Décroissance polynômiale en vitesse et théorème H exponentiel pour l'équation de Boltzmann

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21 décembre 2012

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Plan

Introduction

The factorization method

The Boltzmann linearized semigroup

Non-symmetric nonlinear energy methods

The Boltzmann equation (Maxwell 1867, Boltzmann 1872)

$$\underbrace{\partial_t f}_{\text{time change}} + \underbrace{v \cdot \nabla_x f}_{\text{space change}} = \underbrace{Q(f,f)}_{\text{collision operator}} \qquad \text{on } f(t,x,v) \geq 0$$

- Partial differential equation
- Transport term $v \cdot \nabla_x$: straight line along velocity v
- Collision operator $Q(f,f)$: bilinear, acting on v only, integral

$$Q(f,f)(v) = \int_{v_*} \int_{collisions} \Big[\underbrace{f(v')f(v'_*)}_{(v',v'_*)\to(v,v_*)} - \underbrace{f(v)f(v_*)}_{(v,v_*)\to\dots} \Big] B$$

- Balance-sheet of particles with velocity v due to collisions

Structure of the Boltzmann equation (I)

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f)$$

- $Q(f, f)$ bilinear integral operator acting on v only (local in t and x), representing interactions between particles:

$$Q(f,f)(v) := \int_{v_* \in \mathbb{R}^3} \int_{\omega \in \mathbb{S}^2} \Big[\underbrace{f(v_*')f(v')}_{\text{“appearing”}} - \underbrace{f(v)f(v_*)}_{\text{“disappearing”}} \Big] \underbrace{B(v - v_*, \omega)}_{\text{collision kernel } (\geq 0)} \, \mathrm{d}\omega \, \mathrm{d}v_*$$

- Velocity collision rule ($(d-1)$ free parameters $\to \omega$):

$$v' := v - (v - v_*, \omega)\omega, \qquad v_*' := v_* + (v - v_*, \omega)\omega$$

- One has (microscopic conservation laws)

$$v' + v_*' = v + v_*, \qquad |v_*'|^2 + |v'|^2 = |v|^2 + |v_*|^2$$

Structure of the Boltzmann equation (II)

- For $\omega \in \mathbb{S}^{d-1}$ the map $(v, v_*) \mapsto (v', v'_*)$ has Jacobian -1
- We deduce for a test function $\varphi(v)$

$$\int_{\mathbb{R}^d} Q(f,f)\varphi(v)\,\mathrm{d}v$$
$$= \frac{1}{4}\int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} [f'f'_* - ff_*] B(v-v_*,\omega)(\varphi+\varphi_*-\varphi'-\varphi'_*)\,\mathrm{d}\omega\,\mathrm{d}v_*\,\mathrm{d}v$$

- Choosing correctly φ we deduce

$$\int_{\mathbb{R}^d} Q(f,f)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix}\mathrm{d}v = 0$$

- This implies formally

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} f \begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix}\mathrm{d}x\,\mathrm{d}v = 0$$

Structure of the Boltzmann equation (III)

- Choosing $\varphi = \log f$ we obtain the *H*-theorem
$$\frac{\mathrm{d}}{\mathrm{d}t} H(f) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^{2d}} f \log f \, \mathrm{d}x \, \mathrm{d}v = -D(f) \leq 0$$

- The entropy production is
$$D(f) = -\int_{\mathbb{R}^{2d}} Q(f,f) \log f \, \mathrm{d}x \, \mathrm{d}v$$
$$= \int_{\mathbb{R}^{2d} \times \mathbb{S}^{d-1}} [f' f'_* - f f_*] \log \frac{f' f'_*}{f f_*} B(v - v_*, \omega) \, \mathrm{d}x \, \mathrm{d}v \geq 0$$

- Cancellation only at Maxwellian local equilibrium satisfying $f f_* = f' f'_*$ everywhere
$$M_f = \frac{\rho}{(2\pi T)^{d/2}} e^{-|v-u|^2/2T}, \quad \rho \geq 0, \; u \in \mathbb{R}^d, \; T > 0$$

- Time-irreversible equation and mathematical basis for studying relaxation to equilibrium (2-d law of thermodynamic)

Collision kernel from physics

- Inverse power-law interaction potentials and hard spheres:

$$B = \Phi(|v - v_*|)\, b(\cos\theta), \quad \Phi(z) = z^\gamma, \quad b \sim_{\theta\sim 0} b_0\, \theta^{-(d-1)-\alpha}$$

  with $\gamma \in (-d, +\infty)$ and $\alpha \in (-\infty, 2)$ (in σ-representation)

- Computations in dimension $d = 3$:

$$\text{Potential } \phi(r) = r^{-(s-1)} \;:\; \gamma = \frac{s-5}{s-1},\; \alpha = \frac{2}{s-1}, \quad s > 2$$

  Hard potentials $s > 5$ (hard spheres roughly $s \to \infty$)
  Maxwell molecules $s = 5$
  Soft potentials $s \in (2,5)$ (Coulomb $s \to 2$)

- Intricate integral operator whose kernel have singularities (when $\alpha \geq 0$) and polynomial growth or decay (cf. γ).

- Comparison with a differential operator (Bobylev):
  $\alpha =$ order of the fractional derivative
  $\gamma =$ order of polynomial growth or decay of coefficients

Exponential H-theorem and Cercignani's conjecture (I)

Quantify H-Theorem for the Boltzmann equation

Old question... Truesdell and Muncaster 1980:

"*Much effort has been spent toward proof that place-dependent solutions exist for all time. [...]* ***The main problem is really to discover and specify the circumstances that give rise to solutions which persist forever****. Only after having done that can we expect to construct proofs that such solutions exist, are unique, and are regular.*"

Mathematically this amounts to prove for a priori smooth solutions

$$H(f_t|M) = \int_{\mathbb{T}^d \times \mathbb{R}^d} f_t \log \frac{f_t}{M} \, \mathrm{d}x \, \mathrm{d}v \xrightarrow{t \to +\infty} 0$$

with the correct timescale (hence requires constructive proofs)

Exponential H-theorem and Cercignani's conjecture (II)

- Cercignani's conjecture 1982

> "*The present contribution is intended as a step toward the solution of the first main problem of kinetic theory, as defined by Truesdell and Muncaster, i.e. 'to discover and specify the circumstances that give rise to solutions which persist forever'.*"

- Linearized semigroup (Ukai'1974) suggests exponential rate
- Conjecture: Linear inequality on the entropy production

$$D(f) = -\frac{\mathrm{d}H(f|M)}{\mathrm{d}t} \geq \lambda H(f|M), \quad \lambda > 0$$

- Kind of nonlinear spectral gap, cf. log-Sobolev inequalities...
- False (Bobylev-Cercignani, Wennberg) for physical B
- Almost true Villani: $D(f) \geq \lambda_\varepsilon H(f|M)^{1+\varepsilon}$ with smoothness and moments... however it gives polynomial time-scales

Exponential H-theorem and Cercignani's conjecture (III)

