Partial regularity and smooth topology-preserving approximations of rough domains

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Abstract

For a bounded domain $\Omega \subset \mathbb{R}^m, m \geq 2$, of class C^0, the properties are studied of fields of 'good directions', that is the directions with respect to which $\partial\Omega$ can be locally represented as the graph of a continuous function. For any such domain there is a canonical smooth field of good directions defined in a suitable neighbourhood of $\partial\Omega$, in terms of which a corresponding flow can be defined. Using this flow it is shown that Ω can be approximated from the inside and the outside by diffeomorphic domains of class C^∞. Whether or not the image of a general continuous field of good directions (pseudonormals) defined on $\partial\Omega$ is the whole of S^{m-1} is shown to depend on the topology of Ω. These considerations are used to prove that if $m = 2, 3$, or if Ω has nonzero Euler characteristic, there is a point $P \in \partial\Omega$ in the neighbourhood of which $\partial\Omega$ is Lipschitz. The results provide new information even for more regular domains, with Lipschitz or smooth boundaries.

1 Introduction

In this paper we study bounded domains $\Omega \subset \mathbb{R}^m$, $m > 1$, of class C^0 and show that they may be approximated both from the inside and the outside by bounded domains Ω_ε of class C^∞ that are diffeomorphic to Ω, and such that $\overline{\Omega}_\varepsilon$ are homeomorphic to $\overline{\Omega}$. Thus the approximating smooth domains preserve topological properties of the rough domain; in particular, for instance, a simply-connected bounded domain of class C^0 can be approximated from the inside and outside by smooth simply-connected domains.

We recall that $\Omega \subset \mathbb{R}^m$, $m > 1$ is *a domain of class* C^0 (respectively *of class* C^r, $r = 1, 2, \ldots, \infty$, Lipschitz) if Ω is a connected open set such that for any point P belonging to the boundary $\partial\Omega$ there exist a $\delta > 0$ and an orthonormal coordinate system $Y \stackrel{\text{def}}{=} (y', y_m) = (y_1, y_2, \ldots, y_m)$ with origin at P, together with a continuous (respectively C^r, Lipschitz) function $f : \mathbb{R}^{m-1} \to \mathbb{R}$, such that

$$\Omega \cap B(P, \delta) = \{y \in \mathbb{R}^m : y_m > f(y'), |y| < \delta\} \tag{1.1}$$

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from which it follows that $\partial\Omega \cap B(P,\delta) = \{y \in \mathbb{R}^m : y_m = f(y'), |y| < \delta\}$ and $f(0) = 0$. We call the unit vector $n(P) = e_m(P)$ in the y_m-direction for this coordinate system a *pseudonormal* at P. More generally, if $P \in \mathbb{R}^m$ is not necessarily a boundary point, but is such that for the coordinate system Y with origin at P we have that (1.1) holds with $\partial\Omega \cap B(P,\delta)$ nonempty, we call the corresponding unit vector $n(P)$ *a good direction at P*. We show (Lemma 2.1) that the set of good directions at a point P is a (geodesically) convex subset of the unit sphere $\mathbb{S}^{m-1}$. Using a partition of unity we deduce (Proposition 2.1) that for a bounded domain of class C^0 there exists a field of good directions $G(P)$, that we call *canonical*, depending *smoothly* on P.

Although good directions and pseudonormals are defined locally, their properties depend on the topology of the domain. If Ω is a bounded domain of class C^1 then the negative Gauss map $\nu_{\partial\Omega} : \partial\Omega \to \mathbb{S}^{m-1}$, defined by $\nu_{\partial\Omega}(P) =$ the inward normal to $\partial\Omega$ at P, is surjective. We show (Theorem 6.1) that the same is true for an arbitrary continuous field of pseudonormals if $m = 2$ or if $m \geq 3$ and Ω has nonzero Euler characteristic. However if $\Omega \subset \mathbb{R}^3$ is a standard solid torus in $\mathbb{R}^3$ then, using an observation of Lackenby [21], we show (Proposition 6.1) that there is a continuous field of pseudonormals with image contained in an arbitrarily small neighbourhood of the great circle in S^2 perpendicular to the axis of cylindrical symmetry of the torus (see Fig. 4).

In our approximation result the approximating domains are given by

$$\Omega_\varepsilon \stackrel{\text{def}}{=} \{x \in \mathbb{R}^m : \rho(x) > \varepsilon\}, \quad 0 < |\varepsilon| < \varepsilon_0, \tag{1.2}$$

where ρ is the regularized signed distance to the boundary defined by Lieberman [22]. The regularized distance has the property that $\nabla\rho(P) \cdot G(P) > 0$ for a canonical good direction $G(P)$ in a suitable neighbourhood of $\partial\Omega$. This enables us to use the *flow of canonical good directions* $S(t)x_0$ defined as the solution of

$$\begin{aligned} &\dot{x}(t) = G(x(t)) \quad \text{for } t \in \mathbb{R}, \\ &x(0) = x_0, \end{aligned}$$

suitably extended so as to be globally defined, to provide the deformation showing that Ω and Ω_ε are C^∞-diffeomorphic and their closures are homeomorphic (Theorem 5.1).

A surprising by-product of our study of the topology of the set Ω and that of the good directions is the fact that in $\mathbb{R}^m$ for $m = 2, 3$ any bounded domain of class C^0 must necessarily have portions of the boundary with better regularity, namely Lipschitz regularity (Theorems 7.2, 7.3). This is true for arbitrary m if Ω has nonzero Euler characteristic (Theorem 7.1). However it is false for unbounded domains of class C^0 (Remark 7.1).

Let us now mention some related literature. Further results on domains of class C^0 and their properties are available in Fraenkel [8]. The relation between domains of class C^0 and their closure is addressed in Grisvard [9, Theorem 1.2.1.5]. Also in [9, Corollary 1.2.2.3], it is noted that a bounded open convex set necessarily has Lipschitz boundary. In a more general framework related types of questions were addressed in the paper of Pugh [30] which studies conditions under which a topological manifold can be smoothed: that is when its (maximal) topological atlas contains a smooth subatlas. Somewhat related domains are the 'cloudy manifolds' defined by Kleiner & Lott [18] which are subsets of an Euclidean space with the property that near each point they look "coarsely close" to an affine subspace of the Euclidean space. It was shown in [18] that any cloudy k-manifold can be well interpolated by a smooth k-dimensional submanifold of the Euclidean space. Another strand of research that can be compared with our partial regularity result for bounded domains of class C^0 is that described by Jones, Katz & Vargas [17], in which the authors prove and generalize a conjecture of Semmes [32] that for a bounded open set $\Omega \subset \mathbb{R}^m$ with $\mathcal{H}^{m-1}(\partial\Omega) = M < \infty$ there exist $\varepsilon > 0$ and a Lipschitz graph Γ with $\mathcal{H}^{m-1}(\Gamma \cap \partial\Omega) \geq \varepsilon$.

Finally in Hofmann, Mitrea & Taylor [16] a definition is given of a continuous vector field transversal to the boundary of an open set with locally finite perimeter, in which it is required that the inner product of the vector field with the normal is bounded away from zero. For the case of bounded domains of class C^0 (which need not have finite perimeter) this is a similar but stronger requirement than being a continuous field of good directions. A result [16, Proposition 2.2] analogous to our Proposition 2.1 is then proved giving conditions under which the existence of a continuous locally tranversal field implies the existence of a global smooth transversal field.

Although the questions addressed in this paper seem natural, we are not aware of any other work that studies the properties of good directions and their spatial variation, or of related approximation results preserving topological properties. We were motivated to consider these questions because in an analysis [1] of orientability for liquid crystals we wanted to use a result of Pakzad & Rivière [29], which assumed Ω to be of class C^∞ and simply-connected, for more general simply-connected domains (though recent work of Bedford [2] gives a way of proving the desired orientability without approximation of the domain). Other potential applications include problems in which the domain is an unknown (such as optimal design), for which the flow of good directions might be used to construct domain variations, and various problems in partial differential equations [14] and potential theory [33].

2 Good directions

Definition 2.1. *Let $\Omega \subset \mathbb{R}^m$ be a domain of class C^0. For a point $P \in \mathbb{R}^m$ we define a* good direction at P, with respect to a ball $B(P, \delta)$, *$\delta > 0$, with $B(P,\delta) \cap \partial\Omega \neq \emptyset$ to be a vector $n \in \mathbb{S}^{m-1}$ such that there is an orthonormal coordinate system $Y = (y', y_m) = (y_1, y_2, \ldots, y_m)$ with origin at the point P and such that $n = e_m$ is the unit vector in the y_m direction, together with a continuous function $f : \mathbb{R}^{m-1} \to \mathbb{R}$ (depending on P and n and δ), such that*

$$\Omega \cap B(P, \delta) = \{y \in \mathbb{R}^m : y_m > f(y'), |y| < \delta\}. \tag{2.1}$$

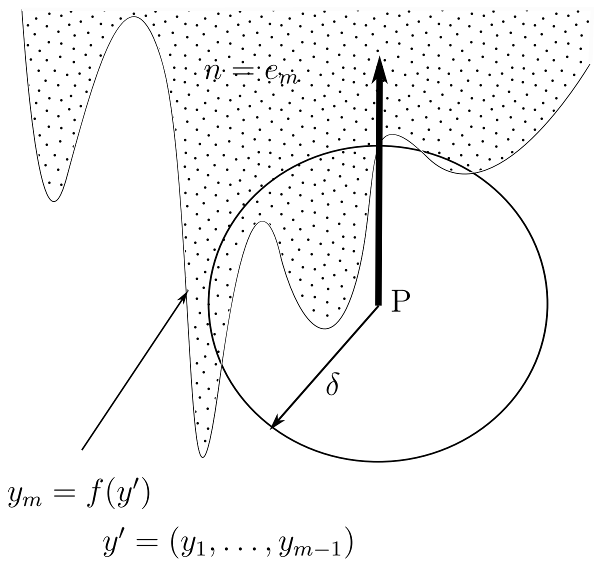


Figure 1: A good direction at the point P

We say that n is a good direction at P *if it is a good direction with respect to some ball $B(P,\delta)$ with $B(P,\delta)\cap\partial\Omega\neq 0$.*

If $P\in\partial\Omega$ then a good direction n at P is called a pseudonormal *at P.*

Remark 2.1. A good direction at a point need not be unique but for a bounded domain of class C^0 there always exists at least one for each point in a (small enough) neighbourhood of $\partial\Omega$. Note also that if both n and $\bar{n}$ are good directions at P then we can choose $\delta=\min\{\delta(P,n),\delta(P,\bar{n})\}$ so that the corresponding two representations (2.1) hold for δ. However a possible choice of $\delta(P,n)$ may not be a possible choice of $\delta(P,\bar{n})$.

Remark 2.2. If one has for instance a domain in $\mathbb{R}^2$ such that part of its boundary can be locally represented as $\{(x,f(x),x\in(-1,1)\}$ with $f(x)=\sqrt{|x|}$ then there is only one good direction at the point $(0,0)$ namely $(0,1)\in\mathbb{S}^1$. This suggests that there exists a connection between the uniqueness of a good direction and the regularity of the boundary, and this topic will be explored in detail in the last section.

We refer to a subset S of a Riemannian manifold M as *geodesically convex* if given any two points $P,Q\in S$ there is a unique shortest curve (minimal geodesic) in M joining P and Q, and this curve lies in S (there are several differing definitions in the literature). The following lemma asserts that the set of good directions at any given point is a geodesically convex subset of S^{m-1}.

