Model-Data Weak Formulation (MDWF): Galerkin Approximation

Recapitulation
Unlimited-Observations Statement
Limited-Observations Statement
Experimentally Observable Spaces
Interpretation: Variational Data Assimilation

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Abstraction

We consider a

physical system

and associated *true* state *field*

$$u^{\mathrm{true}} \in \mathcal{X},$$

which we assume is *deterministic* and stationary.

We wish to predict the field u^{true} based on

a best-knowledge mathematical model $\{A, f\}$, and

M experimental observations,

within an integrated (variational) *weak formulation*.

Model Bias ...

Given the true state of the physical system

$$\text{(field) } u^{\text{true}} \in \mathcal{X} \ ,$$

define the model bias as residual

$$g \equiv Au^{\text{true}} - f \in \mathcal{Y}' \ ,$$

such that

$$Au^{\text{true}} = f + g \ .$$

Recall $Au^{\text{bk}} = f$: $u^{\text{bk}} = u^{\text{true}}$ if and only if $g = 0$.

Given that

$$Au^{\text{true}} = f + g$$

we must *simultaneously* estimate

model bias: $g \in \mathcal{Y}'$, and

state: $u^{\text{true}} \in \mathcal{X}$;

unique decomposition of $f = Au^{\text{true}} - g$ in terms of

g and u^{true}

requires experimental observations.

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- Preliminaries
- Saddle Formulation

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Operators and Sesquilinear Forms

Recall

inner products $(\cdot,\cdot)_{\mathcal{X}}$, $(\cdot,\cdot)_{\mathcal{Y}}$,

duality pairings $\langle\cdot,\cdot\rangle_{\mathcal{X}'\times\mathcal{X}}$, $\langle\cdot,\cdot\rangle_{\mathcal{Y}'\times\mathcal{Y}}$,

representation operators $X\colon \mathcal{X}\to\mathcal{X}'$, $Y\colon \mathcal{X}\to\mathcal{Y}'$,

associated with our spaces $\mathcal{X}(\Omega),\mathcal{Y}(\Omega)$, and

operator $A\colon \mathcal{X}\to\mathcal{Y}'$, *or* equivalently

sesquilinear form $a\colon \mathcal{X}\times\mathcal{Y}\to\mathbb{C}$, and

source $f\in\mathcal{Y}'$,

associated with our best-knowledge model.

Model Bias

Recall the definition of model bias

$$g \equiv Au^{\text{true}} - f \in \mathcal{Y}'$$

in terms of best-knowledge model and true state.

Now introduce the $\mathcal{Y}$-representation of g

$$q^{\text{mod}} \equiv Y^{-1} g \in \mathcal{Y};$$

we shall refer to both g and q^{mod} as model bias.

Recall that model bias can also represent error in the best-knowledge model $\{A, f\}$, $\{\delta A, \delta f\}$.

Observable Field (Assumption: Strong)

Recall the "true state" of the physical system,

$$u^{\text{true}} \in \mathcal{X} \ ,$$

which we assume is deterministic and stationary.

Now express the "observable" state as

$$u^{\text{obs}} \equiv u^{\text{true}} - q^{\text{obs}},$$

for $q^{\text{obs}} \in \mathcal{X}$ deterministic and stationary:

$$\begin{aligned} q^{\text{obs}} &= 0 \quad \text{— perfect observations;} \\ q^{\text{obs}} &\neq 0 \quad \text{— imperfect observations;} \\ q^{\text{obs}} &\to \infty \quad \text{— uninformative observations;} \end{aligned}$$

q^{obs} represents a measurement-induced perturbation.

Weighting Parameter

Introduce non-negative real parameter,

$$\nu \in \mathbb{R}_{+,0} \ ,$$

which can be interpreted (conceptually) as[†]

$$\frac{\text{trust in best-knowledge model}}{\text{trust in experimental observations}} .$$

A priori theory shall suggest

$$\nu^{\mathrm{opt}} \equiv \frac{\|\mathbb{A} q^{\mathrm{obs}}\|_{\mathcal{Y}}}{\|q^{\mathrm{mod}}\|_{\mathcal{Y}}}$$

as a guideline for choice of ν.

Note as approach perfect observations, $\nu \to 0$;
as approach uninformative observations, $\nu \to \infty$.

[†] We can view ν^{-1} as the gain on the data "innovation".

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Statement

Given $\nu \in \mathbb{R}_{+,0}$, find $(u, \psi) \in \mathcal{X} \times \mathcal{Y}$ such that

$$
\begin{aligned}
a(u,v) - (\psi, v)_{\mathcal{Y}} &= \langle f, v \rangle_{\mathcal{Y}' \times \mathcal{Y}}, && \forall v \in \mathcal{Y}, \\
-a(u,\phi) - \nu(\psi, \phi)_{\mathcal{Y}} &= -\, a(u^{\text{true}}, \phi) \\
&\quad - \nu(q^{\text{mod}}, \phi)_{\mathcal{Y}}, && \forall \phi \in \mathcal{Y},
\end{aligned}
$$

or

$$
\begin{aligned}
a(u,v) - (\psi, v)_{\mathcal{Y}} &= \langle f, v \rangle_{\mathcal{Y}' \times \mathcal{Y}}, && \forall v \in \mathcal{Y}, \\
-a(u,\phi) - \nu(\psi, \phi)_{\mathcal{Y}} &= -\, a(u^{\text{obs}}, \phi) - a(q^{\text{obs}}, \phi) \\
&\quad - \nu(q^{\text{mod}}, \phi)_{\mathcal{Y}}, && \forall \phi \in \mathcal{Y},
\end{aligned}
$$

since $u^{\text{true}} = u^{\text{obs}} + q^{\text{obs}}$.

Solution (by Construction)

Proposition 1. The unique solution to our saddle problem is $(u, \psi) = (u^{\text{true}}, q^{\text{mod}}) \in \mathcal{X} \times \mathcal{Y}$.

Sketch of proof:

Satisfaction of second equation: trivial.

Satisfaction of first equation (also trivial):

$$a(u, v) - (\psi, v)_{\mathcal{Y}}$$

$$= \langle f, v \rangle_{\mathcal{Y}' \times \mathcal{Y}},$$

from definitions of q^{mod} and g.

Uniqueness follows from our assumptions on A. □

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$$a(u^{\text{true}}, v) - (q^{\text{mod}}, v)_{\mathcal{Y}} = \langle Au^{\text{true}} - Yq^{\text{mod}}, v\rangle_{\mathcal{Y}' \times \mathcal{Y}}$$

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from definitions of q^{mod} and g.