- Hence one has to turn to the study of semigroup decay properties (*vs.* functional inequality): importance of the Cauchy theory and the natural space for it
- Incompatibility of functional spaces: linearized study $L^2(M^{-1})$ and nonlinear evolution equation $L^1(poly)$ (see later)
- First non-constructive proof of exponential decay in physical space Arkeryd-Esposito-Pulvirenti'87, Arkeryd'88: although the proof can be filled to my opinion, never been used and remained debatted...
- Constructive proof Desvillettes-Villani'05 with polynomial rates for a priori smooth solutions
- Constructive proof CM'06 in the spatially homogeneous case with sharp exponential rate: connection of entropy production estimates with new quantitative linearized estimates

Review on the Cauchy problem (I)

- HILBERT'1909: compactness collision operator
- CARLEMAN'1933: spatially homogeneous solutions in L^∞ with polynomial moments
- CARLEMAN'1957: spectral gap for the collision
- GRAD'1950S-60S: extension
- WANG-CHANG-UHLENBECK'1950S: eigenpairs for Maxwell molecules
- BOBYLEV'1970S: Fourier methods for Maxwell molecules
- UKAI'1974: perturbative smooth solutions for hard spheres in $L^2(M^{-1})$ in the torus, then in the whole space
- ARKERYD 70S-80S: L^1 Cauchy theory in velocity only

Review on the Cauchy problem (II)

- Illner-Shinbrot'1984: Close-to-vacuum solutions
- DiPerna-Lions'1989: renormalized solutions for cutoff interactions (cf. Leray solutions for Navier-Stokes)
- Mischler-Wennberg, Lu'99: optimal Cauchy theory in velocity only (spatially hom. hard spheres)
- Alexandre-Villani'2000s: renormalized sol. non-cutoff
- Baranger-CM'2005: bounds spectral gap collision operator for hard spheres
- CM'2006: explicit coercivity for cutoff collisions
- CM-Strain'2007: explicit spectral gap for non-cutoff collisions
- Gressman-Strain, AMUXY'2011: smooth perturbative solutions in the non-cutoff case

Moment estimates in the spatially homogeneous Boltzmann equation (I)

- Ikenberry-Truesdell'1956: closed eqs for moments (Maxwell molecules)
- Elmroth'1983: polynomial moments bounds (hard pot.)

$$\int_{\mathbb{R}^d} f(t,v)|v|^k \, \mathrm{d}v$$

using inequalities from Povzner'1962 on

$$\int_{\mathbb{S}^{d-1}} \left(|v'|^k + |v'_*|^k - |v|^k - |v_*|^k \right) \, b\left(\frac{(v - v_*) \cdot \sigma}{|v - v_*|} \right) \, \mathrm{d}\sigma$$

$$\leq C(|v||v_*|^{k-1} + |v_*||v|^{k-1}) - K(|v|^k + |v_*|^k)$$

- Desvillettes'1993, Wennberg'1994: appearance polynomial moments (hard pot.), w/ or w/o cutoff

Moment estimates in the spatially homogeneous Boltzmann equation (II)

- BOBYLEV'1996, BOBYLEV-GAMBA-PANFEROV'2004: propagation of L^1 exponential moments

$$\int_{\mathbb{R}^d} f(t,v) e^{a|v|^s} \, \mathrm{d}v, \quad s \in (0,2], \quad a << 1$$

- MISCHLER-CM'2006: appearance of exponential moments for cutoff hard potentials with $s \leq \gamma/2$
- GAMBA-PANFEROV-VILLANI'2009: L^∞ gaussian bounds
- ALONSO-CAÑIZO-GAMBA-CM'2012: propagation + appearance for optimal $s = \gamma$ with new simplified approach
- No results so far show preservation of the $L^2(M^{-1})$ bound, even at spatially homogeneous level (counter-exple BOBYLEV'1988 for cutoff Maxwell molecules):
- The space of symmetry obtained from linearized entropy seems **not** natural at nonlinear level

The main Cauchy results

Main Cauchy result

Boltzmann equation for hard spheres in the torus $x \in \mathbb{T}^d$.
• It is locally well-posed around its global gaussian equilibrium $M = M(v) = Ce^{-|v|^2/2}$ in the space $L^1_v L^\infty_x(1+|v|^k)$ for $k > 2$.
• It is well-posed for weakly inhomogeneous initial data, in the sense: f_{in} is close in $L^1_v L^\infty_x(1+|v|^k)$ to $g_{in} = g_{in}(v)$, where the smallness condition depends on $\|g_{in}\|_{L^1_4}$.
• Constructive constants and rate of convergence $f_t \to M$ in $L^1_v L^\infty_x(1+|v|^k)$ given by $Ce^{-\lambda_k t}$ where λ_k optimal for $k > k_*$.

Remarks:

- Variants with derivatives $W^{\sigma,1}_v W^{s,\infty}$ with $0 \le \sigma \le s$, $s > 6/p$
- Variants with other Lebesgue spaces...
- Variants with stretched exponential weights...
- Spectral decomposition of linearized semigroup in these spaces

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(Hypo)dissipative and (hypo)coercive operators (I)

- $\Lambda - a$ dissipative in $X = (X, \|.\|_X)$ if

$$\forall\, f \in D(\Lambda),\ \forall\, \phi \in F(f), \quad \Re e\langle (\Lambda - a)\, f, \phi\rangle \le 0$$

with

$$\phi \in F(f) \quad \Longleftrightarrow \quad \langle f, \phi\rangle = \|f\|_X^2 = \|\phi\|_{X^*}^2$$

- $\Lambda - a$ hypodissipative in X if

$\Lambda - a$ dissipative in $(X, |\cdot|_X)$ for an equivalent norm $|\cdot|_X \sim \|\cdot\|_X$

- In the case of a Hilbert space structure, this coincides with the notion of coercivity and hypocoercivity
- Notion of (maximal) *m*-(hypo)dissipativity and *m*-(hypo)coercivity if furthermore $\mathrm{Range}(\Lambda - a) = X$

(Hypo)dissipative and (hypo)coercive operators (II)

Lumer-Philipps Theorem

The m-dissipativity implies $\|e^{t\Lambda}\|_{B(X)} \leq e^{at}$

Hille-Yosida Theorem

The m-hypodissipativity implies $\|e^{t\Lambda}\|_{B(X)} \leq C_a e^{at}$

- Add to these definition finite number of discrete eigenvalues

$$\|e^{t\Lambda}(1 - \Pi_{\Lambda,a})\|_{B(X)} \leq C_a\, e^{at},\ \ C_a = 1 \text{ or } C_a > 1$$

$$\text{or} \qquad \begin{cases} \Lambda - a \text{ is } m\text{-hypodissipative on invariant set } X_0 \\ X_0 \text{ closed},\ \operatorname{codim} X_0 < \infty \end{cases}$$

- Stronger notions of self-adjoint and of sectorial operators: $\|R(x+iy)\| \leq C|x+iy-a|^{-1}$ for $y = \pm\mu(x-a)$, $x \leq a$, for some $\mu \in (0, +\infty)$

