $\delta > 0$ and $c > 0$ such that

$$|y'(t)|^2 + |A^{1/2}y(t)|^2 \leq ce^{-\delta t}\left(|y_1|^2 + |A^{1/2}y_0|^2\right),$$
$$\forall\,(y_0, y_1)\textit{(initial data)} \in D(A^{1/2}) \times H\,, \forall\, t \geq 0\,.$$

if and only if all the solutions of the conservative system satisfy the following observabilty inequality

$$|\phi'(0)|^2 + |A^{1/2}\phi(0)|^2 \leq C\int_0^T |B^{1/2}\phi'|^2\,dt\,.$$

for $T > 0$ (sufficiently large) and $C \geq 0$ (not depending on the initial data).

This result is due to Haraux 1989. Extension to linear unbounded damping operators Ammari Tucsnak 2001

We consider the following second order differential equation

$$\begin{cases} \ddot{w}(t) + Aw(t) + a(.)\rho(., \dot{w}) = 0\,, & t \in (0,\infty)\,, x \in \Omega \\ w(0) = w^0\,, \dot{w}(0) = w^1\,. \end{cases}$$

where

- Ω is a bounded open set in $\mathbb{R}^N$, with a boundary Γ sufficiently smooth.
- we set $H = L^2(\Omega)$, with its usual scalar product denoted by $\langle \cdot, \cdot \rangle_H$ and the associated norm $|\cdot|_H$
- $A : D(A) \subset H \to H$ is a densely defined self-adjoint linear operator satisfying

$$\langle Au, u \rangle_H \geq C\|u\|_H^2 \qquad \forall u \in D(A)$$

for some $C > 0$

- $\rho \in \mathcal{C}(\overline{\Omega} \times \mathbb{R}; \mathbb{R})$ is a continuous monotone nondecreasing function with respect to the second variable on Ω such that $\rho(.,0) = 0$ on Ω.
- $\exists$ a continuous strictly increasing odd function g in $\mathcal{C}([-1,1];\mathbb{R})$ continuously differentiable in a neighbourhood of 0 and satisfying $g(0) = g'(0) = 0$ with

$$c_1 g(|v|) \leq |\rho(.,v)| \leq c_2 g^{-1}(|v|)\,, \quad |v| \leq 1\,, \text{a.e. on } \Omega\,,$$

$$c_1 |v| \leq |\rho(.,v)| \leq c_2 |v|\,, \quad |v| \geq 1\,, \text{a.e. on } \Omega\,,$$

  where $c_i > 0$ for $i = 1,2$.
- $a \in L^\infty(\overline{\Omega})$, with $a \geq 0$ on Ω

$$\exists\ a_- > 0 \text{ such that } a \geq a_- \text{ on } \omega\,.$$

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We define the energy of a solution by

$$E_w(t) = \frac{1}{2}\Big(|A^{1/2}w(t)|_H^2 + |\dot{w}(t))\|_H^2\Big)$$

We set $H_{1/2} = D(A^{1/2})$.

The idea is to compare as Haraux the linearly damped equation to the conservative one.

Here we pursue and compare the linearly damped equation to the nonlinearly damped equation. The key point to treat the nonlinear damping is to use once again the optimal-weight convexity method.

Then, we can state the following result.

Theorem (A.-B. Ammari JFA 2011)

Assume that ρ *and a satisfy the above assumptions, that there exists* $r_0 > 0$ *sufficiently small so that the function H is strictly convex on* $[0, r_0^2]$ *and that*

$$\lim_{x \to 0^+} \frac{H'(x)}{\Lambda_H(x)} = 0$$

Moreover assume that there exists $T > 0$ *such that the following observability inequality is satisfied for the linear conservative system*

$$\begin{cases} \ddot{\phi}(t) + A\phi(t) = 0, \\ \phi(0) = \phi^0, \, \dot{\phi}(0) = \phi^1. \end{cases}$$

$$c_T E_\phi(0) \leq \int_0^T |\sqrt{a}\dot{\phi}|_H^2 \, dt \,, \quad \forall (\phi_0, \phi_1) \in H_{1/2} \times H \,.$$

with a certain $c_T > 0$*. Then, the energy of the solution of the damped equation satisfies*

Theorem (continued)

$$E_w(t) \leq \beta T L\Big(\frac{1}{\psi_r^{-1}(\frac{t-T}{T_0})}\Big)\,, \qquad \textit{for } t \textit{ sufficiently large}.$$