Uniqueness follows from our assumptions on A. □

Interpretation

We interpret the first equation

$$
\begin{aligned}
a(u,v) - (\psi, v)_{\mathcal{Y}} &= \langle f, v\rangle_{\mathcal{Y}'\times\mathcal{Y}}, && \forall v \in \mathcal{Y},\\
-a(u,\phi) - \nu(\psi,\phi)_{\mathcal{Y}} &= -a(u^{\text{obs}},\phi) - a(q^{\text{obs}},\phi)\\
&\quad - \nu(q^{\text{mod}},\phi)_{\mathcal{Y}}, && \forall \phi \in \mathcal{Y},
\end{aligned}
$$

as the "model" equation: best-knowledge *plus* model bias.

We interpret the second equation

$$
\begin{aligned}
a(u,v) - (\psi, v)_{\mathcal{Y}} &= \langle f, v\rangle_{\mathcal{Y}'\times\mathcal{Y}}, && \forall v \in \mathcal{Y},\\
-a(u,\phi) - \nu(\psi,\phi)_{\mathcal{Y}} &= -a(u^{\text{obs}},\phi) - a(q^{\text{obs}},\phi)\\
&\quad - \nu(q^{\text{mod}},\phi)_{\mathcal{Y}}, && \forall \phi \in \mathcal{Y},
\end{aligned}
$$

as the "data" equation: model-observation connection.

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Unlimited-Observations: Impractical

Our infinite-observations saddle

$$a(u,v) - (\psi, v)_{\mathcal{Y}} = \langle f, v\rangle_{\mathcal{Y}'\times\mathcal{Y}}, \quad \forall v \in \mathcal{Y},$$

$$-a(u,\phi) - \nu(\psi,\phi)_{\mathcal{Y}} = -\overbrace{a(u^{\text{obs}},\phi)}^{(\mathbb{A}u^{\text{obs}},\phi)_{\mathcal{Y}}} - \overbrace{a(q^{\text{obs}},\phi)}^{(\mathbb{A}q^{\text{obs}},\phi)_{\mathcal{Y}}} - \nu(q^{\text{mod}},\phi)_{\mathcal{Y}}, \quad \forall \phi \in \mathcal{Y},$$

is not actionable since we do not know

1. the observable field, u^{obs}, *everywhere*;
2. (precisely) the *observational imperfection*, q^{obs};
3. the *model bias*, q^{mod}.

We can not evaluate the "knowns" for the second equation.

Limited-Observations: Discretization Procedure

To construct our *limited-observations* saddle we

1a. project $\mathbb{A}u^{\rm obs}$ over *appropriate* approximation space $\mathcal{Y}_M \subset \mathcal{Y}$ of *finite* dimension $M \geq 0$;

1b. search for $\psi_{M,\nu} \approx \psi$ in $\mathcal{Y}_M$ to ensure uniqueness of discrete solution $(u_{M,\nu}, \psi_{M,\nu})$;

2. assume that $\|\mathbb{A}q^{\rm obs}\|_{\mathcal{Y}}$ is negligibly small;

3. assume that $\nu\|q^{\rm mod}\|_{\mathcal{Y}}$ is negligibly small.

The latter two assumptions reflect the achievable accuracy (for $\nu \approx \nu^{\rm opt}$).

Limited-Observations Saddle

Given $\nu \in \mathbb{R}_{+,0}$, find $(u_{M,\nu}, \psi_{M,\nu}) \in \mathcal{X} \times \mathcal{Y}_M$ such that

$$a(u_{M,\nu}, v) - (\psi_{M,\nu}, v)_{\mathcal{Y}} = \langle f, v \rangle_{\mathcal{Y}' \times \mathcal{Y}}, \qquad \forall v \in \mathcal{Y},$$
$$-a(u_{M,\nu}, \phi) - \nu(\psi_{M,\nu}, \phi)_{\mathcal{Y}} = -a(u^{\text{obs}}, \phi), \ \ \forall \phi \in \mathcal{Y}_M;$$

recall that u^{obs} is the experimentally observable field.

Note for $M = 0$ ($\mathcal{Y}_M \equiv 0$): $\psi_{M,\nu} = 0$ and hence

$$u_{M,\nu} = u^{\text{bk}}$$

since

$$a(u^{\text{bk}}, v) = \langle f, v \rangle_{\mathcal{Y}' \times \mathcal{Y}}, \ \ \forall v \in \mathcal{Y};$$

in absence of data, recover best-knowledge model.

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Observation Functionals [F]

We introduce $M_{\max}$ observation functionals

$$\ell_m^{\mathrm{o}} \in \mathcal{X}', \quad m = 1, \ldots, M_{\max},$$

which reflect

transducer position or focus, and

transducer filter characteristics,

associated with the data acquisition procedure.

We define a single observation as

$$m \in \{1, \ldots, M_{\max}\} \to \ell_m^{\mathrm{o}}(u^{\mathrm{obs}}) \in \mathbb{C} \ ,$$

where recall $u^{\mathrm{obs}} = u^{\mathrm{true}} - q^{\mathrm{obs}}$ is the observable field.

Example: Gaussian Observation Functionals

Consider $\Omega \subset \mathbb{R}^d$, $H_0^1(\Omega) \subset \mathcal{X} \subset H^1(\Omega)$, and define

$$\ell_m^{\mathrm{o}}(v) = \mathrm{Gauss}\,(v; x_m^{\mathrm{c}}, \sigma) \equiv \int_\Omega \frac{1}{(2\pi)^{d/2}\sigma^d}\, e^{-\frac{|x - x_m^{\mathrm{c}}|^2}{2\sigma^2}} v(x)dx$$

for

observation *centers* $x_m^{\mathrm{c}} \in \Omega$, $m = 1, \ldots, M_{\max}$,
observation *filter width* $\sigma \in \mathbb{R}_+$;

we apply corrections near the domain boundary $\partial\Omega$.

In theory, we may consider $\sigma \to 0$ but only for $d = 1$;
for *real* transducers, σ will be finite for any d.