Spectral mapping theorem

- General problem in semigroup theory

$$\text{Prove that} \quad \Sigma(e^{tL}) = e^{\Sigma(tL)}$$

- When true, spectral localization implies semigroup decay
- In general hard problem for non-self-adjoint operators
- Hörmander operators type I: $X_0^* X_0 + X_1^* X_1$ self-adjoint, already hard problem for regularity and semigroup decay
- Hörmander operators type II: $X_0 + X_1^* X_1$ non self-adjoint but still strong symmetry structure, semigroup decay solved recently Hérau-Nier, Villani'2000s
- Here non-self-adjoint and non-symmetric situations, in Banach spaces, with non-diffusive operators (not "Hörmander form")
- We shall use theories in a small space E in order to get results in a larger Banach space $\mathcal{E}$

Assumptions in the small space E

Abstract method that can be used for many other equations

Notation: half complex plane $\Delta_a := \{z \in \mathbb{C};\, \Re e\, z > a\}$

1. Localization of the spectrum
   - $\Sigma(L) \cap \Delta_a = \{\xi_1, \dots, \xi_k\}$
   - $\xi_1, \dots, \xi_k :=$ discrete eigenvalues
   - $\Pi_{L,\xi_j} :=$ eigenspace projector (finite dimensional eigenspace)
2. Growth estimate on the semigroup e^{tL}
   - $\Pi_{L,a} := \Pi_{L,\xi_1} + \cdots + \Pi_{L,\xi_k}$
   - $\|e^{tL}(1 - \Pi_{L,a})\|_{B(E)} \le C_a\, e^{a\,t}\ \forall\, t \ge 0$

Simpler example:
- L self-adjoint non-positif with compact resolvent
- $\Sigma(L) \subset \mathbb{R}_-$ discrete with 0 eigenvalue
- $\|e^{tL} f_0 - \langle f_0 \rangle\, M\|_E \le \|f_0\|_E\, e^{a\,t}$ with $\langle f \rangle := \int_{\mathbb{R}^d} f\, dv = \langle f, M \rangle_E$

Idea of the result

Setting: $\mathcal{E}$ Banach space including E
$\mathcal{L}$ generator of C_0-semigroup s.t. $\mathcal{L}_{|E} = L$

- If $\mathcal{L}$ decomposes as $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with
  - $\mathcal{A} : \mathcal{E} \to E$ ("regularizing" term)
  - $\mathcal{B} - a$ is dissipative (coercive term $\Rightarrow$ good spectral localization)

- Then $\mathcal{L}$ inherits the spectral properties of L
  - $\Sigma(\mathcal{L}) \cap \Delta_a = \{\xi_1, \dots, \xi_k\}$, with ξ_j discrete eigenvalue
  - $\Pi_{\mathcal{L},\xi_j}|_E = \Pi_{L,\xi_j}$ = spectral projector

- And $e^{t\,\mathcal{L}}$ inherits the growth estimate of e^{tL}

$$\forall\, t \geq 0,\ \forall\, a' > a, \quad \|e^{t\mathcal{L}} - e^{tL}\Pi_{\mathcal{L},a}\|_{\mathcal{E}\to\mathcal{E}} \leq C_{a'}\, e^{a'\,t}$$

Remark: Quantitative partial spectral mapping theorem

$$\Sigma(e^{\mathcal{L}\,t}) \cap \Delta_{e^{a'\,t}} = e^{\Sigma(L)\,t} \cap \Delta_{e^{a'\,t}}$$

Main difficulties and strategy (I)

Difficulties

- L and $\mathcal{L}$ may be non symmetric
- $\mathcal{E}$ may not have a Hilbert space structure
- we want constructive estimates

Use of resolvent estimates for relating decays on e^{tL} and $e^{t\mathcal{L}}$?

Semigroup of L $\longrightarrow$ Resolvent of $(L-\xi)^{-1}$

$\downarrow$

Semigroup of $\mathcal{L}$ $\longleftarrow$ Resolvent of $(\mathcal{L}-\xi)^{-1}$

Main difficulties and strategy (II)

- The horizontal arrows:
  Quantitative dictionary between semigroup decay estimates and resolvent estimates
  - homogeneous case: complex integration for sectorial operators
  - inhomogeneous case: inversion of Laplace transforms = complex integration on vertical lines in $\mathbb{C}$:
    in general loss on the norm in Banach spaces
    can be overcome by abstract Plancherel theo. if E Hilbert

- The vertical arrow:
  The factorization method
  - $\mathcal{L} = \mathcal{A} + \mathcal{B} \approx$ smooth + well known
  - $\mathcal{R}_{\mathcal{L}} = \mathcal{R}_{\mathcal{B}} - \mathcal{R}_L \, \mathcal{A} \, \mathcal{R}_{\mathcal{B}}$
    or more generally

$$\mathcal{R}_{\mathcal{L}} = \mathcal{R}_{\mathcal{B}} - \mathcal{R}_{\mathcal{B}}(\mathcal{A}\,\mathcal{R}_{\mathcal{B}}) + \cdots + (-1)^n \mathcal{R}_L \, (\mathcal{A}\,\mathcal{R}_{\mathcal{B}})^n$$

- Additional key ingredient: use the factorization in order to always rely on Plancherel's formula in the small space, in order to use the Hilbertian structure and avoid loss on the norm

A rigorous statement in the simpler case

$E \subset \mathcal{E}$ Banach spaces, L, $\mathcal{L}$ generators s.t. $\mathcal{L}_{|\mathcal{E}} = L$ with for $a < 0$:

(H0) E is a Hilbert space

(H1) **L is coercive:** $\leftarrow$ known

- (i) $\Sigma(L) \cap \Delta_a = \Sigma_d(L) \cap \Delta_a = \{0\}$ (localization of the spectrum)
- (ii) $L - a$ is dissipative on $R(I - \Pi_{L,0})$

(H2) **Decomposition of $\mathcal{L}$:** $\exists\, \mathcal{A}, \mathcal{B}$ s.t. $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and

- (i) $\mathcal{B} - a$ is dissipative ($\Rightarrow \Sigma(\mathcal{B}) \cap \Delta_a = \emptyset$) $\leftarrow$ to be proved
- (ii) $\mathcal{A} \in B(\mathcal{E}, E)$ $\leftarrow$ to be proved

Theorem

(i) $\Sigma(\mathcal{L}) \cap \Delta_a = \Sigma_d(\mathcal{L}) \cap \Delta_a = \{0\}$, $\Pi_{\mathcal{L},0}|_E = \Pi_{L,0}$

(ii) $\forall\, a' > a, \exists\, C_{a'} > 0$ s.t. $\forall\, t \geq 0, \quad \|e^{t\mathcal{L}} - \Pi_{\mathcal{L},0}\|_{B(\mathcal{E})} \leq C_{a'}\, e^{a'\, t}$

