Systèmes de Coulomb, énergie renormalisée, et réseau d'Abrikosov

Sylvia Serfaty

Université P. et M. Curie Paris 6, Laboratoire Jacques-Louis Lions
http://www.ann.jussieu.fr/∼serfaty

Séminaire du LJLL, 30 novembre 2012

The statistical mechanics of Coulomb / log gases

Hamiltonian

$$w_n(x_1, \cdots, x_n) = -\sum_{i \neq j} \log |x_i - x_j| + n \sum_{i=1}^{n} V(x_i),$$

with $x_i \in \mathbb{R}^d$, $d = 1$ or 2.
V smooth enough and grows faster than $\log^2 |x|$ at infinity.

With temperature: Gibbs measure (= probability law)

$$d\mathbb{P}_n^{\beta}(x_1, \cdots, x_n) = \frac{1}{Z_n^{\beta}} e^{-\frac{\beta}{2} w_n(x_1, \cdots, x_n)} dx_1 \cdots dx_n$$

where Z_n^{β} is the associated partition function ($\int d\mathbb{P}_n^{\beta} = 1$).

- In 2D such an ensemble is called a Coulomb gas, also a one-component plasma. In 1D it is called a log gas. Also sometimes β-ensembles. book by Forrester '10
- Minimizers are also maximizers of

$$\prod_{i<j} |x_i - x_j| \prod_{i=1}^{n} e^{-n\frac{V}{2}(x_i)}$$

$\rightarrow$ **weighted Fekete sets** (interpolation, orthogonal polynomials).
- connection with random matrices Wigner '59
- For $d=1$, $\beta=2$, $V(x)=x^2/2 \rightsquigarrow$ **GUE** (= law of eigenvalues of Hermitian matrices with complex Gaussian i.i.d. entries).
- For $d=1$, $\beta=1$, $V(x)=x^2/2 \rightsquigarrow$ **GOE** (real symmetric matrices with Gaussian i.i.d. entries).
- For $d=2$, $\beta=2$ and $V(x)=|x|^2 \rightsquigarrow$ **Ginibre ensemble** (matrices with complex Gaussian i.i.d. entries).
- Global and local statistics known in 1D for V quadratic Valko-Virag, and for general V's ("universality") Bourgade-Erdös-Yau. In 2D only for $\beta=2$ and V quadratic.

- In 2D such an ensemble is called a Coulomb gas, also a one-component plasma. In 1D it is called a log gas. Also sometimes β-ensembles. book by Forrester '10
- Minimizers are also maximizers of

$$\prod_{i<j} |x_i - x_j| \prod_{i=1}^{n} e^{-n\frac{V}{2}(x_i)}$$

$\rightarrow$ **weighted Fekete sets** (interpolation, orthogonal polynomials).

- connection with random matrices Wigner '59
- For $d = 1$, $\beta = 2$, $V(x) = x^2/2 \rightsquigarrow$ **GUE** (= law of eigenvalues of Hermitian matrices with complex Gaussian i.i.d. entries).
- For $d = 1$, $\beta = 1$, $V(x) = x^2/2 \rightsquigarrow$ **GOE** (real symmetric matrices with Gaussian i.i.d. entries).
- For $d = 2$, $\beta = 2$ and $V(x) = |x|^2 \rightsquigarrow$ **Ginibre ensemble** (matrices with complex Gaussian i.i.d. entries).
- Global and local statistics known in 1D for V quadratic Valko-Virag, and for general V's ("universality") Bourgade-Erdös-Yau. In 2D only for $\beta = 2$ and V quadratic.

- In 2D such an ensemble is called a Coulomb gas, also a one-component plasma. In 1D it is called a log gas. Also sometimes β-ensembles. book by Forrester '10
- Minimizers are also maximizers of

$$\prod_{i<j} |x_i - x_j| \prod_{i=1}^{n} e^{-n\frac{V}{2}(x_i)}$$

$\rightarrow$ **weighted Fekete sets** (interpolation, orthogonal polynomials).

- connection with random matrices Wigner '59
- For $d = 1$, $\beta = 2$, $V(x) = x^2/2 \rightsquigarrow$ **GUE** (= law of eigenvalues of Hermitian matrices with complex Gaussian i.i.d. entries).
- For $d = 1$, $\beta = 1$, $V(x) = x^2/2 \rightsquigarrow$ **GOE** (real symmetric matrices with Gaussian i.i.d. entries).
- For $d = 2$, $\beta = 2$ and $V(x) = |x|^2 \rightsquigarrow$ **Ginibre ensemble** (matrices with complex Gaussian i.i.d. entries).
- Global and local statistics known in 1D for V quadratic Valko-Virag, and for general V's ("universality") Bourgade-Erdös-Yau. In 2D only for $\beta = 2$ and V quadratic.

- In 2D such an ensemble is called a Coulomb gas, also a one-component plasma. In 1D it is called a log gas. Also sometimes β-ensembles. book by Forrester '10
- Minimizers are also maximizers of

$$\prod_{i<j} |x_i - x_j| \prod_{i=1}^{n} e^{-n\frac{V}{2}(x_i)}$$

$\to$ **weighted Fekete sets** (interpolation, orthogonal polynomials).

- connection with random matrices Wigner '59
- For $d = 1$, $\beta = 2$, $V(x) = x^2/2 \rightsquigarrow$ **GUE** (= law of eigenvalues of Hermitian matrices with complex Gaussian i.i.d. entries).
- For $d = 1$, $\beta = 1$, $V(x) = x^2/2 \rightsquigarrow$ **GOE** (real symmetric matrices with Gaussian i.i.d. entries).
- For $d = 2$, $\beta = 2$ and $V(x) = |x|^2 \rightsquigarrow$ **Ginibre ensemble** (matrices with complex Gaussian i.i.d. entries).
- Global and local statistics known in 1D for V quadratic Valko-Virag, and for general V's ("universality") Bourgade-Erdös-Yau. In 2D only for $\beta = 2$ and V quadratic.

- In 2D such an ensemble is called a Coulomb gas, also a one-component plasma. In 1D it is called a log gas. Also sometimes β-ensembles. book by Forrester '10
- Minimizers are also maximizers of

$$\prod_{i<j} |x_i - x_j| \prod_{i=1}^{n} e^{-n\frac{V}{2}(x_i)}$$

$\rightarrow$ **weighted Fekete sets** (interpolation, orthogonal polynomials).