Hierarchical Spaces $\mathcal{E}_M^{\{\ell^o\}}$

Given choices of

observation functionals ℓ_m^o, $m = 1, \ldots, M_{\max}$,

define, for $1 \leq M \leq M_{\max}$, $\mathcal{E}_0^{\{\ell^o\}} \equiv 0$

$$\mathcal{E}_M^{\{\ell^o\}} \equiv \mathrm{span}\{A^{-*}\ell_m^o,\ 1 \leq m \leq M\};$$

note spaces are hierarchical, $\mathcal{E}_M^{\{\ell^o\}} \subset \mathcal{E}_{M+1}^{\{\ell^o\}}$.

We say that $\mathcal{E}_M^{\{\ell^o\}}$ is *experimentally observable* with respect to observation functionals $\{\ell_m^o\}_{1 \leq m \leq M}$.

Projection of Observable Field

Recall that $(u_{M,\nu}, \psi_{M,\nu}) \in \mathcal{X} \times \mathcal{Y}_M$ satisfies

$$a(u_{M,\nu}, v) - (\psi_{M,\nu}, v)_{\mathcal{Y}} = \langle f, v \rangle_{\mathcal{Y}' \times \mathcal{Y}}\,, \qquad \forall v \in \mathcal{Y},$$

$$-a(u_{M,\nu}, \phi) - \nu(\psi_{M,\nu}, \phi)_{\mathcal{Y}} = -\overbrace{a(u^{\text{obs}}, \phi)}^{\text{projection of observable field: } (\mathbb{A}u^{\text{obs}}, \phi)_{\mathcal{Y}}}\,, \quad \forall \phi \in \mathcal{Y}_M.$$

We now require $\mathcal{Y}_M = \mathcal{E}_M^{\{\ell^{\text{o}}\}}$: then, for $\phi \in \mathcal{Y}_M$,

$$\Big(\mathbb{A}u^{\text{obs}}, \underbrace{\sum_{m=1}^{M} \alpha_m A^{-*}\ell^{\text{o}}_m}_{\phi \in \mathcal{Y}_M}\Big)_{\mathcal{Y}} = \Big\langle A^* \sum_{m=1}^{M} \alpha_m A^{-*}\ell^{\text{o}}_m, u^{\text{obs}} \Big\rangle_{\mathcal{X}' \times \mathcal{X}} = \sum_{m=1}^{M} \alpha_m \langle \ell^{\text{o}}_m, u^{\text{obs}} \rangle_{\mathcal{X}' \times \mathcal{X}}$$

corresponds to M (realizable) single observations.

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Regularization-Misfit Minimization [ZN] (*restricted to Galerkin approximation*)

The state $u_{M,\nu} \in \mathcal{X}$ satisfies

$$u_{M,\nu} = \arg\min_{w \in \mathcal{X}} \Big(\underbrace{\|f - Aw\|^2_{\mathcal{Y}'}}_{\text{regularization}} + \frac{1}{\nu} \underbrace{\|\Pi_M \mathbb{A}(u^{\text{obs}} - w)\|^2_{\mathcal{Y}}}_{\text{model-observation misfit}} \Big),$$

where the projector $\Pi_M : \mathcal{Y} \to \mathcal{Y}_M$ is given by

$$(\Pi_M w, v)_{\mathcal{Y}} = (w, v)_{\mathcal{Y}}, \ \forall\, v \in \mathcal{Y}_M, \text{ for any } w \in \mathcal{Y}.$$

For *perfect observations* and $\nu \to 0$ we recover a penalty formulation for *constrained estimation*.

Saddle: primal-dual Euler-Lagrange equations.†

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†The primal-only Euler-Lagrange equation constitutes a single-field formulation for $u_{M,\nu} \in \mathcal{X}$ (not well-suited to computation).

MDWF Galerkin: Analysis

A Priori Estimates
Stability Constant: Monotonic Improvement
Approximation Theory: Simple Example
Limit of Small Model Bias

MDWF Galerkin: Analysis

A Priori Estimates

- Perfect Observations (P-O) [QV]
- Imperfect Observations (I-O)

Stability Constant: Monotonic Improvement

Approximation Theory: Simple Example

Limit of Small Model Bias

MDWF Galerkin: Analysis

- A Priori Estimates
  - Perfect Observations (P-O) [QV]
  - Imperfect Observations (I-O)
- Stability Constant: Monotonic Improvement
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Formulation Rappel (I-O) ...

Recall $(u, \psi) \in \mathcal{X} \times \mathcal{Y}$ satisfies

$$a(u, v) - (\psi, v)_{\mathcal{Y}} = \langle f, v \rangle_{\mathcal{Y}' \times \mathcal{Y}}, \quad \forall v \in \mathcal{Y},$$

$$-a(u, \phi) - \nu(\psi, \phi)_{\mathcal{Y}} = -a(u^{\text{obs}}, \phi) - a(q^{\text{obs}}, \phi) - \nu(q^{\text{mod}}, \phi)_{\mathcal{Y}}, \quad \forall \phi \in \mathcal{Y},$$

and $(u_{M,\nu}, \psi_{M,\nu}) \in \mathcal{X} \times \mathcal{Y}_M$ satisfies

$$a(u_{M,\nu}, v) - (\psi_{M,\nu}, v)_{\mathcal{Y}} = \langle f, v \rangle_{\mathcal{Y}' \times \mathcal{Y}}, \quad \forall v \in \mathcal{Y},$$

$$-a(u_{M,\nu}, \phi) - \nu(\psi_{M,\nu}, \phi)_{\mathcal{Y}} = -a(u^{\text{obs}}, \phi), \quad \forall \phi \in \mathcal{Y}_M.$$

Now recall (P-O) $\equiv q^{\text{obs}} = 0 \Rightarrow$ *choose* $\nu = 0$.

Then $(u, \psi) \in \mathcal{X} \times \mathcal{Y}$ satisfies[†]

$$a(u, v) - (\psi, v)_{\mathcal{Y}} = \langle f, v \rangle_{\mathcal{Y}' \times \mathcal{Y}}, \quad \forall v \in \mathcal{Y},$$

$$-a(u, \phi) = -a(u^{\text{obs}}, \phi) - a(q^{\text{obs}}, \phi), \quad \forall \phi \in \mathcal{Y},$$

and $(u_{M,0}, \psi_{M,0}) \in \mathcal{X} \times \mathcal{Y}_M$ satisfies

$$a(u_{M,0}, v) - (\psi_{M,0}, v)_{\mathcal{Y}} = \langle f, v \rangle_{\mathcal{Y}' \times \mathcal{Y}}, \quad \forall v \in \mathcal{Y},$$

$$-a(u_{M,0}, \phi) = -a(u^{\text{obs}}, \phi), \quad \forall \phi \in \mathcal{Y}_M.$$

We henceforth suppress the subscript $\nu (= 0)$.