Proof of the theorem - Step 1: right inverse of $\mathcal{L} - \xi$

Define $\Omega := \Delta_a \backslash \{0\}$ (simpler example) and

$$\mathcal{U}(\xi) := \underbrace{\mathcal{R}_{\mathcal{B}}(\xi)}_{(\mathcal{B}-\xi)^{-1}} - \underbrace{\mathcal{R}_L(\xi)}_{(L-\xi)^{-1}} \; \mathcal{A} \; \underbrace{\mathcal{R}_{\mathcal{B}}(\xi)}_{(\mathcal{B}-\xi)^{-1}}$$

For $\xi \in \Omega$:
- $\mathcal{A}\mathcal{R}_{\mathcal{B}}(\xi)$ bounded from $\mathcal{E}$ to E
- $\mathcal{R}_{\mathcal{B}}(\xi)$ bounded in $\mathcal{E}$
- $\mathcal{R}_L(\xi)$ bounded in E

$\rightarrow$ hence $\mathcal{U}(\xi)$ bounded in $\mathcal{E}$ and

$$\begin{aligned}
(\mathcal{L} - \xi)\,\mathcal{U}(\xi) &= (\mathcal{A} + (\mathcal{B} - \xi))\,\mathcal{R}_{\mathcal{B}}(\xi) - (\mathcal{L} - \xi)\,\mathcal{R}_L(\xi)\,\mathcal{A}\,\mathcal{R}_{\mathcal{B}}(\xi) \\
&= \mathcal{A}\,\mathcal{R}_{\mathcal{B}}(\xi) + \mathrm{Id}_{\mathcal{E}} - (L - \xi)\,\mathcal{R}_L(\xi)\,\mathcal{A}\,\mathcal{R}_{\mathcal{B}}(\xi) \\
&= \mathcal{A}\,\mathcal{R}_{\mathcal{B}}(\xi) + \mathrm{Id}_{\mathcal{E}} - \mathcal{A}\,\mathcal{R}_{\mathcal{B}}(\xi) = \mathrm{Id}_{\mathcal{E}}.
\end{aligned}$$

Step 2: $\mathcal{L} - \xi$ is one-to-one on Ω

- $\mathcal{L}$ generates a semigroup $\Rightarrow \exists\, \xi_0 \in \Delta_a$ s.t. $\mathcal{L} - \xi_0$ is invertible
- $\mathcal{R}_{\mathcal{L}}(\xi_0)$ exists and bounded by some C_R

  $\Rightarrow \mathcal{R}_{\mathcal{L}}(z)$ exists on $B(\xi_0, 1/C_R)$

  $\Rightarrow \mathcal{R}_{\mathcal{L}}(z) = \mathcal{U}(z)$ on $B(\xi_0, 1/C_R)$
- But we have the following a priori bound on $\mathcal{U}(\xi)$

$$\begin{aligned} \|\mathcal{U}(\xi)\|_{B(\mathcal{E})} &\leq \|\mathcal{R}_{\mathcal{B}}(\xi)\|_{B(\mathcal{E})} + \|\mathcal{R}_L(\xi)\|_{B(E)} \, \|\mathcal{A}\|_{B(\mathcal{E},E)} \, \|\mathcal{R}_{\mathcal{B}}(\xi)\|_{B(\mathcal{E})} \\ &\leq C_R \qquad \text{on} \quad \Delta_a \backslash B(0,r) \end{aligned}$$

- Conclusion by a continuation argument:

  Existence of $\quad \mathcal{R}_{\mathcal{L}}(z) = \mathcal{R}_{\mathcal{B}}(z) - \mathcal{R}_L(z)\, \mathcal{A}\, \mathcal{R}_{\mathcal{B}}(z) \quad$ on Ω

  with the bound C_R

Step 3: the discrete spectrum (I)

- On Δ_a spectrum of $\mathcal{L}$ = poles of $\mathcal{R}_{\mathcal{L}}$ = poles of $\mathcal{R}_L = \{0\}$
- First inclusion clear on the algebraic eigenspaces

$$\mathrm{Range}(\Pi_{L,0}) \subset \mathrm{Range}(\Pi_{\mathcal{L},0})$$

- Eigenspaces and eigenprojectors: write the Laurent series

$$\mathcal{R}_L(z) = \sum_{\ell=1}^{\ell_0} z^{-\ell} R_{-\ell} + \sum_{\ell=0}^{\infty} z^{\ell} R_{\ell}, \qquad \mathcal{A}\,\mathcal{R}_{\mathcal{B}}(z) = \sum_{j=0}^{\infty} z^{j}\,\mathcal{C}_j$$

with $R_{-1} = \Pi_{L,0}$ and

$$\mathrm{Range}(R_{-\ell_0}), \ldots, \mathrm{Range}(R_{-2}) \subset \mathrm{Range}(R_{-1})$$

Step 3: the discrete spectrum (II)

Then we have the following formula for the spectral projector

$$\begin{aligned}
\Pi_{\mathcal{L},0} &:= \frac{i}{2\pi}\int_{|z|=r} \mathcal{R}_{\mathcal{L}}(z)\,dz \\
&= \frac{1}{2i\pi}\int_{|z|=r} \mathcal{R}_L(z)\,\mathcal{A}\,\mathcal{R}_{\mathcal{B}}(z)\,dz \\
&= R_{-1}\,\mathcal{C}_0 + R_{-2}\,\mathcal{C}_1 + \cdots + R_{-\ell_0}\,\mathcal{C}_{\ell_0-1} \\
R(\Pi_{\mathcal{L},0}) &\subset \textit{algebraic eigenspace of } L
\end{aligned}$$

Remark

Another proof can be done by assuming additionnally some invertibility of $B-\xi$ in E for $\xi \in \Delta_a$

However it is more convenient not to have to check this in the applications. . .

Step 4: The representation formula

We want to establish and estimate the growth of (for $a' > a$)

$$\forall\, f_0 \in \mathcal{D}(\mathcal{L}),\ \forall\, t \geq 0, \quad e^{t\mathcal{L}}\, f_0 = \Pi_{\mathcal{L},0}\, f_0 + e^{t\,\mathcal{B}}\, f_0 + T_1(t)\, f_0$$

where

$$T_1(t) := \lim_{M\to\infty} \frac{1}{2i\pi} \int_{a'-iM}^{a'+iM} e^{zt}\, \mathcal{R}_L(z)\, \mathcal{A}\, \mathcal{R}_{\mathcal{B}}(z)\, \mathrm{d}z$$

<u>Sketch of proof</u>:

$$\begin{aligned} e^{t\mathcal{L}}\, f_0 &\approx \Pi_{\mathcal{L},0}\, f_0 + \frac{1}{2i\pi} \int_{a'-i\infty}^{a'+i\infty} e^{zt}\, \mathcal{R}_{\mathcal{L}}(z)\, f_0\ \mathrm{d}z \\ &\approx \Pi_{\mathcal{L},0}\, f_0 + \frac{1}{2i\pi} \int_{a'-i\infty}^{a'+i\infty} e^{zt}\, \mathcal{R}_{\mathcal{B}}(z)\, f_0\ \mathrm{d}z \\ &\quad + \frac{1}{2i\pi} \int_{a'-i\infty}^{a'+i\infty} e^{zt}\, \mathcal{R}_L(z)\, \mathcal{A}\, \mathcal{R}_{\mathcal{B}}(z)\, f_0\ \mathrm{d}z \end{aligned}$$