- connection with random matrices Wigner '59
- For $d = 1$, $\beta = 2$, $V(x) = x^2/2 \rightsquigarrow$ **GUE** (= law of eigenvalues of Hermitian matrices with complex Gaussian i.i.d. entries).
- For $d = 1$, $\beta = 1$, $V(x) = x^2/2 \rightsquigarrow$ **GOE** (real symmetric matrices with Gaussian i.i.d. entries).
- For $d = 2$, $\beta = 2$ and $V(x) = |x|^2 \rightsquigarrow$ **Ginibre ensemble** (matrices with complex Gaussian i.i.d. entries).
- Global and local statistics known in 1D for V quadratic Valko-Virag, and for general V's ("universality") Bourgade-Erdös-Yau. In 2D only for $\beta = 2$ and V quadratic.

- In 2D such an ensemble is called a Coulomb gas, also a one-component plasma. In 1D it is called a log gas. Also sometimes β-ensembles. book by Forrester '10
- Minimizers are also maximizers of

$$\prod_{i<j} |x_i - x_j| \prod_{i=1}^{n} e^{-n\frac{V}{2}(x_i)}$$

$\rightarrow$ **weighted Fekete sets** (interpolation, orthogonal polynomials).

- connection with random matrices Wigner '59
- For $d=1$, $\beta=2$, $V(x)=x^2/2 \rightsquigarrow$ **GUE** (= law of eigenvalues of Hermitian matrices with complex Gaussian i.i.d. entries).
- For $d=1$, $\beta=1$, $V(x)=x^2/2 \rightsquigarrow$ **GOE** (real symmetric matrices with Gaussian i.i.d. entries).
- For $d=2$, $\beta=2$ and $V(x)=|x|^2 \rightsquigarrow$ **Ginibre ensemble** (matrices with complex Gaussian i.i.d. entries).
- Global and local statistics known in 1D for V quadratic Valko-Virag, and for general V's ("universality") Bourgade-Erdös-Yau. In 2D only for $\beta=2$ and V quadratic.

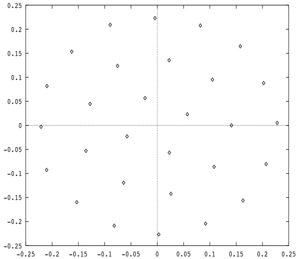
- In 2D such an ensemble is called a Coulomb gas, also a one-component plasma. In 1D it is called a log gas. Also sometimes β-ensembles. book by Forrester '10
- Minimizers are also maximizers of

$$\prod_{i<j} |x_i - x_j| \prod_{i=1}^{n} e^{-n\frac{V}{2}(x_i)}$$

$\rightarrow$ **weighted Fekete sets** (interpolation, orthogonal polynomials).

- connection with random matrices Wigner '59
- For $d = 1$, $\beta = 2$, $V(x) = x^2/2 \rightsquigarrow$ **GUE** (= law of eigenvalues of Hermitian matrices with complex Gaussian i.i.d. entries).
- For $d = 1$, $\beta = 1$, $V(x) = x^2/2 \rightsquigarrow$ **GOE** (real symmetric matrices with Gaussian i.i.d. entries).
- For $d = 2$, $\beta = 2$ and $V(x) = |x|^2 \rightsquigarrow$ **Ginibre ensemble** (matrices with complex Gaussian i.i.d. entries).
- Global and local statistics known in 1D for V quadratic Valko-Virag, and for general V's (“universality") Bourgade-Erdös-Yau. In 2D only for $\beta = 2$ and V quadratic.

Minimizers of w_n



Numerical minimization of w_n **for** $V(x) = |x|^2$ (Gueron-Shafrir), $n = 29$

Equilibrium measure

Define

$$\mathcal{F}(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}^d} -\log|x-y|\, d\mu(x)\, d\mu(y) + \int_{\mathbb{R}^d} V(x)\, d\mu(x).$$

$\mathcal{F}$ has a unique minimizer among probability measures, called the *equilibrium measure*, denoted μ_0. Frostman '55, Saff-Totik '97

Denote $\Sigma = Supp(\mu_0)$. In 2D we assume Σ is compact with C^1 boundary, in 1D it is a finite union of disjoint intervals.

- For GUE: $\mu_0 = \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{\{|x|<2\}}$ "Wigner semicircle law"
- For Ginibre: $\mu_0 = \frac{1}{\sqrt{\pi}}\mathbf{1}_{B_1}$ "circle law", Edelman, Girko, Mehta.

We assume that μ_0 is of this type: in 2D of density bounded above and below by constants on Σ, in 1D of density in $C^{0,\frac{1}{2}}$, increasing like $\sqrt{x}$.

Equilibrium measure

Define

$$\mathcal{F}(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}^d} -\log|x-y|\, d\mu(x)\, d\mu(y) + \int_{\mathbb{R}^d} V(x)\, d\mu(x).$$

$\mathcal{F}$ has a unique minimizer among probability measures, called the *equilibrium measure*, denoted μ_0. Frostman '55, Saff-Totik '97

Denote $\Sigma = Supp(\mu_0)$. In 2D we assume Σ is compact with C^1 boundary, in 1D it is a finite union of disjoint intervals.

- For GUE: $\mu_0 = \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{\{|x|<2\}}$ "Wigner semicircle law"
- For Ginibre: $\mu_0 = \frac{1}{\sqrt{\pi}}\mathbf{1}_{B_1}$ "circle law", Edelman, Girko, Mehta.

We assume that μ_0 is of this type: in 2D of density bounded above and below by constants on Σ, in 1D of density in $C^{0,\frac{1}{2}}$, increasing like $\sqrt{x}$.