[†] This nonsymmetric saddle may be converted to a symmetric saddle.

Equivalently, $(u, \psi) \in \mathcal{X} \times \mathcal{Y}$ satisfies

$$\begin{aligned} (\mathbb{A}u, v)_{\mathcal{Y}} - (\psi, v)_{\mathcal{Y}} &= \langle f, v \rangle_{\mathcal{Y}' \times \mathcal{Y}}, \ \forall v \in \mathcal{Y}, \\ -(\mathbb{A}u, \phi)_{\mathcal{Y}} &= -(\mathbb{A}u^{\mathrm{obs}}, \phi)_{\mathcal{Y}}, \ \forall \phi \in \mathcal{Y}, \end{aligned}$$

and $(u_M, \psi_M) \in \mathcal{X} \times \mathcal{Y}_M$ satisfies

$$\begin{aligned} (\mathbb{A}u_M, v)_{\mathcal{Y}} - (\psi_M, v)_{\mathcal{Y}} &= \langle f, v \rangle_{\mathcal{Y}' \times \mathcal{Y}}, \ \forall v \in \mathcal{Y}, \\ -(\mathbb{A}u_M, \phi)_{\mathcal{Y}} &= -(\mathbb{A}u^{\mathrm{obs}}, \phi)_{\mathcal{Y}}, \ \forall \phi \in \mathcal{Y}_M. \end{aligned}$$

Recall $\mathbb{A} = Y^{-1}A$ such that

$$a(w, v) = \langle Aw, v \rangle_{\mathcal{Y}' \times \mathcal{Y}} = (\mathbb{A}w, v)_{\mathcal{Y}},$$

for any $w \in \mathcal{X}$, $v \in \mathcal{Y}$.

A Priori Estimates (P-O): Definitions

Define observation-constrained spaces

$$\mathcal{Y}_M^{\perp} \equiv \{w \in \mathcal{Y} \colon (w, v)_{\mathcal{Y}} = 0, \ \forall v \in \mathcal{Y}_M\}$$

$$\mathcal{X}_M^{\perp} \equiv \{w \in \mathcal{X} \colon \underbrace{(\mathbb{A}w, v)_{\mathcal{Y}}}_{a(w,v)} = 0, \ \forall v \in \mathcal{Y}_M\}$$

and an associated inf-sup constant† $\nu = 0$

$$\beta_{M,\nu=0} \equiv \inf_{w \in \mathcal{X}_M^{\perp}} \frac{\|\mathbb{A}w\|_{\mathcal{Y}}}{\|w\|_{\mathcal{X}}};$$

note

$$\beta_{M=0,\nu=0} = \beta_0 \equiv \inf_{w \in \mathcal{X}} \sup_{v \in \mathcal{Y}} \frac{\langle Aw, v\rangle_{\mathcal{Y}' \times \mathcal{Y}}}{\|w\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}} > 0.$$

† The norm $\|\cdot\|_{\mathcal{X}}$ can be replaced with a semi-norm $|\cdot|_{\mathcal{X}}$.

A Priori Estimates (P-O)

Proposition 2: The error $e \equiv u^{\text{true}} - u_M$ satisfies

$$\|\mathbb{A}e\|_{\mathcal{Y}} = \inf_{\phi \in \mathcal{Y}_M} \| \underbrace{q^{\text{mod}}}_{\text{model bias}} - \phi \|_{\mathcal{Y}}$$

or in $\mathcal{X}$-norm

$$\|e\|_{\mathcal{X}} \leq \frac{1}{\beta_{M,\nu=0}} \inf_{\phi \in \mathcal{Y}_M} \|q^{\text{mod}} - \phi\|_{\mathcal{Y}};$$

furthermore

$$\|q^{\text{mod}} - \psi_M\|_{\mathcal{Y}} = \inf_{\phi \in \mathcal{Y}_M} \|q^{\text{mod}} - \phi\|_{\mathcal{Y}}.$$

Recall $\nu = \nu^{\text{opt}} = 0$ (for $q^{\text{obs}} = 0$).

Proof of Proposition 2 (P-O): Preliminaries

Error Equation: The error $e \equiv u^{\mathrm{true}} - u_M$ satisfies

$$(\mathbb{A}e, v)_{\mathcal{Y}} - (q^{\mathrm{mod}} - \psi_M, v)_{\mathcal{Y}} = 0,\ \forall v \in \mathcal{Y}, \qquad \text{EQN1}$$

$$-(\mathbb{A}e, \phi)_{\mathcal{Y}} = 0,\ \forall \phi \in \mathcal{Y}_M, \qquad \text{EQN2}$$

since $\psi = q^{\mathrm{mod}}$.

Projection Operator: Recall $\Pi_M : \mathcal{Y} \to \mathcal{Y}_M$ ($\equiv \mathcal{E}_M^{\{\ell^{\mathrm{o}}\}}$)

$$(\Pi_M w, v)_{\mathcal{Y}} = (w, v)_{\mathcal{Y}},\ \forall v \in \mathcal{Y}_M, \text{ for any } w \in \mathcal{Y}.$$

Constrained Spaces: Recall

$$\mathcal{Y}_M^{\perp} = \{w \in \mathcal{Y} \colon (w, v)_{\mathcal{Y}} = 0,\ \forall v \in \mathcal{Y}_M\},$$

$$\mathcal{X}_M^{\perp} = \{w \in \mathcal{X} \colon \mathbb{A}w \in \mathcal{Y}_M^{\perp}\}.$$

Proof of Proposition 2 (P-O) . . .

