# Timelike extremal surfaces in the Minkowski spacetime

Matteo Novaga

University of Pisa

The evolution of a (classical) relativistic string is modelled by the hyperbolic equation

$$\mathrm{a} = (1 - \mathrm{v}^2)\kappa \tag{1}$$

which is known as Born-Infeld equation, in the case of graphs.

Equation (1) also corresponds to the minimal surface equation for timelike submanifolds in the Minkowski spacetime $\mathbb{R}^{1+n}$.

Equation (1) has been considered by many authors, and short-time existence has been established by T. Deck when $n = 2$ (1994) and O. Milbredt (2008) in the general case.

The issue of global existence is more delicate, since the equation develops singularities in finite time.

When the initial datum is almost flat, global existence has been proved by H. Lindblad (2004) and S. Brendle (2002).

# Area functional

We denote by $\mathbb{R}^{1+n}$ the Minkowski spacetime with inner product

$$\langle x, y \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_n y_n.$$

Given a smoothly immeresed surface $\Sigma = \phi(\Omega)$, with $\Omega \subset \mathbb{R}^h$, we let

$$\mathcal{A}(\Sigma) = \int_\Omega \sqrt{|det\, g|}\, du_1 \cdots du_h$$

where $g_{ij} = \langle \phi_i, \phi_j \rangle$.

The surface $\Sigma$ is timelike if $det\, g < 0$.

One can equivalently write

$$\mathcal{A}(\Sigma) = \int_{\Sigma} \#\{\phi^{-1}(x)\} \sqrt{|\langle \nu, \nu \rangle|}\, d\mathcal{H}^n(x)$$

where $\nu$ is a suitable unit normal vector to $\Sigma$. The timelike condition reads $\langle \nu, \nu \rangle > 0$, that is, $\nu$ is a spacelike vector.

The Euler-Lagrange equation of $\mathcal{A}$ is given by

$$\kappa_m = \frac{1}{\sqrt{|det\, g|}} \operatorname{div} \left( \sqrt{|det\, g|}\, g^{ij} \phi_j \right) = 0$$

which is equivalent to (1).

# Approximation with semilinear PDEs

$$\int \epsilon |Du|^2 + \frac{W(u)}{\epsilon} \to c_W \int |Du|$$

$$\text{(DeGiorgi, Modica} - \text{Mortola)}$$

$$-\Delta u + \frac{1}{\epsilon^2} W'(u) = 0 \to \kappa = 0$$

$$\text{(Hutchinson} - \text{Tonegawa, Pacard} - \text{Ritore}' \ldots)$$

$$u_t = \Delta u - \frac{1}{\epsilon^2} W'(u) \to v = \kappa$$

$$\text{(deMottoni} - \text{Schatzman, Barles} - \text{Soner} - \text{Souganidis, Ilmanen} \ldots)$$

As in the elliptic and parabolic setting, one may try to approximate equation (1) by means of the semilinear wave equation

$$u_{tt} - \Delta u + \frac{1}{\epsilon^2} W'(u) = 0 \tag{2}$$

where $u : \mathbb{R}^{1+n} \to \mathbb{R}^k$, with $k \in \{1, 2\}$, and $W$ is a double–well potential.

A formal asymptotic analysis which confirms such expectation has been performed by J. Neu (1990).

The singular perturbation limit has been studied by R. Jerrard and F.H. Lin (1999) when $n = k = 2$ (vortices).

In a recent work (2011) R. Jerrard establishes convergence for small time of solutions of (2) to minimal surfaces, in the case of well–prepared initial data.

Understanding convergence after the singularities and for more general initial data seems to be a difficult task.

It is possible to associate to a solution $u_\epsilon$ of (2) a stationary varifold $V_\epsilon$ which is expected to concentrate in the limit on a minimal surface (and it does it for well–prepared initial data, as proved by R. Jerrard).

Under suitable assumptions on the density and the tangent space of the limit varifold $V = \lim_\epsilon V_\epsilon$, one can show (Bellettini, N., Orlandi, 2010) that $V$ is indeed a stationary rectifiable lorentzian varifold.

This result is the analog of a result by L. Ambrosio and H.M. Soner in the parabolic setting, that is for the mean curvature flow.