Mean-field limit and large deviations

Proposition (Mean-field limit)

$$\frac{w_n}{n^2}\ \Gamma - \textit{converges to } \mathcal{F}: \ \textit{in particular}$$

$$\lim_{n\to\infty} \frac{\min w_n}{n^2} = \mathcal{F}(\mu_0) \qquad \lim_{n\to\infty} \frac{\sum_{i=1}^n \delta_{x_i}}{n} = \mu_0 \quad \textit{for a minimizer}$$

Theorem (Ben Arous-Guionnet $d=1$, Ben Arous-Zeitouni $d=2$)

$\mathbb{P}_n^\beta$ *satisfies a large deviations principle with good rate function* $\mathcal{F}(\cdot)$ *and speed* n^{-2}*: for all* $A \subset \{$*probability measures*$\}$,

$$-\inf_{\mu\in A^\circ}(\mathcal{F}(\mu)-\mathcal{F}(\mu_0)) \leq \liminf_{n\to\infty}\frac{1}{n^2}\log\mathbb{P}_n^\beta(A)$$

$$\leq \limsup_{n\to\infty}\frac{1}{n^2}\log\mathbb{P}_n^\beta(A) \leq -\inf_{\mu\in\bar{A}}(\mathcal{F}(\mu)-\mathcal{F}(\mu_0)).$$

Mean-field limit and large deviations

Proposition (Mean-field limit)

$$\frac{w_n}{n^2}\ \Gamma - \textit{converges to } \mathcal{F} : \ \textit{in particular}$$

$$\lim_{n\to\infty} \frac{\min w_n}{n^2} = \mathcal{F}(\mu_0) \qquad \lim_{n\to\infty} \frac{\sum_{i=1}^n \delta_{x_i}}{n} = \mu_0 \quad \textit{for a minimizer}$$

Theorem (Ben Arous-Guionnet $d = 1$, Ben Arous-Zeitouni $d = 2$)

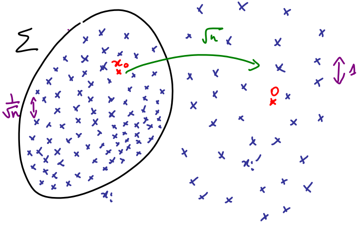
$\mathbb{P}_n^\beta$ satisfies a large deviations principle with good rate function $\mathcal{F}(\cdot)$ and speed n^{-2}: for all $A \subset \{$probability measures$\}$,

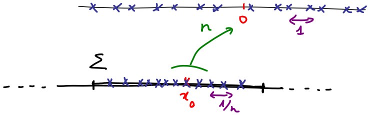
$$- \inf_{\mu\in A^\circ} (\mathcal{F}(\mu) - \mathcal{F}(\mu_0)) \le \liminf_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n^\beta(A)$$
$$\le \limsup_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_n^\beta(A) \le - \inf_{\mu\in \bar{A}} (\mathcal{F}(\mu) - \mathcal{F}(\mu_0)).$$

First objective

Understand minimizers of w_n. We know the global distribution of the points μ_0 and $\min w_n \sim n^2\mathcal{F}(\mu_0)$.

Can we say more about the local distribution of points and the next order terms in $\min w_n$?? For that we want to blow up the points at the scale $n^{1/d}$ to see them at finite distances from each other.





Splitting of w_n

The starting idea is to understand the next order behavior by splitting w_n, writing

$$\nu_n := \sum_{i=1}^n \delta_{x_i} \quad \text{as } n\mu_0 + (\nu_n - n\mu_0).$$

$$w_n(x_1, \cdots, x_n) = \iint_{\triangle^c} -\log|x-y|\, d\underbrace{\nu_n(x)}_{n\mu_0 + (\nu_n - n\mu_0)}\, d\underbrace{\nu_n(y)}_{n\mu_0 + (\nu_n - n\mu_0)} + \int V(x)\, d\underbrace{\nu_n(x)}_{n\mu_0 + (\nu_n - n\mu_0)}.$$

We find *in 2D*

$$w_n(x_1,\cdots,x_n) = n^2\mathcal{F}(\mu_0) + 2n\sum_{i=1}^n \zeta(x_i) + \underbrace{\frac{1}{\pi}W(\nabla H_n, \mathbf{1}_{\mathbb{R}^2})}_{\text{"renormalized" self-interaction of } \nu_n - n\mu_0}$$

Here

$$\zeta = cst + \frac{1}{2}V - \int \log|x-y|\,d\mu_0(y)$$

μ_0 is characterized by the fact that we can take the cst such that

$$\begin{cases} \zeta = 0 & \text{in } \Sigma \\ \zeta > 0 & \text{in } \mathbb{R}^2\backslash\Sigma \end{cases}$$

$$H_n = -2\pi\Delta^{-1}\left(\sum_{i=1}^n \delta_{x_i} - n\mu_0\right) = -\log|x| * \left(\sum_{i=1}^n \delta_{x_i} - n\mu_0\right)$$

and for every function χ, $W(\nabla H_n,\chi) = \int |\nabla H_n|^2\chi$ computed in a renormalized way à la Bethuel-Brezis-Hélein:

$$W(\nabla H_n,\chi) := \lim_{\eta\to 0}\int_{\mathbb{R}^2\backslash\cup_{i=1}^n B(x_i,\eta)} \chi|\nabla H_n|^2 + \pi(\log\eta)\sum_i \chi(x_i).$$

In rescaled coordinates $x' = \sqrt{n}(x - x_0)$ this becomes

$$w_n(x_1, \cdots, x_n) = n^2 \mathcal{F}(\mu_0) - n \log \sqrt{n} + \frac{1}{\pi} W(\nabla H_n', \mathbf{1}_{\mathbb{R}^2}) + 2n \sum_{i=1}^n \zeta(x_i)$$

where H_n' is the solution to

$$H_n'(x') = -2\pi \Delta^{-1} \left(\sum_{i=1}^n \delta_{x_i'} - \mu_0(x_0 + \frac{x'}{\sqrt{n}}) \right)$$

- in the limit $n \to \infty$

$$-\Delta H' = 2\pi \left(\sum_i \delta_{x_i'} - \mu_0(x_0) \right)$$

- remains to take the limit $n \to \infty$ in $W(\nabla H_n', \mathbf{1}_{\mathbb{R}^2})$, “renormalized" Coulomb interaction between the points in a neutralizing background, of slowly varying density $\sim \mu_0$