To recover the first result

(a) from EQN2, $\mathbb{A}e \in \mathcal{Y}_M^{\perp}$, $e \in \mathcal{X}_M^{\perp}$;

(b) from EQN1 tested on $v \in \mathcal{Y}_M^{\perp} \subset \mathcal{Y}$,

$$(\mathbb{A}e, v)_{\mathcal{Y}} = (q^{\mathrm{mod}}, v)_{\mathcal{Y}},\ \forall v \in \mathcal{Y}_M^{\perp};$$

(c) from (a) and (b)

$$\mathbb{A}e = q^{\mathrm{mod}} - \Pi_M q^{\mathrm{mod}};$$

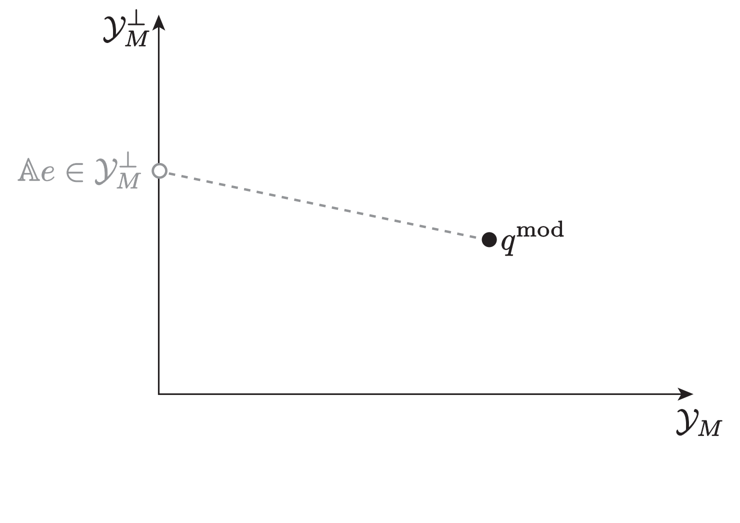
(d) from definition of projection

$$\|\mathbb{A}e\|_{\mathcal{Y}} = \inf_{\phi \in \mathcal{Y}_M} \|q^{\mathrm{mod}} - \phi\|_{\mathcal{Y}}.$$

Error in bias induces error in state.

. . . Proof of Proposition 2 (P-O) . . .

or, in a picture



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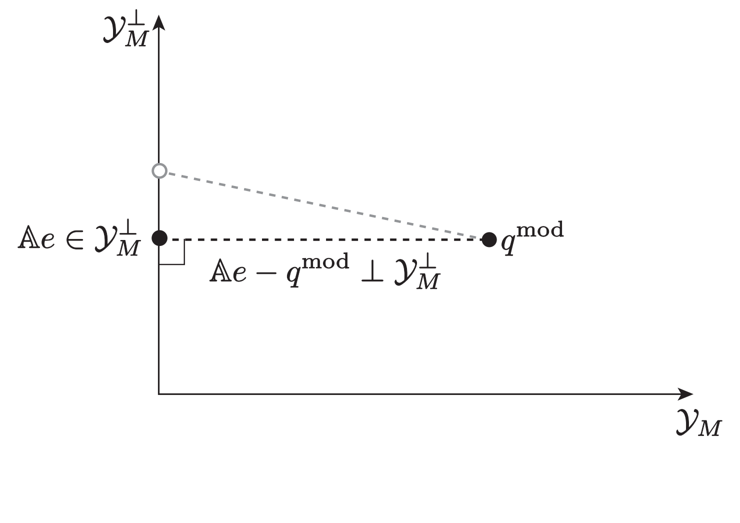

...Proof of Proposition 2 (P-O) ...

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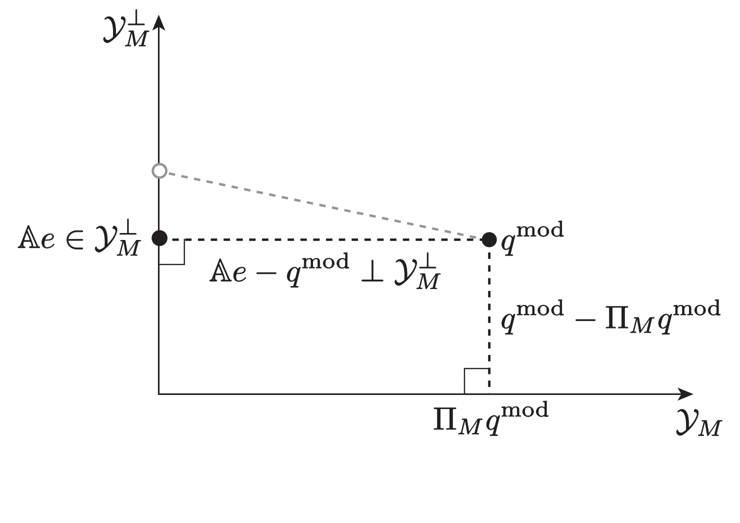

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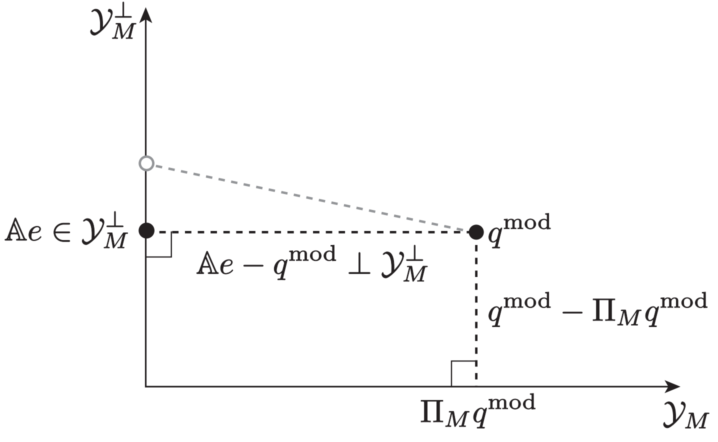

$$\|\mathbb{A}e\|_{\mathcal{Y}} = \|q^{\mathrm{mod}} - \Pi_M\, q^{\mathrm{mod}}\|_{\mathcal{Y}} = \inf_{\phi \in \mathcal{Y}_M} \|q^{\mathrm{mod}} - \phi\|_{\mathcal{Y}}$$

...Proof of Proposition 2 (P-O)

To recover the second result

(e) from (a), (d), and definition of stability constant[†]

$$\beta_{M,\nu=0} \equiv \inf_{w \in \mathcal{X}_M^\perp} \frac{\|\mathbb{A}w\|_\mathcal{Y}}{\|w\|_\mathcal{X}},$$

we obtain

$$\beta_{M,\nu=0}\|e\|_\mathcal{X} \le \inf_{\phi \in \mathcal{Y}_M} \|q^{\text{mod}} - \phi\|_\mathcal{Y}.$$

To recover the third result

(f) from EQN1 tested on $v \in \mathcal{Y}$

$$q^{\text{mod}} - \psi_M = \mathbb{A}e;$$

(g) from (c), $\psi_M = \Pi_M \psi$. □

[†]We may replace $\|w\|_\mathcal{X}$ with any desired semi-norm $|w|_\mathcal{X}$.