Step 5: Control of the remainder term

Cauchy-Schwarz inequality: for any $\phi \in E^* = E$

$$|\langle \phi, T_1(t) f_0 \rangle| = \frac{1}{2\pi} \lim_{M\to\infty} \left| \int_{a'-iM}^{a'+iM} e^{z\,t} \, \langle \mathcal{R}_{L^*}(z)\,\phi, \mathcal{A}\,\mathcal{R}_{\mathcal{B}}(z)\, f_0 \rangle \, \mathrm{d}z \right|$$

$$\leq \frac{e^{a' t}}{2\pi} \int_{-\infty}^{\infty} \|\mathcal{R}_{L^*}(a'+iy)\,\phi\|_{E^*} \, \|\mathcal{A}\,\mathcal{R}_{\mathcal{B}}(a'+iy)\, f_0\|_E \, \mathrm{d}y$$

$$\leq \frac{e^{a' t}}{2\pi} \left(\int_{\mathbb{R}} \|\mathcal{R}_{L^*}(a'+iy)\,\phi\|^2 \, \mathrm{d}y \right)^{1/2} \left(\int_{\mathbb{R}} \|\mathcal{A}\,\mathcal{R}_{\mathcal{B}}(a'+iy)\, f_0\|^2 \, \mathrm{d}s \right)^{1/2}$$

$$\leq \frac{e^{a' t}}{2\pi} \left(K_1 \, \|\phi\|_{E^*} \right) \left(K_2 \, \|f_0\|_E \right) \qquad \leftarrow \text{two estimates}$$

from which we conclude

$$\| T_1(t) f_0 \| \leq \frac{e^{at}}{2\pi} \, K_1 \, K_2 \, \|f_0\|_E$$

Step 5: First estimate

(1) Use the resolvent identity

$$R_{L^*}(a' + iy) = \left(Id_{E^*} + (a' - b)\, R_{L^*}(a' + iy)\right) R_{L^*}(b + iy)$$

(2) Use the uniform bound

$$\sup_{y \in \mathbb{R}} \|R_{L^*}(a' + iy)\|_{B(E^*)} = \sup_{y \in \mathbb{R}} \|R_L(a' + iy)\|_{B(E)} \leq C$$

(3) Use Plancherel's identity in the Hilbert space $E = E^*$

(4) Use the control (for some $b > a'$ large enough)

$$\|e^{tL^*}\|_{B(E^*)} = \|e^{tL}\|_{B(E)} \leq C_b\, e^{b/2\, t}$$

$\rightarrow$ We get
$$\int_{\mathbb{R}} \|\mathcal{R}_{L^*}(a' + iy)\phi\|^2 \,\mathrm{d}y \leq C \int_{\mathbb{R}} \|\mathcal{R}_{L^*}(b + iy)\phi\|^2 \,\mathrm{d}y$$
$$\leq 2\pi\, C \int_0^{+\infty} \left\| e^{-b\,t}\, e^{t\,L^*} \phi \right\|^2 \mathrm{d}t$$
$$\leq 2\pi\, C \left(\int_0^{+\infty} \left\| e^{-b\,t}\, e^{t\,L^*} \right\|^2 \mathrm{d}t \right) \|\phi\|^2 \leq C'\, \|\phi\|^2_{E^*}$$

Step 5: Second estimate

Introduce the C^1 function $\varphi : \mathbb{R}_+ \to E$, $\varphi(t) := \mathcal{A}\, e^{t\,\mathcal{B}}\, f_0$
Its Laplace transform $r(z)$ satisfies

$$\forall\, z \in \Delta_a, \quad \left\{ \begin{aligned} & r(z) = \int_0^\infty e^{-z\,t}\, \varphi(t)\,\mathrm{d}t = \mathcal{A}\,\mathcal{R}_\mathcal{B}(z)\, f_0 \\ & \varphi(t) = \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} e^{zt}\, r(z)\,\mathrm{d}z \end{aligned} \right.$$

The Plancherel's identity in E then gives for $a' > a$

$$\begin{aligned} \int_\mathbb{R} \|\mathcal{A}\,\mathcal{R}_\mathcal{B}(a'+iy) f_0\|_E^2 \,\mathrm{d}y &= \int_\mathbb{R} \left\| r(a'+iy) \right\|_E^2 \,\mathrm{d}y \\ &= 2\pi \int_0^\infty \left\| \varphi(t)\, e^{-a'\,t} \right\|_E^2 \,\mathrm{d}t = 2\pi \int_0^\infty \left\| e^{-a'\,t}\,\mathcal{A}\, e^{t\,\mathcal{B}}\, f_0 \right\|_E^2 \,\mathrm{d}t \\ &\le C \left(\int_0^\infty e^{2(a-a')\,t}\,\mathrm{d}t \right) \|f\|_\mathcal{E}^2 \le C'\, \|f\|_\mathcal{E}^2 \end{aligned}$$

The higher-order factorization method

$E \subset \mathcal{E}$ Banach spaces, L, $\mathcal{L}$ generators s.t. $\mathcal{L}_{|\mathcal{E}} = L$ with for $a < 0$:

(H0) E is a Hilbert space

(H1) **L is coercive:** $\leftarrow$ known

- (i) $\Sigma(L) \cap \Delta_a = \Sigma_d(L) \cap \Delta_a = \{0\}$ (localization of the spectrum)
- (ii) $L - a$ is dissipative on Range$(I - \Pi_{L,0})$

(H2) **Decomposition of $\mathcal{L}$:** $\exists\, \mathcal{A}, \mathcal{B}$ s.t. $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and

- (i) $\mathcal{B} - a$ is hypodissipative : $\|e^{\mathcal{B}t}(t)\|_{B(\mathcal{E})} \leq C_a\, e^{a\,t}$
- (ii) $\mathcal{A} \in B(\mathcal{E})$
- (iii) $T_n := (\mathcal{A}\, \mathcal{S}_{\mathcal{B}})^{(*n)}$ satisfies $\|\, T_n(t)\|_{B(\mathcal{E},E)} \leq C_a\, e^{a\,t}$ for $n \in \mathbb{N}^*$

Theorem

(i) $\Sigma(\mathcal{L}) \cap \Delta_a = \Sigma_d(\mathcal{L}) \cap \Delta_a = \{0\},\ \Pi_{\mathcal{L},0}|_E = \Pi_{L,0}$

(ii) $\forall\, a' > a, \exists\, C_{a'} > 0$ s.t. $\forall\, t \geq 0, \quad \|e^{t\mathcal{L}} - \Pi_{\mathcal{L},0}\|_{B(\mathcal{E})} \leq C_{a'}\, e^{a'\,t}$