In rescaled coordinates $x' = \sqrt{n}(x - x_0)$ this becomes

$$w_n(x_1, \cdots, x_n) = n^2 \mathcal{F}(\mu_0) - n \log \sqrt{n} + \frac{1}{\pi} W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2}) + 2n \sum_{i=1}^{n} \zeta(x_i)$$

where H'_n is the solution to

$$H'_n(x') = -2\pi \Delta^{-1} \left(\sum_{i=1}^{n} \delta_{x'_i} - \mu_0(x_0 + \frac{x'}{\sqrt{n}}) \right)$$

- in the limit $n \to \infty$

$$-\Delta H' = 2\pi \left(\sum_i \delta_{x'_i} - \mu_0(x_0) \right)$$

- remains to take the limit $n \to \infty$ in $W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2})$, "renormalized" Coulomb interaction between the points in a neutralizing background, of slowly varying density $\sim \mu_0$

Analogue in dimension 1

It suffices to embed the 1D situation into 2D! $x_i \rightsquigarrow (x_i, 0)$

- $$H_n = -2\pi\Delta^{-1}\left(\sum_{i=1}^n \delta_{x_i} - n\mu_0(x)\delta_{\mathbb{R}}\right) \quad \text{in} \quad \mathbb{R}^2$$
  where $\delta_{\mathbb{R}}$ is the measure of length on the real axis $\int \varphi \delta_{\mathbb{R}} = \int \varphi(x,0)\,dx$.
- H_n is defined in $\mathbb{R}^2$, and harmonic outside the real axis. (Of course $H_n = -2\pi\Delta^{-1/2}(\sum \delta_{x_i} - \mu_0)$ on $\mathbb{R}$.)
- $W(\nabla H_n, \mathbf{1}_{\mathbb{R}})$ is defined as before.
- Rescaled coordinates $x' = nx$.
  $$H_n'(x') = -2\pi\Delta^{-1}\left(\sum_{i=1}^n \delta_{x_i'} - \mu_0(x_0 + \frac{x'}{n})\delta_{\mathbb{R}}\right)$$
- Final splitting formula

$$\boxed{w_n(x_1,\cdots,x_n) = n^2\mathcal{F}(\mu_0) - \frac{n}{d}\log n + \frac{1}{\pi}W(\nabla H_n', \mathbf{1}_{\mathbb{R}^2}) + 2n\sum_{i=1}^n \zeta(x_i).}$$

Complete definition of W

Let $m > 0$ given. Let Λ be a discrete set in $\mathbb{R}^d$, and $\mathcal{E}(= \nabla H)$ a vector field (= electric field generated by the charges) such that

$$\mathrm{div}\ \mathcal{E} = 2\pi(\nu - m\,\delta_{\mathbb{R}}) \quad \text{and} \quad \mathrm{curl}\,\mathcal{E} = 0, \quad \text{where} \quad \nu = \sum_{p\in\Lambda} \delta_p.$$

We say such a $\mathcal{E}$ belongs to the class $\mathcal{A}_m$.

Definition (Sandier-S)

For any smooth positive χ, let

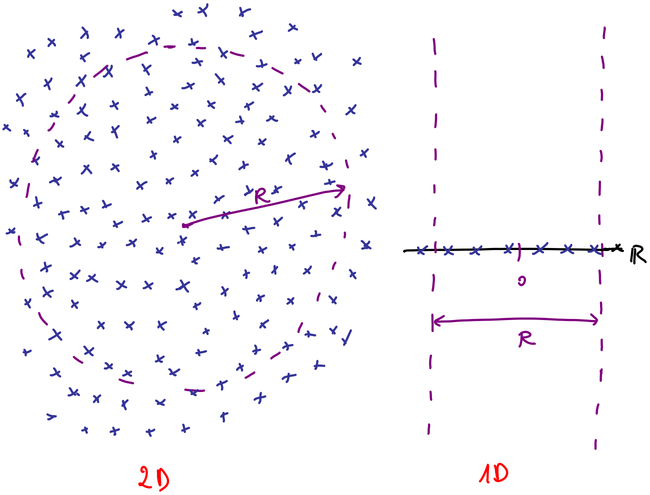
$$W(\mathcal{E},\chi) = \lim_{\eta\to 0}\left(\frac{1}{2}\int_{\mathbb{R}^2\setminus\cup_{p\in\Lambda}B(p,\eta)} \chi|\mathcal{E}|^2 + \pi\log\eta\sum_{p\in\Lambda}\chi(p)\right).$$

We define the **renormalized energy** *W for $\mathcal{E}\in\mathcal{A}_m$ by*

$$W(\mathcal{E}) = \limsup_{R\to\infty}\frac{W(\mathcal{E},\chi_R)}{|B_R|}\ 2D \qquad = \limsup_{R\to\infty}\frac{W(\mathcal{E},\chi_R)}{R}\ 1D,$$

where χ_R is any cutoff function supported in B_R with $\chi_R = 1$ in B_{R-1} and $|\nabla\chi_R| \le C$ in 2D, and same with $[-R/2, R/2]\times\mathbb{R}$ in 1D.

Computing W



The case of the torus

In 2D: Assume Λ is $\mathbb{T}$-periodic. Then W can be written as a function of Λ" = "$\{a_1, \ldots, a_n\}$.

$$W(a_1, \cdots, a_n) = \frac{\pi}{|\mathbb{T}|} \sum_{j \neq k} G(a_j - a_k) + \pi \lim_{x \to 0} \left(G(x) + \log |x| \right),$$

where G= Green's function of the torus ($-\Delta G = \delta_0 - 1/|\mathbb{T}|$).

In 1D: Assume Λ is $\mathbb{T}$-periodic, $\mathbb{T} = R/(N\mathbb{Z})$. Then W can be written

$$W(a_1, \ldots, a_n) = -\frac{\pi}{N} \sum_{i \neq j} \log \left| 2 \sin \frac{\pi(a_i - a_j)}{N} \right| - \pi \log \frac{2\pi}{N}$$

Corol: the minimum among periodic configurations in 1D is achieved by the lattice $\mathbb{Z}$ (convexity argument).

The case of the torus

In 2D: Assume Λ is $\mathbb{T}$-periodic. Then W can be written as a function of $\Lambda`` = " \{a_1, \dots, a_n\}$.

$$W(a_1, \cdots, a_n) = \frac{\pi}{|\mathbb{T}|} \sum_{j \neq k} G(a_j - a_k) + \pi \lim_{x \to 0} \left(G(x) + \log |x| \right),$$

where G= Green's function of the torus ($-\Delta G = \delta_0 - 1/|\mathbb{T}|$).