A Priori Estimates (P-O): Contributions

(Recall) Proposition 2:

$$\|u^{\text{true}} - u_M\|_{\mathcal{X}} \leq \frac{1}{\beta_{M,\nu=0}} \inf_{\phi \in \mathcal{Y}_M} \|q^{\text{mod}} - \phi\|_{\mathcal{Y}} \ .$$

The state error depends on $\mathcal{Y}_M \subset \mathcal{Y}_{M+1}$

1. the stability constant:

   $\beta_{M,\nu=0}$ (↑) as M (↑);

2. the model-bias best-fit error:

   $\inf_{\phi \in \mathcal{Y}_M} \|q^{\text{mod}} - \phi\|_{\mathcal{Y}}$ (↓) as M (↑).

Note implicit dependence on regularity of q^{mod}.

Output Error Estimation (P-O)

Proposition 3: for any $\ell^{\text{out}} \in \mathcal{X}$,

$$|\ell^{\text{out}}(e)| \leq \inf_{\zeta \in \mathcal{Y}_M} \|A^{-*}\ell^{\text{out}} - \zeta\|_{\mathcal{Y}} \inf_{\phi \in \mathcal{Y}_M} \|q^{\text{mod}} - \phi\|_{\mathcal{Y}}.$$

The output error depends on

1. the model-bias best-fit error:

$$\inf_{\phi \in \mathcal{Y}_M} \|q^{\text{mod}} - \phi\|_{\mathcal{Y}} \ (\downarrow) \text{ as } M \ (\uparrow);$$

2. the adjoint best-fit error:

$$\inf_{\zeta \in \mathcal{Y}_M} \|A^{-*}\ell^{\text{out}} - \zeta\|_{\mathcal{Y}} \ (\downarrow) \text{ as } M \ (\uparrow).$$

Note for $\ell^{\text{out}} = \ell^{\text{o}}_{\cdot}$, $|\ell^{\text{out}}(e)| = 0$.

MDWF Galerkin: Analysis

A Priori Estimates

- Perfect Observations (P-O) [QV]
- Imperfect Observations (I-O)

Stability Constant: Monotonic Improvement

Approximation Theory: Simple Example

Limit of Small Model Bias

Energy Norm (I-O)

Define an energy norm $\Pi_M : \mathcal{Y} \to \mathcal{Y}_M$

$$|||w|||_{M,\nu} \equiv \left(\|\mathbb{A}w\|^2_{\mathcal{Y}} + \nu^{-1}\|\Pi_M \mathbb{A}w\|^2_{\mathcal{Y}}\right)^{1/2}$$

and a stability constant or semi-norm $|w|_{\mathcal{X}}$

$$\beta_{M,\nu} \equiv \inf_{w \in \mathcal{X}} \frac{|||w|||_{M,\nu}}{\|w\|_{\mathcal{X}}};$$

note consistency in the $\nu \to 0$ limit,

$$\beta_{M,\nu\to 0} = \inf_{w \in \mathcal{X}} \frac{|||w|||_{M,\nu\to 0}}{\|w\|_{\mathcal{X}}} \to \inf_{w \in \mathcal{X}_M^{\perp}} \frac{\|\mathbb{A}w\|_{\mathcal{Y}}}{\|w\|_{\mathcal{X}}} \equiv \beta_{M,\nu=0}$$

since $\|\Pi_M \mathbb{A}w\|_{\mathcal{Y}}$ is penalized by $\nu^{-1} \to \infty$.

A Priori Estimate (I-O): Contributions

Proposition 4: The state estimate satisfies $\qquad \nu = \nu^{\text{opt}}$

$$\|u^{\text{true}} - u_{M,\nu}\|_{\mathcal{X}} \leq$$

$$\frac{1}{\beta_{M,\nu^{\text{opt}}}}\left(\inf_{\phi \in \mathcal{Y}_M} \|q^{\text{mod}} - \phi\|_{\mathcal{Y}}^2 + \underbrace{4\|q^{\text{mod}}\|_{\mathcal{Y}}\|\mathbb{A}q^{\text{obs}}\|_{\mathcal{Y}}}_{\text{dictates error as } M \to \infty} \right)^{\frac{1}{2}}.$$

The state error depends on

1. the stability constant: $\beta_{M,\nu^{\text{opt}}}$ (↑) as M (↑);
2. the model-bias best-fit error:

   $\inf_{\phi \in \mathcal{Y}_M} \|q^{\text{mod}} - \phi\|_{\mathcal{Y}}$ (↓) as M (↑);
3. the observation imperfection: $\|\mathbb{A}q^{\text{obs}}\|_{\mathcal{Y}}$.

Note $\|\mathbb{A}q^{\text{mod}}\|_{\mathcal{Y}}$ does not depend on M.

A Priori Estimate (I-O): Observations $\to$ Perfect

As $\|\mathbb{A} q^{\mathrm{obs}}\|_{\mathcal{Y}} / \|q^{\mathrm{mod}}\|_{\mathcal{Y}} \to 0$ (and $\nu^{\mathrm{opt}} \to 0$),

$$\|u^{\mathrm{true}} - u_{M,\nu=0}\|_{\mathcal{X}} \leq \frac{1}{\beta_{M,\nu=0}} \inf_{\phi \in \mathcal{Y}_M} \|q^{\mathrm{mod}} - \phi\|_{\mathcal{Y}};$$

if $\inf_{\phi \in \mathcal{Y}_M} \|q^{\mathrm{mod}} - \phi\|_{\mathcal{Y}} \to 0$ as $M \to \infty$ then

$$\|u^{\mathrm{true}} - u_{M,\nu=0}\|_{\mathcal{X}} \to 0 \text{ as } M \to \infty.$$

Note $\{A, f\}^{\mathrm{bk}}$ will still play role in convergence:

inf-sup constant, $\beta_{M,\nu=0}$;

magnitude and regularity of q^{mod};

experimentally observable spaces $\mathcal{Y}_M$.

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Proposition 5. The inf-sup constant

$$\beta_{M,0} \equiv \inf_{w \in \mathcal{X}_M^\perp} \frac{\|\mathbb{A}w\|_{\mathcal{Y}}}{\|w\|_{\mathcal{X}}}$$

is a *non-decreasing* function of M.