# Generalized minimal surfaces

A Radon measure $V$ on $\mathbb{R}^{1+n} \times G_{m,n+1}$ is a rectifiable lorentzian varifold of dimension $m$ if

$$V = \theta \sigma^m \llcorner \Sigma \otimes \delta_{P_\Sigma} \qquad \theta > 0$$

where $G_{m,n+1}$ is the Grassmannian of $m$-dimensional planes in $\mathbb{R}^{1+n}$, $\sigma^m$ is the lorentzian $m$-dimensional surface measure, $\Sigma$ is a $m$-dimensional rectifiable set whose tangent space $T_x\Sigma$ is timelike almost everywhere, and $P_\Sigma$ is the lorentzian orthogonal projection onto $T_x\Sigma$.

The function $\theta \in L^1(\Sigma, \sigma^m)$ is called multiplicity of the varifold $V$.

Problem: the Grassmannian $G_{m,n+1}$ is not compact!

# Stationarity

The varifold $V$ is stationary if

$$\int_\Sigma \mathrm{tr}\,(P_\Sigma \nabla X)\,\theta\, d\sigma^m \;=\; 0$$

for all $X \in \left(C_c^1(\mathbb{R}^{1+n})\right)^{1+n}$.

When $\Sigma$ is smooth the stationarity condition implies that $\Sigma$ is a minimal submanifold, and $\theta$ is constant on $\Sigma$.

# Energy density and stress-energy tensor

$$
\begin{aligned}
e_\epsilon &= c_m(\epsilon)\left(\frac{|u_{\epsilon t}|^2 + |\nabla u_\epsilon|^2}{2} + \frac{W(u_\epsilon)}{\epsilon^2}\right) \\
\ell_\epsilon &= c_m(\epsilon)\left(\frac{-|u_{\epsilon t}|^2 + |\nabla u_\epsilon|^2}{2} + \frac{W(u_\epsilon)}{\epsilon^2}\right) \\
T_\epsilon^{\alpha\beta} &= -c_m(\epsilon)\eta^{\alpha\gamma}\partial_{x^\gamma} u_\epsilon \cdot \eta^{\beta\delta}\partial_{x^\delta} u_\epsilon + \ell_\epsilon\, \eta^{\alpha\beta}
\end{aligned}
$$

where $c_n(\epsilon) = \epsilon$, $c_{n-1}(\epsilon) = |\log \epsilon|^{-1}$, and $\eta^{\alpha\beta}$ is the standard metric tensor in Minkowski spacetime.

# Assumption on initial data

We assume that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^n} e_\epsilon(0, x)\, dx \le C$$

for all $\epsilon$.

Notice that

$$\int_{\mathbb{R}^n} e_\epsilon(t, x)\, dx = \int_{\mathbb{R}^n} e_\epsilon(0, x)\, dx \qquad \text{(conservation of energy)}$$

for all $t > 0$.

Under this assumption one gets

$$e_\epsilon \, dt \, dx \rightharpoonup e$$

$$\ell_\epsilon \, dt \, dx \rightharpoonup \ell$$

$$c_m(\epsilon) \, \frac{W(u_\epsilon)}{\epsilon^2} \rightharpoonup w$$

$$T_\epsilon \, dt \, dx \rightharpoonup T.$$

# Assumptions on the limit varifold

(A1) For $\ell$-a.e. $(t,x)$ it holds

$$0 < \lim_{\rho \to 0} \frac{\ell(B_\rho(t,x))}{\rho^m} < +\infty\,.$$

This ensures that the lagrangian $\ell$ concentrates on a rectifiable set $\Sigma$ of codimension $k$, that is, $\ell = \theta\sigma^m \llcorner \Sigma$.

(A2) $\xi - \dfrac{\mathrm{d}T}{\mathrm{d}\ell}\,\xi$ is spacelike for all $\xi \neq 0$.

This assumption implies that the measure $T$ takes values in the lorentzian orthogonal projections onto timelike spaces.

(A3) When $m = n$ (codimension one case) we also assume

$$\frac{d\,w}{d\,\ell} = \frac{1}{2}\,.$$

This is the so-called equipartition of energy, which holds true in the elliptic and parabolic setting (see T. Ilmanen), but could fail in the hyperbolic case.