In 1D: Assume Λ is $\mathbb{T}$-periodic, $\mathbb{T} = R/(N\mathbb{Z})$. Then W can be written

$$W(a_1, \dots, a_n) = -\frac{\pi}{N} \sum_{i \neq j} \log \left| 2 \sin \frac{\pi(a_i - a_j)}{N} \right| - \pi \log \frac{2\pi}{N}$$

Corol: the minimum among periodic configurations in 1D is achieved by the lattice $\mathbb{Z}$ (convexity argument).

Minimization of W

- Scaling: If $\mathcal{E}$ belongs to $\mathcal{A}_m$, then $\mathcal{E}' = \frac{1}{m^{1/d}}\mathcal{E}(\cdot/m^{1/d})$ belongs to $\mathcal{A}_1$ and

$$W(\mathcal{E}) = m\left(W(\mathcal{E}') - \frac{\pi}{d}\log m\right)$$

  so we can reduce to $\mathcal{A}_1$.
- Proposition: $\min_{\mathcal{A}_1} W$ is the limit as $N \to \infty$ of the min over $\mathbb{T}_N$-periodic configurations. Relies on screening (cf. later).

Theorem

In dimension 1, $\min_{\mathcal{A}_m} W$ is achieved by the perfect lattice $\Lambda = \frac{1}{m}\mathbb{Z}$.

Minimization of W

- Scaling: If $\mathcal{E}$ belongs to $\mathcal{A}_m$, then $\mathcal{E}' = \frac{1}{m^{1/d}}\mathcal{E}(\cdot/m^{1/d})$ belongs to $\mathcal{A}_1$ and
$$W(\mathcal{E}) = m\left(W(\mathcal{E}') - \frac{\pi}{d}\log m\right)$$
so we can reduce to $\mathcal{A}_1$.
- Proposition: $\min_{\mathcal{A}_1} W$ is the limit as $N \to \infty$ of the min over $\mathbb{T}_N$-periodic configurations. Relies on screening (cf. later).

Theorem

In dimension 1, $\min_{\mathcal{A}_m} W$ is achieved by the perfect lattice $\Lambda = \frac{1}{m}\mathbb{Z}$.

Dimension 2 : minimization among lattices

In 2D we can only identify the minimizers of W among perfect lattice configurations, i.e., $\Lambda = \mathbb{Z}\vec{u} + \mathbb{Z}\vec{v}$, with unit volume.

Theorem (Sandier-S.)

The minimum of $\Lambda \mapsto W(\Lambda)$ over perfect lattice configurations is achieved uniquely, modulo rotations, by the triangular lattice (with 60^o angles).

Theorem (Cassels, Rankin, Ennola, Diananda, 50's)

Let

$$\zeta(s) = \sum_{p \in \Lambda} \frac{1}{|p|^s},$$

with Λ a lattice, be the Epstein zeta function. For $s > 2$, it is uniquely minimized among lattices of volume one, by the triangular lattice.

Conjecture

This "Abrikosov" triangular lattice is a global minimizer of W.

Dimension 2 : minimization among lattices

In 2D we can only identify the minimizers of W among perfect lattice configurations, i.e., $\Lambda = \mathbb{Z}\vec{u} + \mathbb{Z}\vec{v}$, with unit volume.

Theorem (Sandier-S.)

The minimum of $\Lambda \mapsto W(\Lambda)$ over perfect lattice configurations is achieved uniquely, modulo rotations, by the triangular lattice (with 60^o angles).

Theorem (Cassels, Rankin, Ennola, Diananda, 50's)

Let

$$\zeta(s) = \sum_{p\in\Lambda} \frac{1}{|p|^s},$$

with Λ a lattice, be the Epstein zeta function. For $s > 2$, it is uniquely minimized among lattices of volume one, by the triangular lattice.

Conjecture

This "Abrikosov" triangular lattice is a global minimizer of W.

Taking the limit $n \longrightarrow \infty$

We would like to obtain W as the limit $n \to \infty$ of the $W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2})$ term in the splitting formula

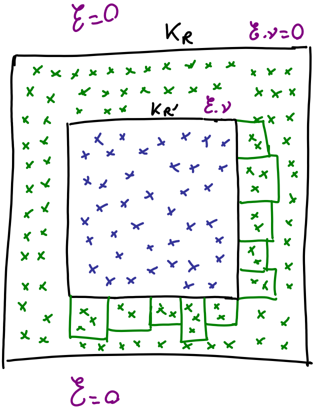
$$w_n(x_1, \cdots, x_n) = n^2 \mathcal{F}(\mu_0) - \frac{n}{d} \log n + \frac{1}{\pi} W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2}) + 2n \sum_{i=1}^{n} \zeta(x_i),$$

where H'_n tends after blow-up around $x_0 \in \Sigma$ to a solution of

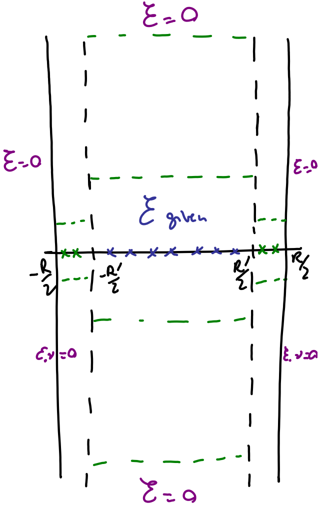
$$-\Delta H' = 2\pi \left(\sum_i \delta_{x'_i} - \mu_0(x_0) \delta_{\mathbb{R}} \right)$$

Two difficulties:

- lower bound needs an averaged formulation, since $W(\nabla H')$ depends on the blow-up center
- "screening" or "truncation" procedure is needed to "copy paste" together vector fields and build test configurations



$\xi=0$
K_R
$\xi\cdot\nu=0$
$K_{R'}$
$\xi\cdot\nu$
$\xi=0$



$\xi=0$
$\xi=0$
$\xi=0$
ξ given
$-\frac{R}{2}$
$-\frac{R'}{2}$
$\frac{R'}{2}$
$\frac{R}{2}$
$\xi\cdot\nu=0$
$\xi\cdot\nu=0$
$\xi=0$