Lemma 2.1. *Let $\Omega\subset\mathbb{R}^m$ be a (possibly unbounded) domain of class C^0. If $p,q\in\mathbb{S}^{m-1}$ are good directions at a point $P\in\mathbb{R}^m$ with respect to the ball $B(P,\delta)$ then $p\neq -q$ and for any $\lambda\in(0,1)$ the vector $\frac{\lambda p+(1-\lambda)q}{|\lambda p+(1-\lambda)q|}$ is a good direction at P with respect to the ball $B(P,\delta)$.*

Proof. Let p,q be good directions at P, and $0<\lambda<1$. We can assume that $P=0$. Let $B\overset{\text{def}}{=}B(0,\delta)$. If $p=q$ there is nothing to prove. So assume $p\neq q$. Then Definition 2.1 implies that $p\neq -q$, and so $\lambda p+(1-\lambda)q\neq 0$. Let $N=\frac{\lambda p+(1-\lambda)q}{|\lambda p+(1-\lambda)q|}$. We choose coordinates such that $e_m=N$. Take any $\xi\in B\backslash\Omega$. Then since p and q are good directions the intersections of B with the open half-lines $\{\xi+tp:t<0\}$ and $\{\xi+tq:t<0\}$ lie in $\mathbb{R}^m\backslash\overline{\Omega}$. Similarly, if $\xi\in B\cap\overline{\Omega}$ the intersections of B with the open half-lines $\{\xi+tp:t>0\}$ and $\{\xi+tq:t>0\}$ lie in Ω.

Given any $\xi'\in B_{m-1}(0,\delta)=\{z\in\mathbb{R}^{m-1}:|z|<\delta\}$ define the line $L(\xi')=\{(\xi',t):t\in\mathbb{R}\}$, and let $S=\{\xi'\in B_{m-1}(0,\delta):L(\xi')\cap\partial\Omega\cap B\text{ is nonempty}\}$. We claim that if $\xi'\in S$ then $L(\xi')$ intersects $B\cap\partial\Omega$ in a unique point $(\xi',t(\xi'))$, and that $\{(\xi',t):t>t(\xi')\}\cap B\subset\Omega$ and $\{(\xi',t):t<t(\xi')\}\cap B\subset\mathbb{R}^m\backslash\overline{\Omega}$. To prove the claim let $\xi=(\xi',t(\xi'))\in B\cap\partial\Omega$, and suppose $\xi-he_m\in B$ for some $h>0$. Then for some $\varepsilon>0$, $\operatorname{dist}(\xi,\partial B)>\varepsilon$, $\operatorname{dist}(\xi-he_n,\partial B)>\varepsilon$. Choose a positive integer $k>\frac{h}{\varepsilon|\lambda p+(1-\lambda)q|}$, and divide the interval $(0,h)$ into k subintervals of length h/k. Then $\bar{\xi}=\xi-\frac{h}{k}\cdot\frac{(1-\lambda)q}{|\lambda p+(1-\lambda)q|}\in B$, and so $\bar{\xi}\in\mathbb{R}^m\backslash\overline{\Omega}$. Thus $\operatorname{dist}(\bar{\xi},\partial B)>\varepsilon$ and $\xi-\frac{h}{k}e_m=\bar{\xi}-\frac{h}{k}\cdot\frac{\lambda p}{|\lambda p+(1-\lambda)q|}\in\mathbb{R}^m\backslash\overline{\Omega}$. Proceeding inductively, after k steps we find that $\xi-he_m\in\mathbb{R}^m\backslash\overline{\Omega}$, so that and $\{(\xi',t):t<t(\xi')\}\cap B\subset\mathbb{R}^m\backslash\overline{\Omega}$. It follows similarly that $\{(\xi',t):t>t(\xi')\}\cap B\subset\Omega$, establishing the claim.

Now define

$$f(\xi')=\begin{cases}0 & \text{if } |\xi'|=\delta\\ t(\xi') & \text{if } \xi'\in S\\ -\sqrt{\delta^2-|\xi'|^2} & \text{if } L(\xi')\cap B=L(\xi')\cap B\cap\Omega\\ \sqrt{\delta^2-|\xi'|^2} & \text{if } L(\xi')\cap B=L(\xi')\cap B\cap(\mathbb{R}^m\backslash\overline{\Omega}).\end{cases}\tag{2.2}$$

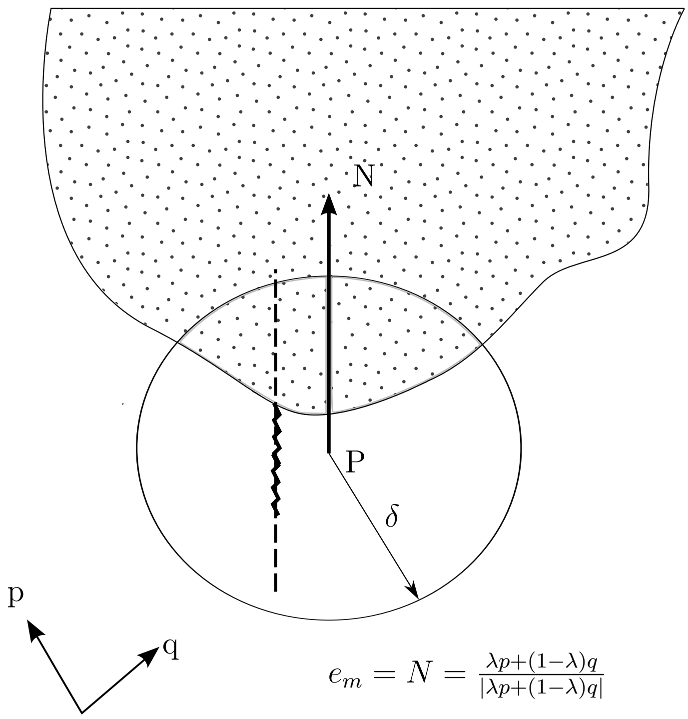


Figure 2: Geodesically convex combination of two good directions

Note that $(\xi', f(\xi')) \in \bar{B}$ for all $\xi' \in B_{m-1}(0,\delta)$. Clearly $\Omega \cap B = \{(\xi', \xi_m) \in B : |\xi'| < \delta, \xi_m > f(\xi')\}$, and it remains to prove that f is continuous, since then we can extend f by zero for $|\xi'| > \delta$ to get a suitable continuous $f : \mathbb{R}^{m-1} \to \mathbb{R}$. Let $\xi'^{(j)} \to \xi'$ in $\overline{B_{m-1}(0,\delta)}$. If $|\xi'| = \delta$ then $|(\xi'^{(j)}, f(\xi'^{(j)}))| \leq \delta$ implies $|\xi'^{(j)}|^2 + f^2(\xi'^{(j)}) \leq \delta^2$ and so $f(\xi'^{(j)}) \to 0 = f(\xi')$. If $\xi' \in S$ then $(\xi', f(\xi')) \in B \cap \partial\Omega$ and so for any sufficiently small $\varepsilon > 0$ the points $x_\varepsilon^+ = (\xi', f(\xi') + \varepsilon)$ and $x_\varepsilon^- = (\xi', f(\xi') - \varepsilon)$ belong to $\Omega \cap B$ and to $(\mathbb{R}^m \backslash \overline{\Omega}) \cap B$ respectively. Since Ω and $\mathbb{R}^m \backslash \overline{\Omega}$ are open, there exists $\delta \in (0,\varepsilon)$ such that $B(x_\varepsilon^+, \delta) \subset \Omega \cap B$ and $B(x_\varepsilon^-, \delta) \subset (\mathbb{R}^m \backslash \overline{\Omega}) \cap B$. Hence for sufficiently large j, the line $L(\xi'^{(j)})$ has points in both $B(x_\varepsilon^+, \delta)$ and $B(x_\varepsilon^-, \delta)$ and thus intersects $\partial\Omega$ in B at the unique point $(\xi'^{(j)}, f(\xi'^{(j)}))$, where $|f(\xi'^{(j)}) - f(\xi')| < \varepsilon$. Since ε is arbitrarily small, $f(\xi'^{(j)}) \to f(\xi')$. Similarly, if $L(\xi') \cap B = L(\xi') \cap \Omega$ so that $f(\xi') = -\sqrt{\delta^2 - |\xi'|^2}$, then for all sufficiently small $\varepsilon > 0$ there is a $\delta \in (0,\varepsilon)$ such that the ball $B((\xi', f(\xi') + \varepsilon), \delta) \subset \Omega \cap B$. Hence for all sufficiently large j we have $f(\xi'^{(j)}) \leq f(\xi') + \varepsilon$, so that $f(\xi'^{(j)}) \to f(\xi')$. The case when $L(\xi') \cap B = L(\xi') \cap B \cap (\mathbb{R}^m \backslash \overline{\Omega})$ is handled in a similar way. □

The above can be easily extended to the case of an arbitrary number of good directions:

Lemma 2.2. *Let $k = 1, 2, \ldots$. If $n_1, n_2, \ldots, n_k \in \mathbb{S}^{m-1}$ are good directions at a point P with respect to the ball $B(P,\delta)$ and $0 < \lambda_i < 1, i = 1, 2, \ldots, k$, with $\Sigma_{i=1}^k \lambda_i = 1$ then $\Sigma_{i=1}^k \lambda_i n_i \neq 0$ and $\frac{\Sigma_{i=1}^k \lambda_i n_i}{|\Sigma_{i=1}^k \lambda_i n_i|}$ is a good direction at P with respect to $B(P,\delta)$.*

Proof. This follows easily from Lemma 2.1 by induction on k, noting that $\sum_{i=1}^{k} \lambda_i n_i$ is parallel to

$$\mu \frac{\sum_{i=1}^{k-1} \lambda_i n_i}{\left|\sum_{i=1}^{k-1} \lambda_i n_i\right|} + (1-\mu) n_k, \text{ where } \mu = \frac{\left|\sum_{i=1}^{k-1} \lambda_i n_i\right|}{\lambda_k + \left|\sum_{i=1}^{k-1} \lambda_i n_i\right|}.$$

□

Despite the fact that the boundary is just of class C^0 we can easily construct a smooth field of good directions in a neighbourhood of the boundary:

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded, open set with boundary of class C^0. There exists a neighbourhood U of $\partial\Omega$ and a smooth function $G : U \to \mathbb{S}^{m-1}$ so that for each $P \in U$ the unit vector $G(P)$ is a good direction.*

Proof. As Ω is of class C^0, for each point $\bar{P} \in \partial\Omega$ there is a good direction $n_{\bar{P}}$ at $\bar{P}$, with corresponding $\delta = \delta(\bar{P})$. Then $n_{\bar{P}}$ is a good direction at any $P \in B(\bar{P}, \frac{1}{2}\delta(\bar{P}))$ since for such P we have $\bar{P} \in B(P, \frac{1}{2}\delta(\bar{P})) \subset B(\bar{P}, \delta(\bar{P}))$. As $\partial\Omega$ is compact, there exist $P_i, i = 1, \dots, k$, such that $\partial\Omega \subset U \stackrel{\text{def}}{=} \cup_{i=1}^{k} B(P_i, \frac{1}{4}\delta(P_i))$. Consider a partition of unity subordinate to the covering $\{B(P_i, \frac{1}{2}\delta(P_i))\}, i = 1, \dots, k$, of $\bar{U}$, namely functions $\alpha_i \in C^\infty(\mathbb{R}^m, \mathbb{R}_+), i = 1, 2, \dots, k$ with $\operatorname{supp} \alpha_i \subset B(P_i, \frac{1}{2}\delta(P_i))$ and $\Sigma_{i=1}^{k} \alpha_i = 1$ in $\bar{U}$. If $P \in U$ and $i \in S_P \stackrel{\text{def}}{=} \{j \in \{1, 2, \dots, k\} : P \in B(P_j, \frac{1}{2}\delta(P_j))\}$ then n_{P_i} is a good direction at P with respect to the ball $B(P, \Delta(P))$ where $\Delta(P) \stackrel{\text{def}}{=} \frac{1}{2} \min_{i \in S_p} \delta(P_i)$. It then follows from Lemma 2.2 that

$$G(P) \stackrel{\text{def}}{=} \frac{\Sigma_{i=1}^{k} \alpha_i(P) n_{P_i}}{|\Sigma_{i=1}^{k} \alpha_i(P) n_{P_i}|}, \quad \text{for all } P \in U \tag{2.3}$$

has the required property. □

Definition 2.2. *We call a field of good directions, constructed by* (2.3), *a* canonical field of good directions.