Idea of the proof

Definition of the time convolution:

$$(S_1 * S_2)(t) := \int_0^t S_1(s) \circ S_2(t-s)\, ds$$

Remark 1: $S_1 * S_2$ not a semigroup but has the good decay

Remark 2: Convolution behaves well w.r.t. the Laplace transform

Cornerstone of the proof

$$(*) \qquad \mathcal{R}_{\mathcal{L}} = \mathcal{R}_{\mathcal{B}} \sum_{\ell=0}^{n-1} (-1)^{\ell} \left(\mathcal{A}\,\mathcal{R}_{\mathcal{B}}\right)^{\ell} + (-1)^n \mathcal{R}_L \left(\mathcal{A}\,\mathcal{R}_{\mathcal{B}}\right)^n$$

The RHS is bounded thanks to assumption (iii)

(iii) and (*) are related to Dyson-Phillips expansion for semigroups

Plan

The setting

$$\boxed{\partial_t h = Lh := \mathcal{L}h - v \cdot \nabla_x h}$$

for $h = h(t, x, v)$, $x \in \mathbb{T}^3$, $v \in \mathbb{R}^3$ and

$$\mathcal{L}h := \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \Big[M(v'_*)\, h(v') + M(v')\, h(v'_*) - M(v_*)\, h(v) - M(v)\, h(v_*) \Big] \, |v - v_*| \, \mathrm{d}v_* \, \mathrm{d}\sigma$$

Collision frequency

$$\nu(v) := 4\pi \int_{\mathbb{R}^3} M(v_*)\, |v - v_*| \, \mathrm{d}v_* = 4\pi \, (M * |\cdot|)\,(v)$$

which satisfies for some constants $\nu_0, \nu_1 > 0$

$$\forall\, v \in \mathbb{R}^3, \quad 0 < \nu_0 \le \nu_0\,(1 + |v|) \le \nu(v) \le \nu_1\,(1 + |v|)$$

Semigroup decay estimates

Theorem

- *Consider* $\mathcal{E} = W^{\sigma,q}_v W^{s,p}_x(m)$ *with* $s, \sigma \in \mathbb{N}$, $\sigma \leq s$, *and*

(i) $m = M^{-1/2}$, $q = p = 2$

(ii) $m = e^{\kappa |v|^\beta}$, $\kappa > 0$, $\beta \in (0,2)$ *and* $p, q \in [1, +\infty]$

(iii) $m = \langle v \rangle^k$, $k > \bar{k}_q$ *with* $p, q \in [1, +\infty]$, $\bar{k}_q := \frac{2}{q} + 9\left(1 - \frac{1}{q}\right)$

then $\bullet$
$$\begin{cases} \Sigma(L) \subset \{z \in \mathbb{C} \mid \Re e\,(z) \leq -\lambda\} \cup \{0\} \\ N(L) = \mathrm{Span}\left\{\mu,\, v_1\,\mu,\, \ldots,\, v_d\,\mu,\, |v|^2\,\mu\right\} \\ \|f_t - \Pi f_{\mathrm{in}}\|_{\mathcal{E}} \leq C\, e^{-\lambda\, t}\, \|f_{\mathrm{in}} - \Pi f_{\mathrm{in}}\|_{\mathcal{E}} \end{cases}$$

- *Moreover* λ *can be taken as close as wanted to the optimal spectral gap of* L *in* $L^2(M^{-1/2})$ *in the cases (i)-(ii), as well as in the case (iii) when* $k > k_*$ *is big enough (with constructive* k_**)*

Strategy of the proof (I)

- We want to apply the factorization approach described before as in CM'2006, but in the spatially inhomogeneous case
- Decomposition $L = \mathcal{A} + \mathcal{B}$ for suitable operators $\mathcal{A} = \mathcal{A}_\delta$ and $\mathcal{B} = \mathcal{B}^c + \mathcal{B}^r_\delta$ by truncation-mollification (parameter $\delta > 0$)
- $\mathcal{A}_\delta =$ truncated-mollified non-local part of collision operator $\mathcal{L}$
- $\mathcal{B}^c =$ local (coercive) part of $\mathcal{L}$ + transport
- $\mathcal{B}^r_\delta =$ remainder of the non-local part
- Needs smallness estimate on $\mathcal{B}^r_\delta$ in various functional spaces
- Proof of such estimates in $L^1(\langle v \rangle^k)$, $k > 2$ by exploiting carefully Povzner inequalities
- Proof of such estimates in $L^\infty(\langle v \rangle^k)$ using representation of gain term for radially sym. fcts reminiscent of works on the Bosons gas Semikov-Tkachev, Escobedo-Mischler

Strategy of the proof (II)

- Then we prove that $\mathcal{B} - a$ is dissipative with $a < 0$, which follows from the smallness estimates on $\mathcal{B}^r_\delta$
- Then we prove that $\mathcal{A}_\delta$ has smoothing effect in the v-variable: cf. Lions' theorem in linearized form CM'2006, integral operator with C^∞_c kernel
- Lastly we prove some new regularity estimates on iterated velocity averages of a solution to a kinetic transport equation
- We deduce some regularity estimates in position x and velocity v variables on the iterated convolutions $(\mathcal{A}\, e^{\mathcal{B}t})^{*n}$, $n \geq n_0$
- Finally piling these arguments together we extend the semigroup decay from $E = L^2(M^{-1})$ to $\mathcal{E} = L^q_v L^p_x(m)$
- Higher-order regularity by linear combination of derivatives with well-chosen constants (cf. energy methods...)

The mollification-truncation

- For $\delta \in (0,1)$ consider a C^∞ characteristics function Θ_δ which equals one on

$$\Big\{ |v| \le \delta^{-1} \ \text{ and } \ 2\delta \le |v - v_*| \le \delta^{-1} \ \text{ and } \ |\cos\theta| \le 1 - 2\delta \Big\}$$

and whose support is included in

$$\Big\{ |v| \le 2\,\delta^{-1} \ \text{ and } \ \delta \le |v - v_*| \le 2\delta^{-1} \ \text{ and } \ |\cos\theta| \le 1 - \delta \Big\}$$

- Split $\mathcal{L}h = \mathcal{A}_\delta h + \mathcal{B}^r_\delta h$ with

$$\mathcal{A}_\delta h(v) := \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \Theta_\delta \Big[M(v'_*)\, h(v') + M(v')\, h(v'_*) - h(v_*)\, M(v) \Big] \, |v - v_*|$$

- Defining the corresponding remainder operator $\mathcal{B}^r_\delta$ with kernel $(1 - \Theta_\delta)$ we have therefore the decomposition

$$L = \mathcal{A}_\delta + \mathcal{B}_\delta$$

$$\text{with } \mathcal{B}_\delta = \mathcal{B}^c + \mathcal{B}^r_\delta \text{ and } \mathcal{B}^c = -\nu(v) - v \cdot \nabla_x$$

L^1 polynomial estimate on the remainder term (I)

Theorem

For any $k > 2$, $\delta > 0$, the remainder collision operator $\mathcal{B}^r_\delta$ satisfies

$$\forall\, h \in L^1(\langle v \rangle^{k+1}), \quad \|\mathcal{B}^r_\delta h\|_{L^1(\langle v \rangle^k)} \leq \left(\frac{4}{k+2} + \varepsilon_k(\delta) \right) \|h\|_{L^1(\nu\, \langle v \rangle^k)},$$

where $\varepsilon_k(\delta)$ is some constant (depending on k) going to zero as δ goes to zero.