The screening construction

Question: let $\mathcal{E} \in \mathcal{A}_1$ with $W(\mathcal{E}) < +\infty$. For $R \to \infty$ find $\mathcal{E}_R$ and R' such that $R' < R$ and $R'/R \to 1$ as $R \to \infty$ with

- $\mathcal{E}_R = \mathcal{E}$ in $K_{R'}$ (resp. near $[-R'/2, R'/2]$ in 1D)
- $\mathcal{E}_R = 0$ outside K_R (resp. outside $[-R'/2, R'/2] \times \mathbb{R}$ in 1D)
- $\mathrm{div}\, \mathcal{E}_R = 2\pi(\sum_{p \in \Lambda_R} \delta_p - \mathbf{1}_{K_R} \delta_{\mathbb{R}})$ (but NOT $\mathrm{curl}\, \mathcal{E}_R = 0$).
- $$\frac{1}{|K_R|} W(\mathcal{E}_R, \mathbf{1}_{K_R}) \leq W(\mathcal{E}) + o(1)$$

  resp.
  $$\frac{1}{R} W(\mathcal{E}_R, \mathbf{1}_{[-R/2, R/2] \times \mathbb{R}}) \leq W(\mathcal{E}) + o(1).$$

Such vector fields live on K_R (resp $[-R/2, R/2] \times \mathbb{R}$) and can thus be pasted together with other such vector fields. They do not satisfy $\mathrm{curl}\, \mathcal{E}_R = 0$, but this can be later corrected by adding a div-free vector field $\nabla^\perp \xi$ in $\mathbb{R}^2$. The situation is saved by the fact that this can only decrease W: $W(\mathcal{E} + \nabla^\perp \xi) \leq W(\mathcal{E})$!

- In dimension 2, it is possible to achieve this. In dimension 1, it is not! The reason is that there is more geometric freedom in 2D.
- go around this by showing it for a subclass of vector fields in $\mathcal{A}_m$ which show uniform decay away from the real axis.
- Fortunately these are generic in the sense that given an translation-invariant probability P as below we can find a set of $\mathcal{E}$ of almost-full measure for P which satisfy this decay (thanks to the ergodic theorem).
- This suffices to obtain the upper bound construction, including for minimizers of W.

The averaged formulation (Varadhan's method)

- Fix $1 < p < 2$ and let $X = \Sigma \times L^p_{loc}(\mathbb{R}^2, \mathbb{R}^2)$. For $\nu_n = \sum_{i=1}^n \delta_{x_i}$ we define $\mathcal{E}_n = \nabla H'_n$ (blow-up electric field around 0). We let P_n be the probability on X which is the push-forward of $\frac{1}{|\Sigma|} dx_{|\Sigma}$ by

$$x \mapsto \left(x, \mathcal{E}_n(n^{1/d}x + \cdot)\right).$$

- For configurations with suitably bounded energy, up to a subsequence, P_n converges to a probability P on X (tightness).

(i) The first marginal of P is $\frac{1}{|\Sigma|} dx_{|\Sigma}$. P is invariant by $(x, \mathcal{E}) \mapsto (x, \mathcal{E}(\lambda + \cdot))$, for any $\lambda \in \mathbb{R}^d$.

(ii) If the energy is suitably bounded, for $P - a.e. (x, \mathcal{E}), \mathcal{E} \in \mathcal{A}_{\mu_0(x)}$.

- P is like a "Young measure" on the blow-up patterns.
- The multiparameter ergodic theorem allows to obtain precisely

$$\liminf_{n\to\infty} \frac{1}{n} W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2}) \geq |\Sigma| \int W(\mathcal{E})\, dP(x, \mathcal{E}).$$

- general abstract method to obtain lower bounds for 2-scale energies which Γ-converge at small scale

The averaged formulation (Varadhan's method)

- Fix $1 < p < 2$ and let $X = \Sigma \times L^p_{loc}(\mathbb{R}^2, \mathbb{R}^2)$. For $\nu_n = \sum_{i=1}^n \delta_{x_i}$ we define $\mathcal{E}_n = \nabla H'_n$ (blow-up electric field around 0). We let P_n be the probability on X which is the push-forward of $\frac{1}{|\Sigma|} dx_{|\Sigma}$ by

$$x \mapsto \left(x, \mathcal{E}_n(n^{1/d}x + \cdot)\right).$$

- For configurations with suitably bounded energy, up to a subsequence, P_n converges to a probability P on X (tightness).

(i) The first marginal of P is $\frac{1}{|\Sigma|} dx_{|\Sigma}$. P is invariant by $(x, \mathcal{E}) \mapsto (x, \mathcal{E}(\lambda + \cdot))$, for any $\lambda \in \mathbb{R}^d$.

(ii) If the energy is suitably bounded, for $P - a.e.(x, \mathcal{E}), \mathcal{E} \in \mathcal{A}_{\mu_0(x)}$.

- P is like a "Young measure" on the blow-up patterns.
- The multiparameter ergodic theorem allows to obtain precisely

$$\liminf_{n\to\infty} \frac{1}{n} W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2}) \geq |\Sigma| \int W(\mathcal{E})\, dP(x, \mathcal{E}).$$

- general abstract method to obtain lower bounds for 2-scale energies which Γ-converge at small scale

The averaged formulation (Varadhan's method)

- Fix $1 < p < 2$ and let $X = \Sigma \times L^p_{loc}(\mathbb{R}^2, \mathbb{R}^2)$. For $\nu_n = \sum_{i=1}^n \delta_{x_i}$ we define $\mathcal{E}_n = \nabla H_n'$ (blow-up electric field around 0). We let P_n be the probability on X which is the push-forward of $\frac{1}{|\Sigma|} dx_{|\Sigma}$ by

$$x \mapsto \left(x, \mathcal{E}_n(n^{1/d} x + \cdot) \right).$$

- For configurations with suitably bounded energy, up to a subsequence, P_n converges to a probability P on X (tightness).

(i) The first marginal of P is $\frac{1}{|\Sigma|} dx_{|\Sigma}$. P is invariant by $(x, \mathcal{E}) \mapsto (x, \mathcal{E}(\lambda + \cdot))$, for any $\lambda \in \mathbb{R}^d$.