3 A proper generalized distance

For a bounded open set Ω define the signed distance $d(x)$ to the boundary $\partial\Omega$ by

$$d(x) = \begin{cases} \inf_{y \in \partial\Omega} |x - y| & \text{if } x \in \Omega \\ -\inf_{y \in \partial\Omega} |x - y| & \text{if } x \notin \Omega. \end{cases} \tag{3.1}$$

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain of class C^0. There exists a function $\rho \in C^\infty(\mathbb{R}^m \setminus \partial\Omega) \cap C^{0,1}(\mathbb{R}^m)$ such that*

$$\frac{1}{2} \le \frac{\rho(x)}{d(x)} \le 2, \ \textit{for all } x \in \mathbb{R}^m \setminus \partial\Omega \tag{3.2}$$

and

$$|\nabla \rho(x)| \neq 0 \textit{ for all } x \textit{ in a neighbourhood of } \partial\Omega, \quad x \notin \partial\Omega \tag{3.3}$$

Proof. We let ρ be a regularized distance function, as constructed by Lieberman [22] (following related earlier work of Fraenkel [7]). To define it let $\varphi \in C^\infty(\mathbb{R}^m)$ be a nonnegative function, whose support is the unit ball and is such that $\int_{\mathbb{R}^m} \varphi(x)\, dx = 1$. For $x \in \mathbb{R}^m, \tau \in \mathbb{R}$, let

$$G(x,\tau) \stackrel{\text{def}}{=} \int_{|z|<1} d\left(x - \frac{\tau}{2}z\right) \varphi(z)\, dz. \tag{3.4}$$

Since d is 1-Lipschitz, $|\partial G/\partial \tau| \le 1/2$, and so there is a unique solution $\rho(x)$ of the equation $\rho(x) = G(x, \rho(x))$. Thus defined, ρ is a Lipschitz function, smooth outside $\partial\Omega$, that satisfies (3.2) but not necessarily (3.3) (see [22], Lemma 1.1 and the comments following it).

We continue by proving (3.3) for x in a neighbourhood of the boundary and $x \in \Omega$. To this end we consider a point $P \in \partial\Omega$. Without loss of generality we can suppose that $P = 0$ and that in a suitable coordinate system there exist $\delta > 0$ and a continuous $f : \mathbb{R}^{m-1} \to \mathbb{R}$ such that

$$\mathcal{U}_\delta \stackrel{\text{def}}{=} \Omega \cap B(0,\delta) = \{y \in \mathbb{R}^m : y_m > f(y'),\ |y| < \delta\}. \tag{3.5}$$

Denoting by e_m the unit vector in the y_m direction, let $y, y + he_m \in \mathcal{U}_{\delta/4}$ for some $h > 0$. Then $h < \delta/2$. Since $0 \in \partial\Omega$, $d(y) \le |y|$. If $v \in \overline{B(y, d(y))}$ then

$$|v| \le |v - y| + |y| \le d(y) + |y| \le 2|y| \le \delta/2.$$

Hence $\overline{B(y, d(y))} \subset U_\delta$. We claim that

$$d(y + he_m) > d(y). \tag{3.6}$$

If not, there would exist $w \in \partial\Omega$ with $|w - y - he_m| \le d(y)$. Thus $w - he_m \in \overline{B(y, d(y))}$ and so $w - he_m \in U_\delta$ and $w_m - h \ge f(w') = w_m$, a contradiction. Since d is Lipschitz it follows that the derivative $\frac{\partial d}{\partial x_m}$ exists a.e. with strictly positive integral on every line segment in $\mathcal{U}_{\delta/4}$ parallel to e_m. By the definition of weak derivatives

$$\begin{aligned}
\frac{\partial G}{\partial x_m}(x,\tau) &= \frac{\partial}{\partial x_m}\int_{\mathbb{R}^m} d\left(x - \frac{\tau}{2}z\right)\varphi(z)\, dz \\
&= \frac{\partial}{\partial x_m}\int_{\mathbb{R}^m} d(\zeta)\varphi\left(\frac{2}{\tau}(x-\zeta)\right)\left(\frac{2}{\tau}\right)^m d\zeta \qquad (3.7)\\
&= \int_{\mathbb{R}^m} d\left(x - \frac{\tau}{2}z\right)\frac{2}{\tau}\varphi_{,m}(z)\, dz \\
&= \int_{\{|z|<1\}} \frac{\partial d}{\partial x_m}\left(x - \frac{\tau}{2}z\right)\varphi(z)\, dz. \qquad (3.8)
\end{aligned}$$

Suppose now that $x \in \mathcal{U}_{\delta/8}$. Then for $|z| < 1$ we have $x - \frac{\rho(x)}{2}z \in \Omega$ and $|x - \frac{\rho(x)}{2}z| \le \frac{\delta}{8} + d(x) < \frac{\delta}{4}$, where we have used (3.2). Hence, since the partial derivatives equal the weak derivatives almost everywhere in Ω, by Fubini's theorem $\frac{\partial G}{\partial x_m}(x, \rho(x)) > 0$, and so differentiating $\rho(x) = G(x, \rho(x))$ we obtain

$$\frac{\partial \rho}{\partial x_m}(x) = \frac{\frac{\partial G}{\partial x_m}}{1 - \frac{\partial G}{\partial \tau}} > 0. \tag{3.9}$$

Thus every point $P \in \partial\Omega$ has a neighbourhood $\mathcal{U}(P)$ such that $|\nabla\rho(x)| \ne 0$ for $x \in \mathcal{U}(P) \cap \Omega$. By compactness this implies that there is a neighbourhood $\mathcal{U}$ of $\partial\Omega$ such that $|\nabla\rho(x)| \ne 0$ for $x \in \mathcal{U} \cap \Omega$.

The case when x is in a neighbourhood of the boundary $\partial\Omega$ but $x \in \mathbb{R}^m \setminus \overline{\Omega}$ is treated similarly. □

Remark 3.1. For $\Omega \subset \mathbb{R}^m$ bounded, of class C^0, the compactness of $\partial\Omega$ implies that there exists a neighbourhood $\mathcal{U}$ of $\partial\Omega$ such that $\mathcal{U} \subset \cup_{i=1}^k B(P_i, \delta_i)$ where $k \geq 1$ and, for all $i = 1, \ldots, k$, $P_i \in \partial\Omega$ and at P_i there is a good direction $n_i \in \mathbb{S}^{m-1}$ with respect to the ball $B(P_i, 8\delta_i)$. Then relation (3.9) in the previous proof shows that for any $P \in \mathcal{U} \setminus \partial\Omega$ we have

$$\frac{\partial \rho}{\partial n_j}(P) = n_j \cdot \nabla\rho(P) > 0$$

for those $j \in \{1, \ldots, k\}$ such that $P \in B(P_j, \delta_j)$.

Moreover, for any n that is a convex combination of those good directions n_j, $j \in \{1, \ldots, k\}$ with $P \in B(P_j, \delta_j)$, we have

$$\frac{\partial \rho}{\partial n}(P) > 0.$$

Remark 3.2. We claim now that for any $R \in \mathcal{U}$ (with $\mathcal{U}$ as in Remark 3.1) and any n that is a convex combination of those good directions n_j, $j \in \{1, \ldots, k\}$, such that $R \in B(P_j, \delta_j)$, we have that n is also a good direction at R and there exists $\delta_n > 0$ such that

$$\rho(R + sn) < \rho(R + tn)$$

for all $s, t \in (-\delta_n, \delta_n)$ with $s < t$.

If $R \notin \partial\Omega$ then Remark 3.1 suffices for obtaining the claim. If $R \in \partial\Omega$ we consider the function $h : [-1, 1] \to \mathbb{R}$ defined by $h(\tau) = \rho(R + \tau n)$. Then Remark 3.1 ensures that $h'(\tau) > 0$ for $\tau \in (-\delta_n, 0) \cup (0, \delta_n)$ for some $\delta_n > 0$. This fact, together with $h(0) = 0$ and $h(\tau)\tau > 0$ for $\tau \in (-\delta_n, 0) \cup (0, \delta_n)$ (since n is a pseudonormal at R) suffices to obtain the claim in this case as well.

4 The flow of canonical good directions

We continue working with $\Omega \subset \mathbb{R}^m$ a bounded domain of class C^0. We take a function $\gamma \in C^\infty(\mathbb{R}^m, \mathbb{R}_+)$ so that $\operatorname{supp}\gamma \subset U$ (where U is as in Proposition 2.1 and $U \subset \mathcal{U}$ with $\mathcal{U}$ as in Remark 3.1) with $\gamma \equiv 1$ on $\overline{W}$ where $W = \{x \in \mathbb{R}^m : |d(x, \partial\Omega)| < \varepsilon\}$ and $\varepsilon > 0$ is small enough so that $W \subset U$, and $\gamma \leq 1$ on $\mathbb{R}^m \setminus \overline{W}$. Let $G : U \to \mathbb{S}^{m-1}$ be the function from Proposition 2.1, so that $G(P)$ is a good direction at P. Let $S(t)x_0$ denote the solution at time $t \in \mathbb{R}$ of the system:

$$\dot{x}(t) = \begin{cases} \gamma(x(t))G(x(t)) & \text{for } t \in \mathbb{R},\ x(t) \in U \\ 0 & \text{for } t \in \mathbb{R},\ x(t) \notin U \end{cases} \tag{4.1}$$

with initial data $x(0) = x_0$.

From now on we call the globally defined flow $S(\cdot)(\cdot) : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ *the flow of canonical good directions.*

We first show that the regularized distance to the boundary increases along this flow.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain of class C^0. Let $P \in U$, with $\gamma(P) \neq 0$ (where U and γ are defined at the beginning of the section). Let $\rho(R)$ denote the regularized distance from the point $R \in \mathbb{R}^m$ to $\partial\Omega$ as defined in Proposition* 3.1. *Then*

$$\rho\left(S(\mu_1)P\right) < \rho\left(S(\mu_2)P\right), \text{ for all } 0 \leq \mu_1 < \mu_2. \tag{4.2}$$

Proof. We proceed in two steps:

Step 1. We claim that for any point $P \in U$ with $\gamma(P) \neq 0$ we have $\rho(S(\mu_1)P) \leq \rho(S(\mu_2)P)$ for $0 \leq \mu_1 < \mu_2$. Then $\gamma(S(t)P) \neq 0$ for all $t \geq 0$.

We consider the Euler polygonal approximation of the system (4.1), on the interval $[0, \mu_2 + 1]$. This is obtained by linearly interpolating between the points P_k defined recursively by:

$$P_{k+1} = P_k + h\gamma(P_k)G(P_k), \ \ h = \frac{\mu_2 + 1}{l}, \ \ k = 0, \dots, l-1,$$

where $P_0 \stackrel{\text{def}}{=} P$.