Proof. We note that

$$\mathcal{Y}_{M+1} \supset \mathcal{Y}_M \Rightarrow \mathcal{X}_{M+1}^\perp \subset \mathcal{X}_M^\perp;$$

more data implies greater stability. □

Consider a generalized SVD of $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$,

$$(\mathbb{A}\xi_j, \mathbb{A}v)_{\mathcal{Y}} = \sigma_j^2(\xi_j, v)_{\mathcal{X}},\ \forall v \in \mathcal{X},$$

$$\eta_j = \frac{1}{\sigma_j}\mathbb{A}\xi_j,$$

for

- singular values in $\mathbb{R}^+$: $\sigma_1 \leq \sigma_2 \leq \cdots$;
- trial singular functions in $\mathcal{X}$: $(\xi_m, \xi_n)_{\mathcal{X}} = \delta_{mn}$;
- test singular functions in $\mathcal{Y}$: $(\eta_m, \eta_n)_{\mathcal{Y}} = \delta_{mn}$.

Note $\sigma_1 = \beta_0 > 0$ (by assumption).

... Improvement in Stability: Ideal Case

Proposition 6: For P-O $\Rightarrow \nu = 0$,

the choice[†] $\quad \ell^{\rm o}_m = X\xi_m, \quad 1 \leq m \leq M,$

yields

$$\beta_{M,\nu=0} = \sigma_{M+1}\ :$$

the first M singular functions are "deflated."

Proof. Express inf-sup as Rayleigh quotient. □

[†]Note this choice yields non-experimentally observable spaces $\mathcal{Y}_M$, however an SVD Anti-Node computational heuristic will be extracted.

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Monotonicity: Statement (P-O,I-O)

Proposition 7: For *any* $\nu \in \mathbb{R}_0^+$, *hierarchical* $\mathcal{Y}_M$

$$\beta_{M',\nu} \geq \beta_{M'-1,\nu}, \quad M' = 1, \ldots, M;$$

and in particular

$$\beta_{M',\nu} \geq \beta_{M=0,\nu} = \beta_0, \quad M' = 1, \ldots, M,$$

where (recall)

$$\beta_0 \equiv \inf_{w \in \mathcal{X}} \sup_{v \in \mathcal{Y}} \frac{\langle Aw, v \rangle_{\mathcal{Y}' \times \mathcal{Y}}}{\|w\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}}$$

is the "no-data" inf-sup of the best knowledge model.

Proposition 5: Proof Sketch

Define the minimizer *fixed ν*

$$\chi_M \equiv \arg\min_{w \in \mathcal{X}} \frac{|||w|||_M^2}{\|w\|_{\mathcal{X}}^2};$$

for any $w \in \mathcal{X}$, $\|\Pi_{M'} \mathbb{A} w\|_{\mathcal{Y}}^2 \geq \|\Pi_{M'-1} \mathbb{A} w\|_{\mathcal{Y}}^2$, and hence

$$\beta_{M'}^2 \equiv \frac{|||\chi_{M'}|||_{M'}^2}{\|\chi_{M'}\|_{\mathcal{X}}^2} = \frac{\|\mathbb{A}\chi_{M'}\|_{\mathcal{Y}}^2 + \nu^{-1}\|\Pi_{M'}\mathbb{A}\chi_{M'}\|_{\mathcal{Y}}^2}{\|\chi_{M'}\|_{\mathcal{X}}^2}$$

$$\geq \frac{\|\mathbb{A}\chi_{M'}\|_{\mathcal{Y}}^2 + \nu^{-1}\|\Pi_{M'-1}\mathbb{A}\chi_{M'}\|_{\mathcal{Y}}^2}{\|\chi_{M'}\|_{\mathcal{X}}^2} = \frac{|||\chi_{M'}|||_{M'-1}^2}{\|\chi_{M'}\|_{\mathcal{X}}^2}$$

$$\geq \frac{|||\chi_{M'-1}|||_{M'-1}^2}{\|\chi_{M'-1}\|_{\mathcal{X}}^2} \equiv \beta_{M'-1}^2. \qquad \square$$

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One Space Dimension: Definitions

Consider

$\Omega =]0,1[$;

$\mathcal{X} = \mathcal{Y} = H_0^1(\Omega)$;

$\|\cdot\|_{\mathcal{X}} = \|\cdot\|_{\mathcal{Y}} = |\cdot|_{H^1(\Omega)}$;

$A = Y$;

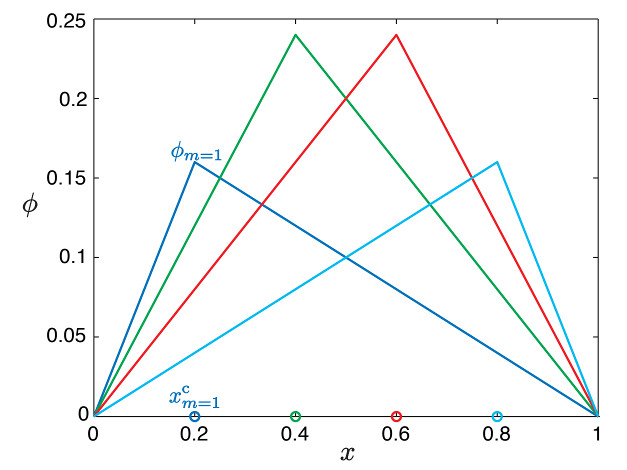
point-wise observation functionals ;

$$\ell_m^{\mathrm{o}} \equiv \delta(\cdot; x_m^{\mathrm{c}}), \quad m = 1, \ldots, M \; ;$$

uniformly spaced observation centers $\{x_m^{\mathrm{c}}\}_{m=1}^M$.

Note ℓ_m^{o} bounded for $\Omega \subset \mathbb{R}^1$.

One Space Dimension: Experimentally Observable Spaces $\mathcal{Y}_M$



Native basis functions for $\mathcal{Y}_M$: $\phi_m = A^{-*}\ell^{\mathrm{o}}_m$, $1 \le m \le M$.

One Space Dimension: Model Bias Approximation

Proposition 8: For

$$q^{\mathrm{mod}} \in H^2(\Omega),$$

we obtain

$$\inf_{\phi \in \mathcal{Y}_M} \|q^{\mathrm{mod}} - \phi\|_{\mathcal{Y}} \leq C M^{-1} \|q^{\mathrm{mod}}\|_{H^2(\Omega)}$$

for C independent of M and q^{mod}.