(ii) If the energy is suitably bounded, for $P - a.e. (x, \mathcal{E}), \mathcal{E} \in \mathcal{A}_{\mu_0(x)}$.

- P is like a "Young measure" on the blow-up patterns.
- The multiparameter ergodic theorem allows to obtain precisely

$$\liminf_{n \to \infty} \frac{1}{n} W(\nabla H_n', \mathbf{1}_{\mathbb{R}^2}) \geq |\Sigma| \int W(\mathcal{E})\, dP(x, \mathcal{E}).$$

- general abstract method to obtain lower bounds for 2-scale energies which Γ-converge at small scale

Γ-convergence of w_n / estimate of ground state energy

Theorem (Sandier-S)

As $n \to \infty$,

$$\frac{1}{n}\left(w_n(x_1,\dots,x_n) - n^2\mathcal{F}(\mu_0) + \frac{n}{d}\log n\right) \xrightarrow{\Gamma} \frac{|\Sigma|}{\pi}\int W(\mathcal{E})\,dP(x,\mathcal{E})$$

in the sense of convergence $P_n \to P$ weakly as probabilities, where P_n is the push-forward of $\frac{1}{|\Sigma|}dx_{|\Sigma}$ by $x \mapsto \left(x, \mathcal{E}_n(n^{1/d}x + \cdot)\right)$. Moreover, if the lhs is bounded, P satisfies (i)-(ii).

The minimum of the Γ-limit is

$$\frac{1}{\pi}\int \min_{\mathcal{A}_{\mu_0(x)}} W\,dx = \frac{1}{\pi}\min_{\mathcal{A}_1} W - \frac{1}{d}\int \mu_0 \log \mu_0 := \alpha_0$$

For minimizers, $P - a.e.$, $\mathcal{E}$ minimizes W over $\mathcal{A}_{\mu_0(x)}$.
"After blowup around a.e. point at scale $n^{1/d}$ we see a minimizer of W"

Improvement for minimizers (2D)

Theorem (Rota Nodari-S)

Let $(x_1, \ldots, x_n)$ *minimize* w_n, *and* $\mathcal{E}_n \to \mathcal{E}$ *the associated blown-up field. Then for any square* K_ℓ, *with* $\ell \geq c > 0$,

$$\left| \frac{W(\mathcal{E}, \chi_{K_\ell})}{|K_\ell|} - \fint_{K_\ell} \min_{\mathcal{A}_{\mu_0'(x)}} W \, dx \right| \leq o(1) \quad \textit{as } \ell \to \infty$$

$$\left| \frac{\nu(K_\ell)}{|K_\ell|} - \fint_{K_\ell} \mu_0'(x) \, dx \right| \leq \frac{C}{\ell^{1-\gamma}}, \quad \gamma \in (0,1).$$

Related to Alberti-Choksi-Otto.

Theorem (Rota Nodari-S)

Let $\mathcal{E}$ *minimize* $W(\mathcal{E}, \mathbf{1}_{K_L})$*, either:*

- among configurations in $\mathcal{A}_1$ *which are* L*-periodic*

- or among configurations in $\mathcal{A}_1$ *with* $\mathcal{E} \cdot \nu = \varphi$ *given on* ∂K_L *and*

$$\int_{\partial K_L} |\varphi|^p \leq CL^{2-\gamma}, \qquad 1 < p < 2, \gamma \in (0,1).$$

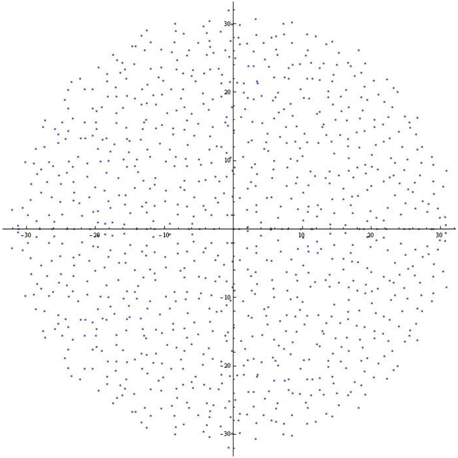
Then for any $\ell \geq c > 0$ *and any subsquare* $K_\ell \subset K_L$

$$\left| \frac{W(\mathcal{E}, \chi_{K_\ell})}{|K_\ell|} - \min_{\mathcal{A}_1} W \right| \leq o(1) \quad \text{as} \quad \ell \to \infty$$

$$\left| \frac{\nu(K_\ell)}{|K_\ell|} - 1 \right| \leq \frac{C}{\ell^{1-\gamma}},$$

uniformly w.r.t to φ *satisfying the bound.*

A simulation



Eigenvalues of 1000-by-1000 matrix with i.i.d Gaussian entries
($\beta = 2$, $\mu_0 = \frac{1}{\sqrt{\pi}}\mathbf{1}_{B_1}$**)** (Borrowed from Benedek Valkó's webpage)

“Large deviations type" result at next order

Theorem (Sandier-S.)

Let $A_n \subset (\mathbb{R}^d)^n$*. Then*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_n^\beta(A_n) \le -\beta\Big(\underbrace{\frac{|\Sigma|}{\pi} \inf_{P\in A} \int W(\mathcal{E})dP(x,\mathcal{E})}_{\text{min of this is } \alpha_0} - \alpha_0 - \frac{C}{\beta}\Big),$$

and A *is the set of probability measures which are limits of blow-ups at rate* $n^{1/d}$ *around a point* x *of the fields* $\mathcal{E}$ *associated to* $\sum_{i=1}^n \delta_{x_i}$ *with* $(x_i) \in A_n$*.*

- For β finite, the average of W lies below a fixed constant $(\alpha_0 + \frac{C}{\beta})$, except with very small probability.
- in dimension 1, it proves **crystallization** as $\beta \to \infty$: $\rightsquigarrow$ after blowing up around a point x in the support of μ_0, at rate $n^{1/d}$, we see (almost surely w.r.t. to x) a configuration which minimizes W. In dimension 2, it proves it modulo the conjecture on minimizers of W

"Large deviations type" result at next order

Theorem (Sandier-S.)