Thus we have the approximate solution $S_l(t) = P_k + (t - k\frac{\mu_2+1}{l})\gamma(P_k)G(P_k)$ for all $t \in [k\frac{\mu_2+1}{l}, (k+1)\frac{\mu_2+1}{l}], k = 0, \dots, l-1$. Note that by the convergence of the Euler approximation, for h small enough $\gamma(P_k) \neq 0$ for all $k = 0, \dots, l-1$. Using then Remark 3.2 we have that for h small enough ρ is an increasing function along $S_l(t), t \in [0, \mu_2 + 1]$. Passing to the limit $l \to \infty$ we have that ρ is a non-decreasing function along the limit function $S(t), t \in [0, \mu_2 + 1]$ that is also a solution of the system (4.1).

Step 2. We claim now that for any $\mu_1 < \mu_2$ we have $\rho(S(\mu_1)P) < \rho(S(\mu_2)P)$. To this end we claim first that for any $\varepsilon < (0, \mu_2 - \mu_1)$ there exists an interval $(a, b) \subset (\mu_1, \mu_1 + \varepsilon)$ and a subset $B \subset \{1, 2, \dots, k\}$ so that $\alpha_i(S(t)P) \neq 0$ for all $t \in (a, b), i \in B$ and $\alpha_i(S(t)P) = 0$ for all $t \in (a, b), i \in \{1, 2, \dots, k\} \setminus B$ (where the functions α_i are those used in the definition of G in the proof of Proposition 2.1).

In order to prove the claim let $M_i \stackrel{\text{def}}{=} \{t \in [\mu_1, \mu_1 + \varepsilon] : \alpha_i(S(t)P) > 0\}$. Then each $M_i, 1 \leq i \leq k$ is relatively open in $[\mu_1, \mu_1 + \varepsilon]$ and the M_i cover $[\mu_1, \mu_1 + \varepsilon]$. Each ∂M_i is closed and nowhere dense. Hence by the Baire category theorem (for a finite number of sets) $\cup_{i=1}^k \partial M_i$ is closed and nowhere dense. Let $(a, b) \subset \left(\cup_{i=1}^k \partial M_i\right)^c$. Then $B(t) \stackrel{\text{def}}{=} \{i \in \{1, \dots, k\} : t \in M_i\}$ is constant in (a, b) and we can take $B = B(t)$, thus finishing the proof of the claim.

We consider now the function $S(t)P$ for $t \in (a, b)$ with $(a, b) \subset (\mu_1, \mu_1 + \varepsilon)$ as in the claim above. Let us denote $R \stackrel{\text{def}}{=} S(a)P$ and let $B = \{i_1, i_2, \dots, i_j\}$. Then $S(a+s)P = S(s)R$ and we have (for $s < b - a$) that

$$\begin{aligned} S(a+s)P &= R + \sum_{r=1}^{j} n_{i_r} \int_0^s \frac{\gamma(S(\tau)R)\alpha_{i_r}(S(\tau)R)}{|\sum_{r=1}^{j} n_{i_r}\alpha_{i_r}(S(\tau)R)|} d\tau \\ &= R + \sum_{r=1}^{j} n_{i_r}\xi_{i_r} = R + \left(\sum_{r=1}^{j} \xi_{i_r}\right)\left(\sum_{p=1}^{j} \frac{\xi_{i_p}}{\left(\sum_{r=1}^{j} \xi_{i_r}\right)} n_{i_p}\right) \end{aligned} \tag{4.3}$$

where we denote $\xi_{i_r} \stackrel{\text{def}}{=} \int_0^s \frac{\gamma(S(\tau)R)\alpha_{i_r}(S(\tau)R)}{|\sum_{r=1}^{j} n_{i_r}\alpha_{i_r}(S(\tau)R)|} d\tau > 0$, $r = 1, \dots, j$. By our choice of the set of indices B, we have that n_{i_p} is a good direction at R for all $i_p \subset B$, $p = 1, \dots, j$, and thus their convex combination $\left(\sum_{p=1}^{j} \frac{\xi_{i_p}}{\left(\sum_{r=1}^{j} \xi_{i_r}\right)} n_{i_p}\right)$ is also a good direction at R. Using then Remark 3.2 we have that $\rho(S(a)P) < \rho(S(a+s)P)$ for $\varepsilon > 0$ small enough and arbitrary $s \in (0, \varepsilon)$, which combined with Step 1 completes the proof. □

We study now the asymptotic properties of the flow of canonical good directions.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain of class C^0. Let*

$$S(\cdot)(\cdot) : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$$

be the flow of canonical good directions as defined at the beginning of the section. Then:

(i) *there exists $t_- \in \mathbb{R}$ such that*

$$\rho(S(t)P) < 0, \text{ for all } t \le t_-, P \in \overline{W}. \tag{4.4}$$

(ii) *there exists $t_+ \in \mathbb{R}$ such that*

$$\rho(S(t)P) > 0, \text{ for all } t \ge t_+, P \in \overline{W}. \tag{4.5}$$

where $W \subset U$ is as defined at the beginning of the section.

Proof. We consider case (i) and show that the flow starting from $P \in \overline{W}$ with $\rho(P) \ge 0$ crosses the boundary uniformly in time, the case (ii) being similar. Let us denote by $Z \subset U$ the α-limit set of the solution $S(t)P$. Then Z is a compact, invariant set that attracts P along the flow S (see, for instance, [11, Lemma 3.1.1]).

We show first that the conclusion is true if we let t_- depend on P. We argue by contradiction and assume that the conclusion is false, so that there exists a sequence $t_k \to -\infty$ such that $\rho(S(t_k)P) \ge 0$. As $\rho(S(t)P)$ is increasing and bounded from below we have that $\lim_{t \to -\infty} \rho(S(t)P) = l \ge 0$ and $\rho(Q) = l$ for all $Q \in Z$. As Z is invariant with respect to the flow of good directions S we have that for $Q \in Z$ we also have $S(t)Q \in Z$ for all $t \le 0$ and $\rho(S(t)Q) = l$ for all $t \le 0$, which contradicts Lemma 4.1. In order to prove that t_- can be chosen independent of P we assume for contradiction that this is not possible, so that there exists a sequence $\{P_k\}_{k \in \mathbb{N}} \subset \overline{W}$ and a corresponding sequence of times $\{t_k\}_{k \in \mathbb{N}}$ so that $\rho(S(t_k)P_k) = 0$ and $t_k \to -\infty$. Using the compactness of $\overline{W}$ we can find a subsequence $P_{k_l} \to P_0 \in \overline{W}$. But for P_0 there exists a time $t_0 < 0$ such that $\rho(S(t_0)P) < 0$ and using the continuity with respect to the initial data for the solution of the system (4.1) we obtain a contradiction. □

Remark 4.1. Relations (4.4) and (4.5) still hold if 0 is replaced by ε for $0 < |\varepsilon| < \varepsilon_0$, for suitable $t_-^\varepsilon, t_+^\varepsilon$.

5 Homeomorphically and C^∞ diffeomorphically equivalent approximations of rough domains

In this section we provide an application of the tools developed in the previous sections by showing that one can approximate from the inside (and also from the outside) domains Ω of class C^0 by smooth domains Ω' such that Ω and Ω' are C^∞-diffeomorphic and their closures are homeomorphic.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^m, m \ge 2$ be a bounded domain of class C^0. Let ρ be a regularized distance as given in Proposition 3.1. There exists $\varepsilon_0 = \varepsilon_0(\Omega) > 0$ such that if $0 < |\varepsilon| < \varepsilon_0$ then*

$$\Omega_\varepsilon = \{x \in \mathbb{R}^m : \rho(x) > \varepsilon\} \tag{5.1}$$

is a bounded domain of class C^∞ satisfying:

(i) $\bigcap_{-\varepsilon_0<\varepsilon<0}\Omega_\varepsilon=\overline{\Omega}$, $\bigcup_{\varepsilon_0>\varepsilon>0}\Omega_\varepsilon=\Omega$,

(ii)*If* $0<|\varepsilon|<\varepsilon_0$ *there is a homeomorphism* f_ε *of* $\overline{\Omega}$ *onto* $\overline{\Omega}_\varepsilon$ *such that* $f_\varepsilon(\partial\Omega)=\partial\Omega_\varepsilon$ *and such that* $f_\varepsilon:\Omega\to\Omega_\varepsilon$ *is a* C^∞ *diffeomorphism.*

(iii) *If* $0<|\varepsilon|<\varepsilon_0, 0<|\varepsilon'|<\varepsilon_0$ *then there is a* C^∞ *diffeomorphism* $f_{\varepsilon,\varepsilon'}$ *of* $\overline{\Omega}_\varepsilon$ *onto* $\overline{\Omega}_{\varepsilon'}$ *such that* $f_{\varepsilon,\varepsilon'}(\partial\Omega_\varepsilon)=\partial\Omega_{\varepsilon'}$.

Proof. We choose $\varepsilon_0>0$ small enough so that $\partial\Omega_{3\varepsilon_0}\subset U$ where U is a neighbourhood of the boundary as defined in Remark 3.1 (and thus we can use in $\Omega_{3\varepsilon_0}$ all the constructions from the previous section). Conclusion (i) is then immediate. We first consider the problem of approximating Ω from the interior, and then describe the modifications necessary for the exterior approximation.

In order to prove (ii) for $\varepsilon>0$ we construct the desired homeomorphism $f=f_\varepsilon$ by assigning to each $x\in\overline{\Omega}$ an $f(x)\in\overline{\Omega}_\varepsilon$ taken to be along the flow $S(\cdot)x$ defined in (4.1), starting at x. However, since the flow is defined to be non-stationary just in a neighbourhood of the boundary, we take $f(x)=x$ for x far enough from the boundary. Thus we define

$$f(x)=\begin{cases} S(t(x))x, & x\in\overline{\Omega}\setminus\Omega_{3\varepsilon},\\ x, & x\in\Omega_{3\varepsilon},\end{cases} \tag{5.2}$$

for a $t(x)$ to be determined. In order to define $t(x)$ let $h:\mathbb{R}_+\to[0,\varepsilon]$ be a smooth function such that $h\equiv\varepsilon$ on $[0,\varepsilon)$ and $h\equiv 0$ on $[5\varepsilon/2,\infty)$ with $-1<h'(s)\le 0$ for all $s\ge 0$. For $x\in\overline{\Omega}\setminus\Omega_{3\varepsilon}$ define $t(x)$ to be the unique $t\ge 0$ such that

$$\rho(S(t)x)=\rho(x)+h(\rho(x)). \tag{5.3}$$

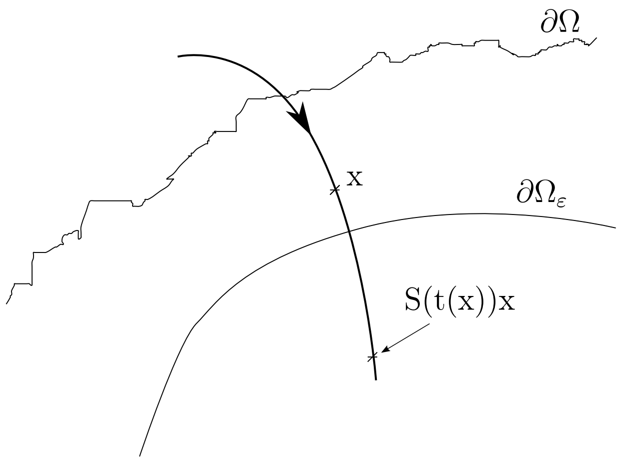


Figure 3: Defining the diffeomorphism along the flow

We claim now that $t(x)$ is well defined. We denote $g(t)\overset{\text{def}}{=}\rho(S(t)x)-\rho(x)-h(\rho(x))$. Then $g(0)\le 0$. But $\rho(S(\bar t)x)=3\varepsilon$ for some $\bar t\ge 0$ because of Remark 4.1 and the intermediate value theorem. Thus $g(\bar t)=3\varepsilon-\rho(x)-h(\rho(x))\ge 0$, since $\rho(S(\bar t)x)\ge\rho(x)$ and $\frac{d}{d\rho}(\rho+h(\rho))>0$. Finally $g(t)$ is strictly increasing by Lemma 4.1. This proves our claim regarding the definition of $t(x)$. Note that the properties of t imply that $f:\partial\Omega\to\partial\Omega_\varepsilon$ and $f:\Omega\to\Omega_\varepsilon$.