If further, $\limsup_{x\to 0^+} \Lambda_H(x) < 1$ *then we have the simplified decay rate*

$$E_w(t) \leq \beta T (H')^{-1}\big(\frac{DT_0}{t-T}\big)\,,$$

for t sufficiently large. Here D is a positive constant which is independent of $E_w(0)$ and T, whereas T_0 depends on T, β is a positive constant chosen so that

$$\beta > \max\Big(\frac{2\alpha T}{C_T}, \frac{E_w(0)}{L(H'(r_0^2))}, \frac{E_w(0)}{\delta}\Big)\,,$$

where the constants $C_T > 0$, α and $\delta > 0$ are explicit positive constants.

The proof of our main result relies on a result important in itself, since it allows to compare discrete energy inequalities to continuous ones.

One can show that if

$$\lim_{x\to 0^+} \frac{H'(x)}{\Lambda_H(x)} = 0$$

then the following property holds:

- The function M defined by

$$M(x) = xL^{-1}(x)\,, \quad x \in [0, r_0^2)\,.$$

is such that $\lim_{x\to 0^+} M'(x) = 0$, where L is defined as above.

Hence thanks to this assumption, the function $x \mapsto x - \kappa M(x)$ is strictly increasing in $[0,\delta] \subset [0, r_0^2]$ for a certain $\kappa > 0$ and $\delta > 0$ sufficiently small.

Key step: used to recognize a discretization through an Euler scheme to prove the following result

Theorem (time discrete inequality, A.-B. Ammari JFA 2011)

Assume that the above assumption holds and let $T > 0$ *and* $\rho_T > 0$ *be given. Let* $\delta > 0$ *be such that the function defined by* $x \mapsto x - \rho_T M(x)$ *is strictly increasing on* $[0, \delta]$. *Assume that* $\widehat{E}$ *is a nonnegative, nonincreasing function defined on* $[0, \infty)$ *with* $\widehat{E}(0) < \delta$ *and satisfying*

$$\widehat{E}((k+1)T) \leq \widehat{E}(kT)\Big(1 - \rho_T L^{-1}(\widehat{E}(kT))\Big), \quad \forall\, k \in \mathbb{N}.$$

Then $\widehat{E}$ *satisfies the upper estimate*

$$\widehat{E}(t) \leq TL\left(\frac{1}{\psi_r^{-1}\left(\frac{(t-T)\rho_T}{T}\right)}\right), \qquad \textit{for } t \textit{ sufficiently large},$$

Theorem (continued)

If moreover $\limsup_{x\to 0^+} \Lambda_H(x) < 1$, *then we have the simplified decay rate*

$$\widehat{E}(t) \leq T(H')^{-1}\Big(\frac{D\,T}{\rho_T(t-T)}\Big)\,,$$

for t *sufficiently large and where* D *is a positive constant independent of* $\widehat{E}(0)$ *and of* T.

$\rightsquigarrow$ **The indirect method allows to "hide" the geometric arguments in the assumption made on the "damping" operator**, namely that the solution of the conservative equation satisfies an observability inequality (case of bounded damping operator)

$\rightsquigarrow$ **Allows us to derive optimal energy decay rates under optimal geometric conditions on the damping region:** (GCC) the Geometric Control Condition of Bardos Lebeau Rauch 1992

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Sommaire

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Joint work with Yannick Privat and Emmanuel Trélat

- We give a very general framework for first order nonlinear dissipative equations with bounded damping operators that include wave, Schrödinger, Boltzmann-type equations (non local dampings in space) ...
- We extend the results of A.-B.-Ammari for the continuous framework
- We study semi-discretization and show which type of numerical viscosity we should add in the usual schemes to obtain an optimal uniform energy decay with respect to the discretization parameter. The proof relies on the optimal-weight convexity method combined with Ervedoza and Zuazua's approach for linear dampings.
- We study time discretization of the continuous abstract equation and prove convergence
- We study the full discretisation (time and space)

Sommaire

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Joint work in progress with Jean-Michel Coron, Vincent Perrollaz and Emmanuel Trélat

- Optimality results for certain classes of infinite dimensional systems
- For large classes of initial data
- Studying the influence of the initial data
- Extensions to larger classes of infinite dimensional systems

Merci de votre attention