# Stationarity of the limit varifold

Under assumptions (A1)–(A3), the limit varifold

$$V = \theta\sigma^m \llcorner \Sigma \otimes \delta_{P_\Sigma}$$

is a stationary rectifiable varifold.

Differently from the elliptic and the parabolic case, the result cannot be true without any assumption, since in general the limit varifold $V$ is not rectifiable, as rectifiable varifolds are not closed under varifold convergence.

# The case of strings

When $m = 2$ a minimal surface, which we also call a relativistic string, can be (locally) parametrized by a function $\gamma : [0, T] \times [0, L] \to \mathbb{R}^{1+n}$ solving the linear wave equation

$$\gamma_{tt} = \gamma_{xx},$$

with constraints

$$\gamma_t \cdot \gamma_x = 0 \qquad |\gamma_t|^2 + |\gamma_x|^2 = 1.$$

In particular, one has the explicit solution

$$\gamma(t, x) = \frac{a(x+t) + b(x-t)}{2},$$

where $a, b : [0, L] \to \mathbb{R}^n$ satisfy $|a'| = |b'| = 1$.

# Wiggly strings

This representation shows that minimal surfaces are not closed under Hausdorff convergence.
Indeed, there are sequences $a_\epsilon \to a$ and $b_\epsilon \to b$ such that $|a'| \leq 1$, $|b'| \leq 1$, but the equality does not hold, so that

$$\gamma_\epsilon(t,x) = \frac{a_\epsilon(x+t) + b_\epsilon(x-t)}{2} \to \gamma(t,x) = \frac{a(x+t) + b(x-t)}{2}$$

where $\gamma$ still satisfies the wave equation but not the minimal surface equation.

Such phenomenon is known in the physical literature under the name of wiggly strings (see Vilenkin & Shellard 1994).
In particular, it shows that compactness of stationary rectifiable varifolds fails in general.

# A closure result

We call a wiggly string a surface which can be parametrized by a function $\gamma$ solving the linear wave equation and such that $|a'| \leq 1$, $|b'| \leq 1$.

Theorem [Bellettini, Hoppe, N., Orlandi]

Wiggly strings are the closure of the relativistic strings under Hausdorff convergence.

An similar result has been obtained by Y. Brenier in the case of graphs, using a different parametrization.

# Singularities

The formation of singularities is discussed in Vilenkin & Shellard (and further analyzed in recent work by J. Eggers and J. Hoppe) where it is shown that a singularity is generically a cusp, with the asymptotic profile $y \sim x^{\frac{2}{3}}$ in graph coordinates.

The previous representation can be used to define weak solutions to the minimal surface equation, after the onset of singularities.

It is controversial if such solutions are reasonable from a physical point of view (selfintersections, persistence of cusps).

# Convex strings

Cusps do not develop in the evolution of centrally symmetric convex strings.

Theorem [Bellettini, Hoppe, N., Orlandi]
Assume that the initial string is a centrally symmetric convex curve, with zero velocity. Then the evolution remains convex and encounters an extinction singularity (collapse). Moreover, if the initial curve is smooth, the limit shape is a round circle.