Proof. Note $\mathcal{Y}_M \equiv \mathrm{span}\{A^{-*}\ell^{\mathrm{o}}_m\}_{m=1}^M$ is equivalent to

$$\{\text{piecewise linear polynomials over nodes } \{x^{\mathrm{c}}_m\}_{m=1}^M\};$$

now apply standard approximation theory.† □

† Note that $q^{\mathrm{mod}} = \mathcal{Y}^{-1} g$ vanishes at end points of Ω.

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Strategy: Stabilization

Recall *a priori* estimate:

$$\|u^{\text{true}} - u_{M,\nu}\|_{\mathcal{X}} \leq \frac{1}{\beta_{M,\nu^{\text{opt}}}} \Big(\inf_{\phi \in \mathcal{Y}_M} \|q^{\text{mod}} - \phi\|_{\mathcal{Y}}^2 + \underbrace{4\|q^{\text{mod}}\|_{\mathcal{Y}} \|\mathbb{A} q^{\text{obs}}\|_{\mathcal{Y}}}_{\text{dictates error as } M \to \infty} \Big)^{\frac{1}{2}};$$

if $\|q^{\text{mod}}\|_{\mathcal{Y}} \ll \|u^{\text{true}}\|_{\mathcal{X}}$, $\|\mathbb{A} q^{\text{obs}}\|_{\mathcal{Y}} \ll \|u^{\text{true}}\|_{\mathcal{X}}$,

we can control the state error

(solely) by improvements in $\beta_{M,\nu^{\text{opt}}}$.

Best conditions: few small singular values.

"Optimal" Spaces for Stabilization (P-O)

Goal: deflate dangerous modes (E-optimality [FM]),
but remain experimentally observable.

Algorithm SVD Anti-Node: compute

$$w_{\min} \equiv \arg \inf_{w \in \mathcal{X}_M^\perp} \frac{\|\mathbb{A} w\|_{\mathcal{Y}}}{\|w\|_{\mathcal{X}}} \;;$$

locate the most sensitive observation point

$$x^{\mathrm{c}}_{M+1} = \arg \sup_{x \in \Omega} |w_{\min}(x)| \;;$$

and set

$$\mathcal{Y}_{M+1} = \mathcal{Y}_M \oplus (A^{-*}\mathrm{Gauss}(\,\cdot\,; x^{\mathrm{c}}_{M+1}, \sigma)) \;;$$

observe least stable mode.

"Optimal" Spaces for Stabilization (I-O)

Goal: deflate dangerous modes (E-optimality [FM]),
but remain experimentally observable.

Algorithm SVD Anti-Node: compute

$$w_{\min} \equiv \arg\inf_{w \in \mathcal{X}} \frac{|||w|||_{M,\nu}}{\|w\|_{\mathcal{X}}} \;;$$

locate the most sensitive observation point

$$x^{\mathrm{c}}_{M+1} = \arg\sup_{x \in \Omega} |w_{\min}(x)| \;;$$

and set

$$\mathcal{Y}_{M+1} = \mathcal{Y}_M \oplus (A^{-*}\mathrm{Gauss}(\,\cdot\,; x^{\mathrm{c}}_{M+1}, \sigma)) \;;$$

observe least stable mode.

MDWF: Petrov-Galerkin Approach (in brief)

Motivation
Formulation
Analysis

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Conundrum

For *MDWF formulation*, in data equation

$$-a(u_{M,\nu},\phi) - \nu(\psi_{M,\nu},\phi)_{\mathcal{Y}} = \overbrace{-a(u^{\text{obs}},\phi)}^{\text{projection of observable field: } (\mathbb{A}u^{\text{obs}},\phi)_{\mathcal{Y}}}, \quad \forall \phi \in \mathcal{Y}_M,$$

we need $\mathcal{Y}_M \equiv \mathcal{E}_M^{\{\ell^{\text{o}}\}}$ — *experimentally observable*.

For *Galerkin*, in model equation

$$a(u_{M,\nu},v) - (\psi_{M,\nu},v)_{\mathcal{Y}} = \langle f, v\rangle_{\mathcal{Y}'\times\mathcal{Y}}, \quad \forall v \in \mathcal{Y},$$

we need $\psi_{M,\nu} \in \mathcal{Y}_M^{\text{trial=test}} = \mathcal{Y}_M = \mathcal{E}_M^{\{\ell^{\text{o}}\}}$.

Unfortunately, $\mathcal{Y}_M = \mathcal{E}_M^{\{\ell^{\text{o}}\}}$ provides

good stability: small model bias ✓,

but *poor* approximation: large model bias ✗.

MDWF: Petrov-Galerkin Approach (in brief)

Motivation

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Unlimited Observations

Given $\nu \in \mathbb{R}_{+,0}$, find $(u, \psi) \in \mathcal{X} \times \mathcal{Y}$ such that

$$\begin{aligned} a(u,v) - (\psi, v)_{\mathcal{Y}} &= \langle f, v \rangle_{\mathcal{Y}' \times \mathcal{Y}}, && \forall v \in \mathcal{Y}, \\ -a(u,\phi) - \nu(\psi,\phi)_{\mathcal{Y}} &= - a(u^{\mathrm{obs}}, \phi) - a(q^{\mathrm{obs}}, \phi) \\ &\quad - \nu(q^{\mathrm{mod}}, \phi)_{\mathcal{Y}}, && \forall \phi \in \mathcal{Y}. \end{aligned}$$

The unlimited-observations formulation is unchanged from the case of Galerkin approximation.

Limited Observations

Given $\nu \in \mathbb{R}_{+,0}$, find $(u_{M,\nu}, \psi_{M,\nu}) \in \mathcal{X} \times \mathcal{Y}_M^{\text{trial}}$

$$a(u_{M,\nu}, v) - (\psi_{M,\nu}, v)_{\mathcal{Y}} = \langle f, v \rangle_{\mathcal{Y}' \times \mathcal{Y}}, \qquad \forall v \in \mathcal{Y},$$

$$-a(u_{M,\nu}, \phi) - \nu(\psi_{M,\nu}, \phi)_{\mathcal{Y}} = -a(u^{\text{obs}}, \phi), \quad \forall \phi \in \mathcal{Y}_M^{\text{test}};$$

for

$\mathcal{Y}_M^{\text{test}} \equiv \mathcal{E}_M^{\{\ell^{\text{o}}\}}$ experimentally observable

but

$\mathcal{Y}_M^{\text{trial}}$ informed only by approximation considerations.

Strategy [DG,DPW]: choose $\{\ell^{\text{o}}\}$ to
maximize stability from SVD anti-node considerations.