Remark: The important and non trivial aspect here is the fact that the constant decays with the truncation parameters and k

Conjecture that the bound is optimal in terms of k...

For small values of k, the essential spectrum seems to "flood" the discrete non-zero eigenvalues...

L^1 polynomial estimate on the remainder term (II)

Lemma (Sharp Povzner Lemma)

For any $k > 2$, we have

$$\forall\, v, v_* \in \mathbb{R}^3, \quad \int_{\mathbb{S}^2} \left[|v'_*|^k + |v'|^k - |v_*|^k - |v|^k \right] \mathrm{d}\sigma$$

$$\leq C_k \left(|v|^{k-1} |v_*| + |v| \, |v_*|^{k-1} \right) - (4\pi - \gamma_k) \, |v|^k$$

where $\gamma_k := 16\pi/(k+2)$

This allows to show by "completing" the four terms difference $\Delta = |v'|^k + |v'_*|^k - |v|^k - |v_*|^k$ that

$$\|\mathcal{B}^r_\delta h\|_{L^1(m)} \leq \mathcal{O}(\delta) \int_{\mathbb{R}^3} |h| \, \langle v \rangle^{k+1} \, \mathrm{d}v + \frac{\gamma_k}{4\pi} \int_{\mathbb{R}^3} \nu(v) \, |h| \, |v|^k \, \mathrm{d}v$$

Dissipativity estimate on the coercive part (I)

Consider a solution h_t to the linear equation

$$\partial_t h_t = \mathcal{B}_\delta \, h_t = \mathcal{B}_\delta^r h_t - \nu(v) \, h_t - v \cdot \nabla_x h_t$$

with given initial datum h_0 and $1 \le p, q < +\infty$

Denote $\Phi'(z) := |z|^{p-1} \, \mathrm{sign}(z)$ and compute

$$\frac{\mathrm{d}}{\mathrm{d}t} \|h_t\|_{L^q_v L^p_x(m)} = \|h\|^{1-q}_{L^q_v L^p_x(m)} \times \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{T}^3} (\mathcal{B}_\delta h) \, \Phi'(h) \, \mathrm{d}x \right) \left(\int_{\mathbb{T}^3} |h|^p \, \mathrm{d}x \right)^{\frac{q}{p}-1} m^q \, \mathrm{d}v \right)$$

with

$$\int_{\mathbb{T}^3} (\mathcal{B}_\delta h) \, \Phi'(h) \, \mathrm{d}x = \int_{\mathbb{T}^3} \left[(\mathcal{B}_\delta^r h) \, \Phi'(h) - \nu \, |h|^p - \frac{1}{p} \, v \cdot \nabla_x \left(|h|^p \right) \right] \mathrm{d}x$$

$$\le \left(\int_{\mathbb{T}^3} |\mathcal{B}_\delta^r h|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \left(\int_{\mathbb{T}^3} |h|^p \, \mathrm{d}x \right)^{1-\frac{1}{p}} - \nu \int_{\mathbb{T}^3} |h|^p \, \mathrm{d}x$$

Dissipativity estimate on the coercive part (II)

$$\frac{\mathrm{d}}{\mathrm{d}t}\|h_t\|_{L^q_v L^p_x(m)}$$
$$\leq \|h\|^{1-q}_{L^q_v L^p_x(m)} \left[\left(\int_{\mathbb{R}^3} \|\mathcal{B}^r_\delta h\|_{L^p_x} \, \|h\|^{q-1}_{L^p_x} \, m^q \, \mathrm{d}v\right) - \left(\int_{\mathbb{R}^3} \nu \, \|h\|^q_{L^p_x} \, m^q \, \mathrm{d}v\right)\right]$$

Denoting $H = \|h\|_{L^p_x}$, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\|h_t\|_{L^q_v L^p_x(m)}$$
$$\leq \|H\|^{1-q}_{L^q_v(m)} \left[\left(\int_v \mathcal{B}^r_\delta(H)\, \nu^{-1/q'} \, m \, H^{q-1} \, m^{q-1} \, \nu^{1/q'}\right) - \int_v \nu \, H^q \, m^q\right]$$
$$\leq \|H\|^{1-q}_{L^q_v(m)} \left[\|\mathcal{B}^r_\delta(H)\|_{L^q_v(m\nu^{-1/q'})} \, \|H\|^{q-1}_{L^q_v(m\nu^{1/q})} - \int_v \nu \, H^q \, m^q\right]$$

$$\left.\begin{cases} \|H\|_{L^q_v(m)} \leq \nu_0^{-1/q} \|H\|_{L^q_v(m\nu^{1/q})} \\ \|\mathcal{B}^r_\delta(H)\|_{L^q_v(m\nu^{-1/q'})} \leq \Lambda_{m,q} \|H\|_{L^q_v(m\nu^{1/q})} \end{cases}\right\} \text{imply finally}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|h_t\|_{L^q_v L^p_x(m)} \leq -\nu_0^{-1}\left[1 - \Lambda_{m,q}(\delta)\right] \|h_t\|_{L^q_v L^p_x(m)}$$

Iterated averaging lemma (I)

Recall: $T_n(t) := (\mathcal{A}\,\mathcal{S}_{\mathcal{B}}(\cdot))^{(*n)}$, $n \in \mathbb{N}$ (iterated time convolution)

Lemma

Consider $s \in \mathbb{R}_+$ and a weight m so that $\mathcal{B}_\delta$ is dissipative in $W^{s',1}(m)$ for $s' \in [0, s+4]$.

Then there are some constructive constants $C_\delta > 0$ and R_δ such that for any $t \geq 0$ and $n \geq 1$

$$\operatorname{supp} T_n(t)h \subset K := B(0, R_\delta)$$

and

$$\begin{cases} \| T_1(t)h\|_{W^{s+1,1}_{x,v}(K)} \leq C\,\dfrac{e^{-\lambda_0' \, t}}{t}\,\|h\|_{W^{s,1}_{x,v}(m)}, & \text{if } s \geq 1 \\ \| T_2(t)h\|_{W^{s+1/2,1}_{x,v}(K)} \leq C\,e^{-\lambda_0' \, t}\,\|h\|_{W^{s,1}_{x,v}(m)}, & \text{if } s \geq 0 \end{cases}$$

Iterated averaging lemma (II)

Consider $\partial_t f + v \cdot \nabla_x f = 0$ and $\rho_\varphi(t,x) := \int_{\mathbb{R}^d} f(t,x,v)\,\varphi(v)\,\mathrm{d}v$

Then ρ_φ satisfies

$$\|\rho_\varphi(t,\cdot)\|_{W^{1,1}_x} \le \left(1 + \frac{1}{t}\right) \|\varphi\|_{W^{1,\infty}} \left(\|f_0\|_{L^1_{x,v}} + \|\nabla_v f_0\|_{L^1_{x,v}}\right)$$