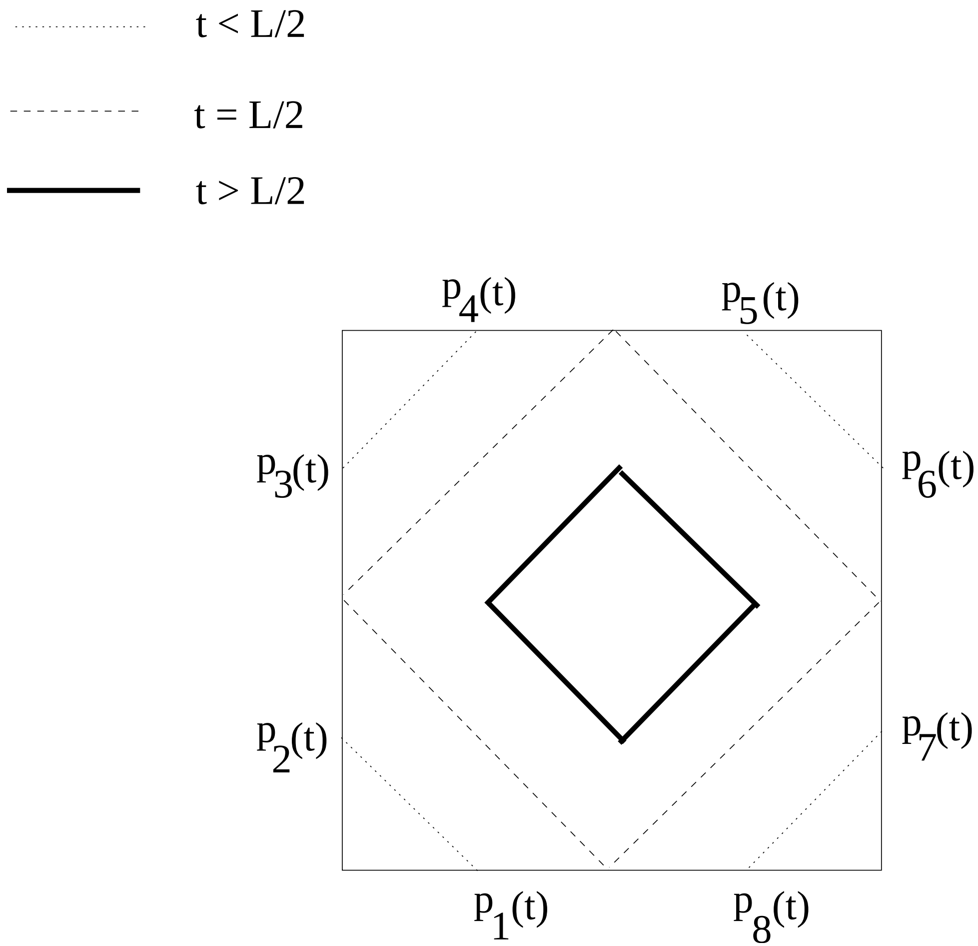
Example: $\gamma(t,x) = R(\cos(x/R), \sin(x/R))\cos(t/R)$ (Kink).

## Remarks

– A similar result has been established by D. Kong, L. Kefeng and Z. Wang for the hyperbolic curvature flow of planar curves:

$$a = k\,.$$

– If the initial convex curve is not smooth, for ex. a square, then the limit shape is not necessarily a circle.




# Figure 1: an evolving square.

# Conservation of energy

The quantity

$$\int_{\Sigma(t)} \frac{1}{\sqrt{1-\mathrm{v}^2}}\, d\mathcal{H}^n \qquad (energy)$$

is conserved along (smooth) solutions to (1).

The evolution of the square is somewhat singular since it dissipates this quantity. One may argue that there is a concentration of energy at the corners of the evolving square.

One can suitably extend the definition of rectifiable varifold in order to take into account wiggly strings (weakly rectifiable varifolds).

# Dimension of the singular set

In the case of strings, one can estimate the Hausdorff dimension of the singular set $\mathrm{Sing}(\gamma) \subset \mathbb{R}^{1+n}$ (Nguyen-Tian 2012, Jerrard-N.-Orlandi 2013):

1. If $\gamma_0 \in C^k$ and $(\gamma_0)_x \in C^{k-1}$, then $\dim_{\mathcal{H}} \mathrm{Sing}(\gamma) \leq 1 + \frac{1}{k}$ and $\mathcal{H}^{1+\frac{1}{k}}(\mathrm{Sing}(\gamma)) = 0$, with optimal bound.

2. If $n = 2$ and $\gamma_0$ is closed, either $\gamma_x(t_0, x) \equiv 0$ (collapse) for some $t_0 > 0$, or $\mathrm{Sing}(\gamma)$ contains an interval.

# Generic regularity

The string $\gamma$ is regular if

$$a'(s) + b'(\sigma) \neq 0$$

for all $s, \sigma \in [0, L]$.

In particular,

1. if $n = 2$ then $\gamma$ is never regular;
2. if $n > 3$ then $\gamma$ is generically regular;
3. if $n = 3$ and $\gamma$ regular, then $\tilde{\gamma}$ is also regular provided $\tilde{\gamma}_0$ is sufficientlyclose to $\gamma_0$.

# Open questions

1. Pass to the limit as $\epsilon \to 0$ in semilinear wave equation (2).
2. Which stationary rectifiable varifolds can be obtained as limits of solutions to the (2)?
3. Dimension of the singular set for stationary rectifiable varifolds.