Let $A_n \subset (\mathbb{R}^d)^n$. Then

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_n^\beta(A_n) \le -\beta\Big(\underbrace{\frac{|\Sigma|}{\pi} \inf_{P\in A} \int W(\mathcal{E}) dP(x,\mathcal{E})}_{\text{min of this is } \alpha_0} - \alpha_0 - \frac{C}{\beta}\Big),$$

and A is the set of probability measures which are limits of blow-ups at rate $n^{1/d}$ around a point x of the fields $\mathcal{E}$ associated to $\sum_{i=1}^n \delta_{x_i}$ with $(x_i) \in A_n$.

- For β finite, the average of W lies below a fixed constant $(\alpha_0 + \frac{C}{\beta})$, except with very small probability.
- in dimension 1, it proves **crystallization** as $\beta \to \infty$: $\rightsquigarrow$ after blowing up around a point x in the support of μ_0, at rate $n^{1/d}$, we see (almost surely w.r.t. to x) a configuration which minimizes W. In dimension 2, it proves it modulo the conjecture on minimizers of W

The quantum case (Bosons)

$$\mathbf{H}_N = -N\sum_{i=1}^{N}\Delta_i - \frac{N}{N-1}\sum_{i\neq j}\log|x_i - x_j| + N\sum_{i=1}^{N}V(x_i)$$

- Find

$$\min_{\|\psi\|=1}\langle\psi, \mathbf{H}_N\psi\rangle \quad \text{among } \psi \in \otimes_{\text{sym}} L^2(\mathbb{R}^2).$$

- Trial wave function $\psi(x_1,\ldots,x_N) = u(x_1)\ldots u(x_N)$ leads to the Hartree energy

$$\mathbf{E}_{Har}(u) = \int_{\mathbb{R}^2}|\nabla u|^2$$
$$-\iint_{\mathbb{R}^2\times\mathbb{R}^2}\log|x-y||u|^2(x)|u|^2(y)\,dx\,dy + \int_{\mathbb{R}^2}V(x)|u|^2(x)\,dx$$

- $\mathbf{E}_{Har}$ has a unique minimizer u_0 with $\|u_0\|_{L^2} = 1$.
- In fact it is true that (Bose-Einstein condensation)

$$\min_{\|\psi\|=1}\langle\psi, \mathbf{H}_N\psi\rangle = N^2\mathbf{E}_{Har}(u_0) + o(N^2) \qquad \psi_0 \simeq u_0^{\otimes N}$$

The quantum case (Bosons)

$$\mathbf{H}_N = -N\sum_{i=1}^{N}\Delta_i - \frac{N}{N-1}\sum_{i\neq j}\log|x_i - x_j| + N\sum_{i=1}^{N}V(x_i)$$

- Find

$$\min_{\|\psi\|=1}\langle\psi, \mathbf{H}_N\psi\rangle \quad \text{among } \psi \in \otimes_{\mathrm{sym}} L^2(\mathbb{R}^2).$$

- Trial wave function $\psi(x_1, \ldots, x_N) = u(x_1)\ldots u(x_N)$ leads to the Hartree energy

$$\mathbf{E}_{Har}(u) = \int_{\mathbb{R}^2}|\nabla u|^2 - \iint_{\mathbb{R}^2\times\mathbb{R}^2}\log|x-y||u|^2(x)|u|^2(y)\,dx\,dy + \int_{\mathbb{R}^2}V(x)|u|^2(x)\,dx$$

- $\mathbf{E}_{Har}$ has a unique minimizer u_0 with $\|u_0\|_{L^2} = 1$.
- In fact it is true that (Bose-Einstein condensation)

$$\min_{\|\psi\|=1}\langle\psi, \mathbf{H}_N\psi\rangle = N^2\mathbf{E}_{Har}(u_0) + o(N^2) \qquad \psi_0 \simeq u_0^{\otimes N}$$

Next order term by Bogoliubov theory

Joint work with M. Lewin, P. Nam, JP Solovej.

- The next order term is in N and not $N \log N$
- The factor is given by the bottom of the spectrum of the Bogoliubov Hamiltonian $\mathbb{H}$, as predicted by Bogoliubov theory

$$\min_{\|\psi\|=1} \langle \psi, \mathbf{H}_N \psi \rangle = N^2 \mathbf{E}_{Har}(u_0) + N \inf \sigma(\mathbb{H}) + o(N)$$

- $\mathbb{H}$ is the "second quantization" of $\frac{1}{2} Hess\, \mathbf{E}_{Har}(u_0)(v, v)$
- in fact the whole spectrum converges (and eigenfunctions too)

- Any ψ can be decomposed uniquely as

$$\psi = \psi_0 u_0^{\otimes N} + \psi_1 \otimes_s u_0^{\otimes N-1} + \psi_2 \otimes_s u_0^{\otimes N-2} + \cdots + \psi_N$$

  where each ψ_k has k variables and $\in (\{u_0\}^\perp)^{\otimes k}$.

- $$U_N : \psi \mapsto \psi_0 \oplus \psi_1 \oplus \cdots \oplus \psi_N$$

  is a unitary operator into the "Fock space"
- in fact only the first few ones (excited states) really count
- the convergence of eigenfunctions (and $\mathbb{H}$) is expressed in terms of U_N
- we prove this in a more general setting (general two-body interaction, any dimension), as long as Bose-Einstein condensation happens and the Hessian is nondegenerate. Previous more restrictive results by Seiringer, Grech-Seiringer.
- works also with finite temperature

Some references

- E. Sandier, S. Serfaty, From the Ginzburg-Landau Model to Vortex Lattice Problems, *Comm. Math. Phys. 313* (2012), 635–743.
- E. Sandier, S. Serfaty, 2D Coulomb Gases and the Renormalized Energy, `arXiv`.
- E. Sandier, S. Serfaty, 1D log gases and the Renormalized Energy: Crystallisation at vanishing temperature, to appear.
- S. Rota Nodari, S. Serfaty, Renormalized Energy Equidistribution and Local Charge Balance in 2D Coulomb Systems, to appear.
- A. Borodin, S. Serfaty, Renormalized Energy Concentration in Random Matrices, to appear in *Comm. Math. Phys.*
- M. Lewin, P.T. Nam, J.P. Solovej, S. Serfaty, Bogoliubov Spectrum of Interacting Bose Gases, `arXiv`.