We continue by noting that t is a smooth function on $\Omega\setminus\overline{\Omega}_{3\varepsilon}$. This follows from the implicit function theorem applied to

$$F(t,x)\overset{\text{def}}{=}\rho(S(t)x)-\rho(x)-h(\rho(x)) \tag{5.4}$$

(the non-degeneracy condition needed for applying the implicit function theorem is a consequence of the relation

$$\frac{d}{dt}\rho(S(t)x) = \nabla\rho(S(t)x)\cdot\frac{d}{dt}S(t)x = [(\nabla\rho\cdot G)\gamma](S(t)x) > 0, \tag{5.5}$$

where for the last inequality we used Remark 3.1 with $x = P$).

Since $S(t)x$ is smooth in x and t we deduce that f is smooth on $\Omega\setminus\overline{\Omega}_{3\varepsilon}$. Since $t(x) = 0$, and thus $f(x) = x$, if $\rho(x)\in[5\varepsilon/2, 3\varepsilon]$, it follows that f is smooth in Ω. To show that $f:\overline{\Omega}\to\overline{\Omega}_\varepsilon$ is continuous it is enough to show that $t:\overline{\Omega}\setminus\overline{\Omega}_\varepsilon\to\mathbb{R}_+$ is continuous up to the boundary. Assume for contradiction that t is not continuous at some $\tilde{x}\in\partial\Omega$, so that there exists a sequence $x_k\to\tilde{x}$, $x_k\in\overline{\Omega}$ such that $t(x_k)$ does not converge to $t(\tilde{x})$. By the uniformity in time in Lemma 4.2 and Remark 4.1 we may assume that $t(x_k)\to\tau\neq t(\tilde{x})$. Replacing x with x_k in (5.3) and passing to the limit $k\to\infty$ we obtain that $\rho\left(S(\tau)\tilde{x}\right) = \rho\left(S(t(\tilde{x}))\tilde{x}\right)$, and hence $\tau = t(\tilde{x})$, a contradiction which proves our assertion that t is continuous up to the boundary.

Next we check that f is one-to-one. Suppose $f(x) = f(y)$ for $x, y\in\overline{\Omega}$. If $\rho(x) > 3\varepsilon$ and $\rho(y) > 3\varepsilon$ then $f(x) = x$, $f(y) = y$ and so $x = y$. If $\rho(x)\le 3\varepsilon$ and $\rho(y) > 3\varepsilon$ then $\rho(f(x)) = \rho(x) + h(\rho(x))\le 3\varepsilon + h(3\varepsilon) = 3\varepsilon < \rho(y) = \rho(f(y))$ so this case cannot occur. Finally if both $\rho(x)$ and $\rho(y)$ are in $[0, 3\varepsilon]$ then we have $S(t(x))x = S(t(y))y$ and hence $\rho(x) = \rho(y)$. If $t(x) = t(y)$ there is nothing to prove, so we assume without loss of generality that $t(x) > t(y)$. Then $S(t(x) - t(y))x = y$, hence $\rho(x) < \rho(y)$ since $\rho(S(t)x)$ is strictly increasing in t, giving a contradiction.

We next show that f is onto. To this end we take $z\in\overline{\Omega}_\varepsilon$. If $\rho(z)\ge 3\varepsilon$ then $f(z) = z$, so we suppose $\rho(z) < 3\varepsilon$. First note that by Lemma 4.2 there exists $\alpha(z)\le 0$ with $\rho(S(\alpha(z))z) = 0$. We look for x of the form $x = S(\beta(z))z$ with $\alpha(z)\le\beta(z)\le 0$. Denoting

$$\bar{g}(\tau) \stackrel{\text{def}}{=} \rho(z) - \rho(S(\tau)z) - h(\rho(S(\tau)z)), \tag{5.6}$$

we have that $\bar{g}(0)\le 0, \bar{g}(\alpha(z)) = \rho(z) - \varepsilon\ge 0$. Since $\bar{g}$ is strictly decreasing in τ it follows that there exists $\beta(z)\in[\alpha(z), 0]$ with $\bar{g}(\beta(z)) = 0$, that is

$$\rho(S(-\beta(z))S(\beta(z))z) = \rho(S(\beta(z))z) + h(\rho(S(\beta(z))z)).$$

Also, since $\beta(z)\ge\alpha(z)$ we have that $\rho(S(\beta(z))z)\ge\rho(S(\alpha(z))z) = 0$, so that $S(\beta(z))z\in\overline{\Omega}$. Hence $t(S(\beta(z))z) = -\beta(z)$ and $f_\varepsilon(S(\beta(z))z) = S(-\beta(z))S(\beta(z))z = z$.

Noting that if $z\in\Omega_\varepsilon$ then $\beta(z) > \alpha(z)$, and that (5.5) implies that $\bar{g}'(\beta(z)) < 0$, we deduce from the implicit function theorem that $\beta(z)$ is a smooth function of z for $z\in\Omega_\varepsilon$, so that $f^{-1}(z) = S(\beta(z))z$ is also smooth in Ω_ε.

This completes the proofs of (i) and (ii) for the case $\varepsilon > 0$. In particular, since Ω is by hypothesis connected, so is Ω_ε. To show that Ω_ε is of class C^∞ let $\bar{x}\in\partial\Omega_\varepsilon$, so that $\rho(\bar{x}) = \varepsilon$. As $|\nabla\rho(\bar{x})|\neq 0$ at least one of the partial derivatives $\frac{\partial\rho}{\partial x_i}(\bar{x})$ is nonzero. Without loss of generality we may assume that $\frac{\partial\rho}{\partial x_m}(\bar{x}) > 0$. By the implicit function theorem there exist $\delta > 0$ and a function $\tilde{f}\in C^\infty(\mathbb{R}^{m-1})$ such that

$$\{x\in B(\bar{x},\delta) : \rho(x) = \varepsilon\} = \{(x', x_m)\in B(\bar{x},\delta) : x_m = \tilde{f}(x')\}.$$

Since for $\delta > 0$ sufficiently small $\frac{\partial\rho}{\partial x_m}(x) > 0$ for all $x\in B(\bar{x},\delta)$ it follows that $\Omega_\varepsilon\cap B(\bar{x},\delta) = \{(x', x_m)\in\Omega_\varepsilon : x_m > \tilde{f}(x')\}$ as required.

The case of the exterior approximation by the domains $\Omega_{-\varepsilon}$ with $\varepsilon > 0$ cannot be handled in exactly the same way, via an analogue of the definition (5.2), because we would then not be able to prove smoothness via the implicit function theorem at points on, or with images on, $\partial\Omega$ (since ρ is not smooth there). Instead we proceed by proving (iii), from which (ii) for $\varepsilon < 0$ follows immediately. To this end it is sufficient to consider the case when $\varepsilon' < 0, \varepsilon > 0$. We define the desired homeomorphism $f = f_{\varepsilon,\varepsilon'}$ from $\overline{\Omega}_\varepsilon$ to $\overline{\Omega}_{\varepsilon'}$ by

$$f(x) = \begin{cases} S(\eta(x))x, & x \in \overline{\Omega}_\varepsilon \setminus \Omega_{3\varepsilon}, \\ x, & x \in \Omega_{3\varepsilon}, \end{cases} \tag{5.7}$$

where $\eta(x)$ is to be determined. Given $x \in \overline{\Omega}_\varepsilon \setminus \Omega_{3\varepsilon}$ there are unique numbers $r_2(x) < r_1(x) \leq 0$ such that $S(r_1(x))x \in \partial\Omega_\varepsilon$ and $S(r_2(x))x \in \partial\Omega_{\varepsilon'}$. Thus $r(x) = r_2(x) - r_1(x) < 0$ is the (negative) time the orbit of the flow of good directions through x takes to cross from $\partial\Omega_\varepsilon$ to $\partial\Omega_{\varepsilon'}$. Next let $\sigma : [\varepsilon, \infty) \to [0, 1]$ be smooth, nonincreasing, and such that $\sigma(\varepsilon) = 1, \sigma(s) = 0$ for $s \geq 2\varepsilon$. We now define

$$\eta(x) = r(x)\sigma(\rho(x)). \tag{5.8}$$

Then $f : \Omega_\varepsilon \to \Omega_{\varepsilon'}$ and $f : \partial\Omega_\varepsilon \to \partial\Omega_{\varepsilon'}$. Note that $r_1(x)$ and $r_2(x)$ are smooth functions of $x \in \overline{\Omega}_\varepsilon \setminus \Omega_{3\varepsilon}$. This follows by the implicit function theorem applied to the equations $F_1(r, x) \stackrel{\text{def}}{=} \rho(S(r)x) - \varepsilon = 0$ and $F_2(r, x) \stackrel{\text{def}}{=} \rho(S(r)x) - \varepsilon' = 0$, using (5.5) and the fact that ρ is smooth in neighbourhoods of $\partial\Omega_\varepsilon$ and $\partial\Omega_{\varepsilon'}$. Hence $r(x)$ is also smooth, and since $\eta(x) = 0$ for $\rho(x) > 2\varepsilon$ it follows that f is smooth on $\overline{\Omega}_\varepsilon$.

We now show that f is one-to-one, using a similar argument to that for the interior approximation. Suppose $f(x) = f(y)$ for $x, y \in \overline{\Omega}_\varepsilon$. If $x, y \in \Omega_{3\varepsilon}$ then $x = y$ is immediate from (5.7). If $x \in \overline{\Omega}_\varepsilon \backslash \Omega_{3\varepsilon}, y \in \Omega_{3\varepsilon}$ then $\rho(f(x)) \leq \rho(x) < \rho(y) = \rho(f(y))$, so this case cannot occur. If $x, y \in \overline{\Omega}_\varepsilon \backslash \Omega_{3\varepsilon}$ then $S(\eta(x))x = S(\eta(y))y$. If $\eta(x) = \eta(y)$ then $x = y$. On the other hand, if say $\eta(x) > \eta(y)$ then $S(\eta(x) - \eta(y))x = y$ and so $\rho(y) > \rho(x)$. But $r(x) = r(y)$ and so (since $r(x) < 0$) $\sigma(\rho(x)) < \sigma(\rho(y))$, which implies that $\rho(x) > \rho(y)$, a contradiction.