Introduce $D_t := t\nabla_x + \nabla_v$ and observe that

$$\partial_t(D_t f) + v \cdot \nabla_x (D_t f) = 0.$$

$$\|f(t,\cdot)\|_{L^1} = \|f_0\|_{L^1}, \qquad \|D_t f(t,\cdot)\|_{L^1} = \|D_0 f_0\|_{L^1} = \|\nabla_v f_0\|_{L^1}$$

$$\text{and } \nabla_x \rho_\varphi(t,x) = \int_{\mathbb{R}^d} \left(\frac{D_t}{t} - \nabla_v\right) f(t,x,v)\,\varphi(v)\,\mathrm{d}v$$

$$= \frac{1}{t}\int_{\mathbb{R}^d} (D_t f)(t,x,v)\,\varphi(v)\,\mathrm{d}v + \int_{\mathbb{R}^d} f(t,x,v)\,\nabla_v\varphi(v)\,\mathrm{d}v$$

Then interpolate to $W^{1/2,1}$ and iterate twice

Plan

Introduction

The factorization method

The Boltzmann linearized semigroup

Non-symmetric nonlinear energy methods

The perturbative setting

Consider now the nonlinear Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = Q(f, f)$$

and write it for fluctuations $f = M + h$:

$$\partial_t h = Lh + Q(h, h)$$

We restrict now to $\mathcal{E} = L^1_v L^\infty_x(\langle v \rangle^k)$, $k > 2$ for the Cauchy theory

At a nonlinear level:
- close-to-equilibrium perturbative solutions
- weakly inhomogeneous solutions
- exponential H-theorem by connecting with Desvillettes-Villani

Main difficulty: No symmetry (sign) structure for energy methods!
I.e. decay of L not given by non-positive Dirichlet form...

Main ingredients

(1) Case L^∞ for x always obtained (even for the linearized semigroup) from L^p_x estimate with $p \to +\infty$

(2) Bilinear estimates of the form

$$\|Q(g,f)\|_{\mathcal{E}} \leq C\ \left(\|g\|_{\mathcal{E}_1}\|f\|_{\mathcal{E}} + \|g\|_{\mathcal{E}}\|f\|_{\mathcal{E}_1}\right)$$

for some constant $C > 0$, where $\mathcal{E}_1 := L^1_v \cap L^\infty_x(\nu\, m)$

(3) Introduction of a dissipative Banach norm

$$|||h|||_{\mathcal{E}} := \eta\,\|h\|_{\mathcal{E}} + \int_0^{+\infty} \|S_L(\tau)h\|_{\mathcal{E}}\ \mathrm{d}\tau, \quad 0 < \eta << 1$$

Equivalent to ambient norm $\|\cdot\|_E$ thanks to the decay of S_L

The dissipative Banach norm (I)

Example of non-symmetric energy estimate

Consider $\mathcal{E} = L^1_v L^\infty_x(m)$ with norm $\|\cdot\|_{\mathcal{E}}$, and define ($\eta > 0$)

$$|||h|||_{\mathcal{E}} := \eta \, \|h\|_{\mathcal{E}} + \int_0^{+\infty} \|S_L(\tau) h\|_{\mathcal{E}} \, \mathrm{d}\tau.$$

Then $\quad \forall \, t \geq 0, \quad \frac{\mathrm{d}}{\mathrm{d}t} |||S_L(t)h|||_{\mathcal{E}} \leq -\lambda_1 \, |||S_L(t)h|||_{\mathcal{E}_1}$

where

$$\mathcal{E}_1 := L^1_v L^\infty_x(\nu \, m) \subset \mathcal{E}$$

and $|||\cdot|||_{\mathcal{E}_1}$ is defined as

$$|||h|||_{\mathcal{E}_1} := \eta \, \|h\|_{\mathcal{E}_1} + \int_0^{+\infty} \|S_L(\tau) h\|_{\mathcal{E}_1} \, \mathrm{d}\tau$$

Extensions: general Lebesgue spaces, derivatives. . .

The dissipative Banach norm (II)

From the previous decay estimates:

$$\|S_L(\tau)h\|_{\mathcal{E}} \le C\, e^{-\lambda\, t}\, \|h\|_{\mathcal{E}} \quad \text{which implies} \quad \|\cdot\|_{\mathcal{E}} \sim |||\cdot|||_{\mathcal{E}}$$

and denoting again $\Phi'(z) := |z|^{p-1}\,\mathrm{sign}(z)$

$$\frac{\mathrm{d}}{\mathrm{d}t}|||h_t|||_{\mathcal{E}} = \eta\, \|h_t\|_{\mathcal{E}}^{1-q} \int_{\mathbb{R}^3} \left(\int_{\mathbb{T}^3} L(h_t)\, \Phi'(h_t)\, dx \right) \|h_t\|_{L^p_x}^{q-p}\, m^q\, \mathrm{d}v$$
$$+ \int_0^{+\infty} \frac{\partial}{\partial t} \|h_{t+\tau}\|_{\mathcal{E}}\, \mathrm{d}\tau =: I_1 + I_2$$

with (arguing as before)

$$I_1 \le \eta \left(C\, \|h\|_{\mathcal{E}} - K\, \|h\|_{\mathcal{E}_{1/q}} \right)$$

and

$$I_2 = \int_0^{+\infty} \frac{\partial}{\partial t} \|h_{t+\tau}\|_{\mathcal{E}}\, \mathrm{d}\tau = \int_0^{+\infty} \frac{\partial}{\partial \tau} \|h_{t+\tau}\|_{\mathcal{E}}\, \mathrm{d}\tau = -\|h\|_{\mathcal{E}}$$

Remarks and open problems I

- Used in other contexts:
  - kinetic Fokker-Planck: w/ Mischler
  - Wigner-Fokker-Planck equation: AGGMMS, Stürzer-Arnold
  - Smoluchowski equation: Cañizo-Lods...
- Possible to study singularities and pointwise estimates on the Green's function (fluid-kinetic) with this approach:
  First step (revisit Liu-Yu's results) w/ Wu in Cambridge
- Useful framework for the clustering problem in granular gases:
  Work in progress w/ Mischler & Rey

Remarks and open problems II

- Whole space $x \in \mathbb{R}^3$: space enlargement? Perturbative solutions with polynomial tails?
  Difficulty: the semigroup has no spectral gap (essential spectrum touches zero)
- Non cutoff interactions?
  Difficulty: the decomposition should be modified
- Landau equation?
  Difficulty: idem + no spectral gap for Landau-Coulomb...
- Uniform estimates in the incompressible hydrodynamic limit
  First step: Briant
- Critical space in velocity $L^1_v(1+|v|^2)$ can (very likely) be obtained by technical refinement of the method
- In x variable L^∞ probably not critical in view of the Navier-Stokes theory: can one obtain L^p_x with $p < \infty$ by exploiting better averaging lemma?