Now we check that f is onto. Let $z \in \overline{\Omega}_{\varepsilon'}$. If $z \in \Omega_{3\varepsilon}$ then $f(z) = z$, so we suppose that $z \in \overline{\Omega}_{\varepsilon'} \setminus \Omega_{3\varepsilon}$. We seek x of the form $x = S(\tau)z$ with $f(x) = z$, i.e.

$$S(\eta(S(\tau)z))S(\tau)z = z. \tag{5.9}$$

But (5.9) holds if and only if

$$\eta(S(\tau)z) + \tau = 0. \tag{5.10}$$

Let $\zeta(\tau) = \eta(S(\tau)z) + \tau$. There exist τ_1 with $\rho(S(\tau_1)z) = \varepsilon$ and $\tau_2 > \max(\tau_1, 0)$ with $\rho(S(\tau_2)z) = 3\varepsilon$. Note that $r(S(\tau_1)z) = r(S(\tau_2)z) \stackrel{\text{def}}{=} \tilde{r}(z)$. Then $\zeta(\tau_1) = \tilde{r}(z) + \tau_1 \leq 0$, $\zeta(\tau_2) = \tau_2 > 0$. Hence $\zeta(\tau) = 0$ for some (unique) $\tau(z) \in (\tau_1, \tau_2)$, so that f is onto.

Since

$$\frac{d}{d\tau}\zeta(\tau) = \tilde{r}(z)\sigma'(\rho(S(\tau)z))\frac{d}{d\tau}\rho(S(\tau)z) + 1 \geq 1$$

it follows by the implicit function theorem that $\tau(z)$ is smooth for $z \in \overline{\Omega}_{\varepsilon'} \setminus \overline{\Omega}_{3\varepsilon}$, and since $\tau(z) = 0$ for $z \in \overline{\Omega}_{2\varepsilon}$ it follows that $\tau : \overline{\Omega}_{\varepsilon'} \to \mathbb{R}$ is smooth, and hence $f^{-1}(z) = S(\tau(z))z$ is smooth. This completes the proof. □

Remark 5.1. As observed by Fraenkel [8, Section 5] the image under a diffeomorphism of a bounded domain of class C^0 need not be a domain of class C^0, since a cusp such as in Remark 2.2 can be bent by the diffeomorphism so that the boundary is not locally a graph. However Theorem 5.1 immediately implies a corresponding smooth approximation result for the larger class of bounded domains $\Omega \subset \mathbb{R}^m$ which are the image under a C^∞ diffeomorphism $\varphi : U \to \mathbb{R}^m$ of a bounded domain $\Omega' \subset \mathbb{R}^m$ of class C^0, where U is an open neighbourhood of Ω'. If $\Omega'_\varepsilon, 0 < |\varepsilon| < \varepsilon_0$, are the approximating domains given by the theorem for Ω' for $\varepsilon_0 > 0$ sufficiently small, then the open sets $\Omega_\varepsilon = \varphi(\Omega'_\varepsilon)$ are a family of bounded domains of class C^∞ such that $\bigcap_{-\varepsilon_0<\varepsilon<0} \Omega_\varepsilon = \bar{\Omega}, \bigcup_{0<\varepsilon<\varepsilon_0} \Omega_\varepsilon = \Omega$.

Remark 5.2. If Ω is Lipschitz, then the homeomorphism between $\overline{\Omega}_\varepsilon$ and $\overline{\Omega}$ defined in the proof of Theorem 5.1 is a bi-Lipschitz map (with Lipschitz constants bounded independently of ε for $0 < |\varepsilon| < \varepsilon_0$). In order to check this it suffices to show that the functions t and β (for the interior approximation, say) are Lipschitz. This can be seen in the case of t, for example, by applying $\frac{\partial}{\partial x_i}$ to $F(t(x), x) = 0$ with F as in (5.4), obtaining thus that $\frac{\partial t}{\partial x_i} = -\frac{\partial F}{\partial x_i} / \frac{\partial F}{\partial t}$. Given $x_0 \in \partial\Omega$ with a corresponding good direction n, there exist $\delta = \delta(x_0)$ and $c_0(\delta), c_1(\delta)$ such that $0 < c_0(\delta) < \nabla\rho(x) \cdot n$, $|\nabla\rho(x)| < c_1(\delta)$ for all $x \in B(x_0, \delta) \cap \Omega$ (see [23, p. 63, relation (A.3), Lemma A.1 and the line after (A.7)]). We can then apply compactness and Remark 3.1 to show that for some $\delta_1 > 0$ and constants c_1, c_2 depending only on δ_1 we have $0 < c_0 < \nabla\rho(x) \cdot G(x), |\nabla\rho(x)| \leq c_1$ for all $x \in \Omega$ with $\mathrm{dist}\,(x, \partial\Omega) < \delta_1$. Hence $\frac{\partial F(t(x),x)}{\partial t}$ is bounded away from zero, and $\frac{\partial F(t(x),x)}{\partial x_i}$ is bounded, for all $x \in \Omega$ with $\mathrm{dist}\,(x, \partial\Omega) < \delta_1$. Thus $|\nabla f(x)|$ is bounded for such x and hence for all $x \in \Omega$. Hence $f \in W^{1,\infty}(\Omega, \mathbb{R}^m)$. Since Ω is Lipschitz this implies that f is Lipschitz. This follows, for example, by noting that by Stein [36, Chapter VI, Theorem 5] f may be extended to a function $\tilde{f} \in W^{1,\infty}(\mathbb{R}^m, \mathbb{R}^m)$ so that $\tilde{f}$ is Lipschitz (a more general result can be found in [13, Theorem 4.1]).

Remark 5.3. That Ω and Ω_ε are diffeomorphic does not follow in general from the fact that they are homeomorphic. If $U \subset \mathbb{R}^m, V \subset \mathbb{R}^m$ are homeomorphic open sets, then if $m = 1, 2, 3$ it is known that U and V are diffeomorphic, while if $m \geq 5$ and U is the whole of $\mathbb{R}^m$ (or equivalently U is an open ball) then U and V are also diffeomorphic, since in these cases U has a unique differential structure up to diffeomorphism. These results are due to [27], [28], [35] and are surveyed in [25]. This is not true if $m = 4$ because of the existence of 'small exotic $\mathbb{R}^4$s' (surveyed in [31], a crucial ingredient being the work of Donaldson [5]), which implies that there is a bounded open subset of $\mathbb{R}^4$ which is homeomorphic but not diffeomorphic to an open ball in $\mathbb{R}^4$. Whether two arbitrary homeomorphic open subsets of $\mathbb{R}^m$, $m \geq 5$, are diffeomorphic seems not to be known in general.

6 The topology of Ω and the properties of the map of good directions

In Section 2 we constructed special smooth fields of good directions, that we called canonical. In this section we study the properties of arbitrary continuous fields of good directions that are not necessarily canonical.

We start with an illustrative case that provides significant insight into more general situations. We consider a standard solid torus in $\mathbb{R}^3$ given by $\Omega_T = T([0, 2\pi] \times [0, 2\pi] \times [0, 1))$, where

$$T : [0, 2\pi] \times [0, 2\pi] \times [0, 1) \to \mathbb{R}^3, \quad T(\theta, \varphi, r) = (\cos\theta\,(2 + r\cos\varphi), \sin\theta\,(2 + r\cos\varphi), r\sin\varphi),$$

whose boundary is $\partial\Omega_T = \mathbb{T}^2 \stackrel{\text{def}}{=} T_b([0, 2\pi]^2)$, where

$$T_b : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3, \quad T_b(\theta, \varphi) = (\cos\theta\,(2 + \cos\varphi), \sin\theta\,(2 + \cos\varphi), \sin\varphi). \tag{6.1}$$

A continuous field $C : \mathbb{T}^2 \to \mathbb{S}^2$ is a field of pseudonormals with respect to Ω_T if and only if

$$C(P) \cdot \nu_{\mathbb{T}^2}(P) > 0 \text{ for all } P \in \mathbb{T}^2, \tag{6.2}$$

where $\nu_{\mathbb{T}^2}(P)$ denotes the interior normal to Ω_T at $P \in \mathbb{T}^2$. The geometrical condition (6.2) imposes a constraint on the image of the field C:

Proposition 6.1. *For any continuous field $C : \mathbb{T}^2 \to \mathbb{S}^2$ satisfying the geometrical condition* (6.2) *there exists a band of size 2δ around the equator on the unit sphere, namely*

$$E_\delta \stackrel{\text{def}}{=} \{n \in \mathbb{S}^2 : |n \cdot e_3| < \delta\}, \tag{6.3}$$

such that $E_\delta \subset C(\mathbb{T}^2)$, where $e_3 = (0,0,1)$.

Conversely, for any given $\gamma > 0$ there exists a continuous field $C_\gamma : \mathbb{T}^2 \to \mathbb{S}^2$ satisfying (6.2) *such that $C_\gamma(\mathbb{T}^2) \subset E_\gamma$.*

Proof. In order to prove the first claim we use degree theory for the map C, on a domain $\omega \subset \mathbb{T}^2$ with respect to the point $n \in \mathbb{S}^2$, denoted $d(C,\omega,n)$. Since the domain ω we use is diffeomorphic to a bounded open subset of $\mathbb{R}^2$ we can use the theory of degree for subsets of Euclidean space as described, for example, in [6]. It suffices to show that $d(C,\omega,n) = 1$ for all $n \in E_\delta$ with suitable $\delta > 0$ and $\omega \subset \mathbb{T}^2$ (because this implies that $E_\delta \subset C(\omega)$). In order to show this we use the homotopy invariance of the degree [6, Theorem 2.3] for the homotopy $H : [0,1] \times \omega \to \mathbb{S}^2$ defined by

$$H(\lambda, P) = \frac{\lambda C(P) + (1-\lambda)\nu_{\mathbb{T}^2}(P)}{|\lambda C(P) + (1-\lambda)\nu_{\mathbb{T}^2}(P)|}$$

that connects the smooth negative Gauss map $\nu_{\mathbb{T}^2}$ with the field C. We choose

$$\omega \stackrel{\text{def}}{=} T_b([0,2\pi] \times \{(0,\pi/2) \cup (3\pi/2, 2\pi]\})$$

to be the 'exterior part' of the torus. We first note that $d(\nu_{\mathbb{T}^2}, \omega, n) = 1$ for all $n \in \mathbb{S}^2 \setminus \{\pm e_3\}$. This is because the Jacobian of $\nu_{\mathbb{T}^2}$ equals the Gaussian curvature of the torus which is positive in ω, and because such n do not belong to $\nu_{\mathbb{T}^2}(\partial\omega)$ and have exactly one inverse image in ω under $\nu_{\mathbb{T}^2}$. Next observe that the condition $C(P)\cdot\nu_{\mathbb{T}^2}(P) > 0$ and the continuity of C ensures that there exists a $\delta > 0$ so that $C(P) \notin E_\delta$ for $P \in \partial\omega$. Thus for $n \in E_\delta$ the condition $n \notin \{H(\lambda,P), \lambda \in [0,1], P \in \partial\omega\}$ is satisfied and we can apply the homotopy invariance of the degree to conclude that $d(C,\omega,n) = 1$ for $n \in E_\delta$, and hence $E_\delta \subset C(\mathbb{T}^2)$.

To prove the second claim of the proposition we consider a modification of an example suggested to us by Marc Lackenby [21]. We denote by $\tilde{n}(\theta,\varphi) \in \mathbb{S}^2$ the interior normal at $T_b(\theta,\varphi)$ (where T_b was defined in (6.1)). For $\gamma \in (0, \frac{\pi}{2})$ we define the continuous map $\tilde{n}^\gamma : [0,2\pi] \times [0,2\pi] \to \mathbb{S}^2$ as follows (see also Figure 4):

$$\tilde{n}^\gamma(\theta,\varphi) = \begin{cases} \tilde{n}(\theta,\varphi), & \text{if } \theta \in [0,2\pi], \varphi \notin (\frac{\pi}{2}-\gamma, \frac{\pi}{2}+\gamma) \cup (\frac{3\pi}{2}-\gamma, \frac{3\pi}{2}+\gamma) \\ \tilde{n}(\theta + \frac{1}{2\gamma}(\varphi - \frac{\pi}{2} + \gamma)\pi, \frac{\pi}{2} - \gamma), & \text{if } \theta \in [0,2\pi], \varphi \in (\frac{\pi}{2}-\gamma, \frac{\pi}{2}+\gamma) \\ \tilde{n}(\theta + \frac{1}{2\gamma}(\varphi - \frac{3\pi}{2} - \gamma)\pi, \frac{3\pi}{2} + \gamma), & \text{if } \theta \in [0,2\pi], \varphi \in (\frac{3\pi}{2}-\gamma, \frac{3\pi}{2}+\gamma) \end{cases} \tag{6.4}$$

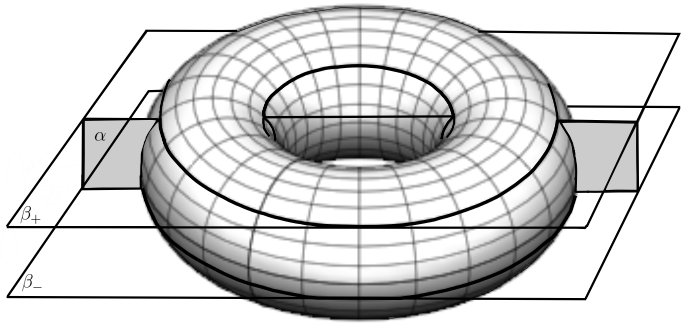
We claim now that for any $\gamma \in (0, \frac{\pi}{2})$ and any point on $\mathbb{T}^2$ the field $\tilde{n}^\gamma$ makes an angle strictly less than $\frac{\pi}{2}$ with the normal field $\tilde{n}$, that is $\tilde{n}(\theta,\varphi) \cdot \tilde{n}^\gamma(\theta,\varphi) > 0$, and thus is a field of pseudonormals.

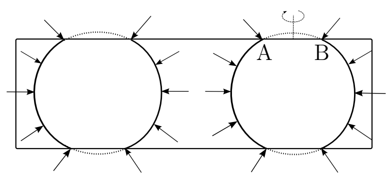
The field of interior normals is given explicitly by

$$\tilde{n}(\theta,\varphi) \stackrel{\text{def}}{=} -(\cos\theta\cos\varphi, \sin\theta\cos\varphi, \sin\varphi), \text{ for } \theta,\varphi \in [0,2\pi],$$

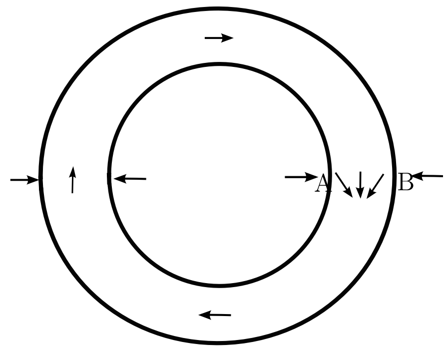
so that

$$\tilde{n}^\gamma(\theta,\varphi) = \begin{cases} -(\cos\theta\cos\varphi, \sin\theta\cos\varphi, \sin\varphi), & \text{if } \theta \in [0,2\pi], \varphi \notin (\frac{\pi}{2}-\gamma, \frac{\pi}{2}+\gamma) \cup (\frac{3\pi}{2}-\gamma, \frac{3\pi}{2}+\gamma) \\ -(\cos\theta_\gamma\cos\varphi_\gamma, \sin\theta_\gamma\cos\varphi_\gamma, \sin\varphi_\gamma) & \text{if } \theta \in [0,2\pi], \varphi \in (\frac{\pi}{2}-\gamma, \frac{\pi}{2}+\gamma) \\ -(\cos\bar{\theta}_\gamma\cos\bar{\varphi}_\gamma, \sin\bar{\theta}_\gamma\cos\bar{\varphi}_\gamma, \sin\bar{\varphi}_\gamma) & \text{if } \theta \in [0,2\pi], \varphi \in (\frac{3\pi}{2}-\gamma, \frac{3\pi}{2}+\gamma) \end{cases}$$





(a) Section through the α plane



(b) Section through the β_+ plane

Figure 4: The field of pseudonormals for the torus as defined in (6.4). Between the two planes β_+ and β_- the field of pseudonormals is identical to the negative Gauss map, so that the field on the intersection with a typical vertical plane α is as shown in (a). To obtain the pseudonormal field on the dotted arc joining A and B the inward normal at A is continuously rotated about the dotted vertical axis, reaching an angle of rotation of π at B. The projections of the field onto the plane β_+ are schematically shown in (b).

where $\theta_\gamma \stackrel{\text{def}}{=} \theta + \frac{1}{2\gamma}(\varphi - \frac{\pi}{2} + \gamma)\pi$, $\varphi_\gamma \stackrel{\text{def}}{=} \frac{\pi}{2} - \gamma$, $\bar{\theta}_\gamma \stackrel{\text{def}}{=} \theta + \frac{1}{2\gamma}(\varphi - \frac{3\pi}{2} - \gamma)\pi$ and $\bar{\varphi}_\gamma \stackrel{\text{def}}{=} \frac{3\pi}{2} + \gamma$.

Noting that $\tilde{n}(\theta,\varphi) = -\tilde{n}(\theta,\varphi+\pi)$, $\tilde{n}^\gamma(\theta,\varphi) = -\tilde{n}^\gamma(\theta,\varphi+\pi)$ and that $\tilde{n}(\theta,\varphi) = \tilde{n}^\gamma(\theta,\varphi)$ for $\theta \in [0,2\pi] \times \big([0,\pi] - (\frac{\pi}{2} - \gamma, \frac{\pi}{2} + \gamma)\big)$, we just need to check that

$$\tilde{n}(\theta,\varphi) \cdot \tilde{n}^\gamma(\theta,\varphi) > 0 \text{ for all } (\theta,\varphi) \in [0,2\pi] \times (\tfrac{\pi}{2} - \gamma, \tfrac{\pi}{2} + \gamma). \tag{6.5}$$

Let $R(\theta) \stackrel{\text{def}}{=} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then

$$\begin{aligned} \tilde{n}(\theta,\phi) \cdot \tilde{n}^\gamma(\theta,\phi) &= R(\theta)\tilde{n}(\theta,\phi) \cdot R(\theta)\tilde{n}^\gamma(\theta,\phi) \\ &= \cos\left(\tfrac{\pi}{2\gamma}(\varphi - \tfrac{\pi}{2} + \gamma)\right) \cos\left(\tfrac{\pi}{2} - \gamma\right) \cos\varphi + \sin\varphi \sin\left(\tfrac{\pi}{2} - \gamma\right) \\ &= -\cos\varphi \cos\left(\tfrac{\pi}{2} - \gamma\right) \sin\left(\tfrac{\varphi - \frac{\pi}{2}}{2\gamma}\right) + \sin\varphi \sin\left(\tfrac{\pi}{2} - \gamma\right). \end{aligned}$$

Thus in order to prove (6.5) it suffices to check that

$$g(\varphi) \stackrel{\text{def}}{=} -\cos\varphi \sin\left(\frac{\varphi-\frac{\pi}{2}}{2\gamma}\right) \geq 0 \text{ for all } \varphi \in \left(\tfrac{\pi}{2}-\gamma, \tfrac{\pi}{2}+\gamma\right). \tag{6.6}$$

Case I. $\varphi \in (\frac{\pi}{2}-\gamma, \frac{\pi}{2}]$. Then $-\frac{1}{2} < \frac{\varphi-\frac{\pi}{2}}{2\gamma} \leq 0$ so that both $\sin\left(\frac{\varphi-\frac{\pi}{2}}{2\gamma}\right)$ and $-\cos\varphi$ are nonpositive, and hence $g(\varphi) \geq 0$.

Case II. $\varphi \in (\frac{\pi}{2}, \frac{\pi}{2}+\gamma)$. Then $0 \leq \frac{\varphi-\frac{\pi}{2}}{2\gamma} < \frac{1}{2}$ so that both $\sin\left(\frac{\varphi-\frac{\pi}{2}}{2\gamma}\right)$ and $-\cos\varphi$ are nonnegative, and hence $g(\varphi) \geq 0$.

We have thus proved that $\tilde{n}^\gamma$ is a field of pseudonormals. Furthermore the image of $\tilde{n}^\gamma$ is contained in a band around the equator $\{(\cos\theta, \sin\theta, 0) : \theta \in [0, 2\pi]\}$ on the sphere $\mathbb{S}^2$, whose size is proportional to $\pi - 2\gamma$ and thus by taking γ suitably close to $\frac{\pi}{2}$ we obtain a field of pseudonormals whose image is contained in an arbitrarily small neighbourhood of the equator. □

We now explore what happens for more general bounded domains.

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain of class C^0.*

(i) *If $m \geq 3$, and the Euler characteristic of Ω is non-zero then any continuous pseudonormal field $n : \partial\Omega \to \mathbb{S}^{m-1}$ is surjective.*

If $m = 3$ and the Euler characteristic of Ω is zero then a continuous pseudonormal field is not necessarily surjective.

(ii) *If $m = 2$ any continuous pseudonormal field $n : \partial\Omega \to \mathbb{S}^1$ is surjective.*

Proof. (i) We first prove that if the Euler characteristic of Ω is non-zero then $n : \partial\Omega \to \mathbb{S}^{m-1}$ is surjective. We assume for contradiction that n is not surjective. Thus there exists a $\delta > 0$ small enough so that any continuous field $\bar{n} : \partial\Omega \to \mathbb{S}^{m-1}$ such that $\|\bar{n} - n\|_{C(\partial\Omega)} < \delta$ is also not surjective. We claim that there exists a continuous field of good directions $\tilde{n}$ defined on a neighbourhood V of $\partial\Omega$ so that $\|\tilde{n} - n\|_{C(\partial\Omega)} < \delta/2$.

In order to prove the claim we first note that as $n : \partial\Omega \to \mathbb{S}^{m-1}$ is continuous and $\partial\Omega$ is bounded, there exists $\delta' > 0$ such that

$$|n(x) - n(y)| < \frac{\delta}{4} \text{ for all } x, y \in \partial\Omega \text{ with } |x - y| < \delta'. \tag{6.7}$$

As Ω is of class C^0 there exist some $\bar{\delta} > 0$ and points $P_1, \ldots, P_l \in \partial\Omega$ such that $\partial\Omega \subset \cup_{j=1}^l B_{\bar{\delta}}(P_j)$, and such that for any $j = 1, 2, \ldots, l$ and $R \in B_{2\bar{\delta}(P_j)}$ we have that $n(P_j)$ is a good direction with respect to Ω at R. We assume, without loss of generality, that $8\bar{\delta} < \delta'$.

Recalling the definition (5.1) of the sets Ω_ε and from (3.2) that $\frac{1}{2} \leq \frac{\rho(x)}{d(x)} \leq 2$ for all $x \in \mathbb{R}^m$, we have that $\Omega_{-\bar{\delta}/2} \setminus \Omega_{\bar{\delta}/2} \subset \cup_{j=1}^l B_{2\bar{\delta}}(P_j)$. Consider a partition of unity $\alpha_j, j = 1, \ldots, l$, such that $\alpha_j \in C_0^\infty(B_{4\bar{\delta}}(P_j))$ and $\Sigma_{j=1}^l \alpha_j(x) = 1,$ for all $x \in \cup_{j=1}^l B_{2\bar{\delta}}(P_j)$. Let

$$\hat{n}(P) \stackrel{\text{def}}{=} \sum_{j=1}^{l} \alpha_j(P) n(P_j).$$

Then, by Lemma 2.2, $\hat{n}(P) \neq 0$ and $\frac{\hat{n}(P)}{|\hat{n}(P)|}$ is a good direction at P for any $P \in \Omega_{-\bar{\delta}/2} \setminus \Omega_{\bar{\delta}/2}$. Taking into account that $\alpha_j(P) = 0$ for $|P - P_j| > 4\bar{\delta}$, that $4\bar{\delta} < \delta'$, and (6.7) we obtain

$$|n(P) - \hat{n}(P)| \leq \sum_{j=1}^{l} \alpha_j(P)|n(P) - n(P_j)| \leq \frac{\delta}{4} \text{ for all } P \in \partial\Omega. \tag{6.8}$$

On the other hand, for $P \in \partial\Omega$ we have

$$\begin{aligned} \left|\hat{n}(P) - \frac{\hat{n}(P)}{|\hat{n}(P)|}\right| &= ||\hat{n}(P)| - 1| \leq \left|\sum_{j=1}^{l} \alpha_j(P)n(P_j) - \sum_{j=1}^{l} \alpha_j(P)n(R)\right| \\ &= \left|\sum_{j\in\mathcal{J}_P} \alpha_j(P)(n(P_j) - n(R))\right| \leq \sum_{j\in\mathcal{J}_P} \alpha_j(P)|n(P_j) - n(R)| < \frac{\delta}{4}, \end{aligned} \tag{6.9}$$

where $\mathcal{J}_P \stackrel{\text{def}}{=} \{j \in \{1, 2, \ldots, l\}; |P - P_j| \leq 4\bar{\delta}\}$ and $R = P_i$ for some arbitrary $i \in \mathcal{J}_P$. For the last inequality we used that $|R - P_j| \leq 8\bar{\delta} \leq \delta'$, for all $j \in \mathcal{J}_P$ and (6.7) together with $\sum_{j\in\mathcal{J}_P} \alpha_j(P) = 1$. We take $\tilde{n}(P) \stackrel{\text{def}}{=} \frac{\hat{n}(P)}{|\hat{n}(P)|}$ on $V \stackrel{\text{def}}{=} \Omega_{-\bar{\delta}/2} \setminus \Omega_{\bar{\delta}/2}$ and (6.8), (6.9) prove our claim about the existence of $\tilde{n}$.

We denote by $n_\varepsilon(P)$ the interior normal to $\partial\Omega_\varepsilon$ at P, so that $-n_\varepsilon : \partial\Omega_\varepsilon \to \mathbb{S}^{m-1}$ is the Gauss map; note that n_ε is parallel to $\nabla\rho(P)$ and has the same degree as $-n_\varepsilon$ up to (possible) change of sign. As noted above $\tilde{n}|_{\partial\Omega}$ is not surjective. Hence, as $\tilde{n}$ is continuous on V, there exists $\varepsilon_1 > 0$ so that $\Omega_{-\varepsilon_1} \setminus \Omega_{\varepsilon_1} \subset V$ and $\tilde{n}|_{\overline{\Omega_{-\varepsilon_1} \setminus \Omega_{\varepsilon_1}}}$ is also not surjective. Moreover, by Theorem 5.1 the sets Ω_ε and Ω are homeomorphic and thus they have the same Euler characteristic [12]. On the other hand for the smooth domain Ω_ε the Euler characteristic equals the degree of the Gauss map ([3, p. 384]) and hence the Gauss map has non-zero degree. For any $P \in \partial\Omega_\varepsilon$ we have that both $\tilde{n}(P)$ and $n_\varepsilon(P)$ are good directions at P, so that by Lemma 2.1 we have that $\tilde{n}(P) \cdot n_\varepsilon(P) \neq -1$. The homotopy $h : [0, 1] \times \partial\Omega_\varepsilon \to \mathbb{S}^{m-1}$ connecting $\tilde{n}|_{\partial\Omega_\varepsilon}$ and n_ε given by

$$h(t, P) = \frac{t\tilde{n}(P) + (1 - t)n_\varepsilon(P)}{|t\tilde{n}(P) + (1 - t)n_\varepsilon(P)|}$$

is thus well defined. Hence n_ε has the same non-zero degree as $\tilde{n}|_{\partial\Omega_\varepsilon}$ and thus $\tilde{n}|_{\partial\Omega_\varepsilon}$ is surjective (see [15, pp.123,125]), a contradiction. Hence n is surjective.

If $m = 3$ and the Euler characteristic of Ω is zero the second part of Proposition 6.1 provides the required counterexample.

(ii) In the same way as for part (i) it suffices to show that $\tilde{n}|_{\partial\Omega_\varepsilon}$ is surjective for nonzero $|\varepsilon|$ sufficiently small. We observe that for $\varepsilon \neq 0$ the set Ω_ε is a smooth 2D manifold with boundary. By the classification theorem for 1D connected, compact smooth manifolds (see for instance [24]) we have that each connected component Γ of $\partial\Omega_\varepsilon$ is diffeomorphic to $\mathbb{S}^1$. Thus Γ can be parametrized as a smooth closed curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ with constant speed $|\dot{\gamma}(t)| = s > 0$. Let $N(t) = n_\varepsilon(\gamma(t))$ for $t \in \mathbb{S}^1$. Then $\Delta(t) = \dot{\gamma}_1(t)N_2(t) - \dot{\gamma}_2(t)N_1(t)$ equals $\pm s$ for each $t \in \mathbb{S}^1$, and since $\Delta(t)$ is continuous the sign of $\Delta(t)$ is independent of $t \in \mathbb{S}^1$. The *Umlaufsatz* theorem of Hopf (see for instance [4, p. 275]) guarantees that $\dot{\gamma}(t)/|\dot{\gamma}(t)|$ has degree ± 1 when regarded as a function from $\mathbb{S}^1$ into $\mathbb{S}^1$. The fact that $\Delta(t)$ has constant sign implies that $N : \mathbb{S}^1 \to \mathbb{S}^1$ is homotopic to $\dot{\gamma}(t)/|\dot{\gamma}(t)|$, and so it has degree ± 1 as well. Since N and $\tilde{n}|_\Gamma$ are also homotopic (by the same argument as before), we have that $\tilde{n}|_\Gamma$ has non-zero degree and hence is surjective. □

Remark 6.1. The proof of Part (i) of Theorem 6.1 shows that if m is odd, then any continuous pseudonormal field is surjective, provided that *a connected component of the boundary* has non-zero Euler characteristic. This is due to the fact that for m odd the degree of the Gauss map of a closed, smooth $(m-1)$-dimensional hypersurface in $\mathbb{R}^m$ is half of its Euler characteristic (see for example [10, p. 196]).

We continue by investigating properties of the *multivalued* map of *all* pseudonormals. For $\Omega \subset \mathbb{R}^m$ a bounded domain of class C^0 and $P \subset \partial\Omega$ we let

$$\mathbb{G}(P) \stackrel{\text{def}}{=} \{n \in S^{m-1} : n \text{ is a pseudonormal at } P\},$$

and for any $E \subset \partial\Omega$ let $\mathbb{G}(E) \stackrel{\text{def}}{=} \cup_{P\in E}\mathbb{G}(P)$. We denote by $\mathcal{P}(S^{m-1})$ the set of all subsets of S^{m-1} and begin by noting that the map $\mathbb{G} : \partial\Omega \to \mathcal{P}(S^{m-1})$ is lower semicontinuous. Indeed, by definition this means that given any $P \in \partial\Omega$ and $n \in \mathbb{G}(P)$, for any neighbourhood V of n in S^{m-1} there is a neighbourhood U of P in $\partial\Omega$ such that $\mathbb{G}(Q) \cap V \neq \emptyset$ for all $Q \in U$; this is obvious since $n \in \mathbb{G}(Q)$ for all Q in a neighbourhood of P.

We use the following topological fact.

Lemma 6.1. *If $\Omega \subset \mathbb{R}^m$ is a connected open set and $\mathcal{U}$ is a connected component of $\mathbb{R}^m \setminus \overline{\Omega}$ then $\partial\mathcal{U}$ is connected.*

Proof. This is a consequence of [20, 49.VI,Theorem 2 and 57.I.9(i),57.III.1] (also noted in [19, Lemmas 4(i), 5]). ☐

In addition we have the following structural result.

Lemma 6.2. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain of class C^0. Then $\mathbb{R}^m \setminus \bar{\Omega}$ has a single unbounded connected component $\mathcal{D}$ and finitely many bounded connected components $\mathcal{U}_i$, $i = 1, \dots, k$, each of which is a bounded domain of class C^0 with $\mathbb{R}^m \setminus \bar{\mathcal{U}}_i$ connected. Furthermore $\partial\Omega$ can be written as the disjoint union*

$$\partial\Omega = \partial\mathcal{D} \cup \partial\mathcal{U}_1 \cup \cdots \cup \partial\mathcal{U}_k \tag{6.10}$$

and the connected components of $\partial\Omega$ are the sets $\partial\mathcal{D}, \partial\mathcal{U}_i$, $i = 1, \dots, k$.

Proof. Since Ω is bounded, $\mathbb{R}^m \setminus \bar{\Omega}$ has a single unbounded connected component $\mathcal{D}$. If there were infinitely many bounded connected components $\mathcal{U}_i$ then we would have $x_i \to x \in \partial\Omega$ for some sequence with $x_i \in \mathcal{U}_i$, which is easily seen to contradict that Ω is of class C^0. Since Ω is of class C^0 we have that $\partial\Omega = \partial(\mathbb{R}^m \setminus \bar{\Omega})$, from which (6.10) follows. The fact that all the sets in the union are disjoint follows easily from Ω being of class C^0. Also $\mathbb{R}^m \setminus \overline{\mathcal{U}}_i = \cup_{j\neq i}\overline{\mathcal{U}_j} \cup \overline{\mathcal{D}} \cup \Omega$, which is connected because all the sets in the union are connected and, for example, each point of $\partial\mathcal{D}$ (resp. $\partial\mathcal{U}_j, j \neq i$) has a neighbourhood consisting of points in $\bar{\mathcal{D}} \cup \Omega$ (resp. $\bar{\mathcal{U}_j} \cup \Omega$). Finally, by Lemma 6.1 each of the disjoint compact sets in (6.10) is connected, and so they are the connected components of $\partial\Omega$. ☐

We can now provide some properties of the image of $\mathbb{G}$:

Proposition 6.2. *Let $\Omega \subset \mathbb{R}^m, m \geq 2$ be a bounded domain of class C^0.Let $\mathcal{C}$ be a connected component of $\partial\Omega$. Then*

(i) $\operatorname{Span}\mathbb{G}(\mathcal{C}) = \mathbb{R}^m$,

(ii) $\mathbb{G}(\mathcal{C})$ *is connected.*

Proof. (i) We assume for contradiction that this is not true. Then Span $\mathbb{G}(\mathcal{C})$ is contained in an $(m-1)$-dimensional affine subspace of $\mathbb{R}^m$ and thus $\mathbb{G}(\mathcal{C}) \subset S^{m-1} \cap \{z \in \mathbb{R}^m : z \cdot N \geq 0\}$ for some $N \in S^{m-1}$. By Lemma 6.2 either $\mathcal{C} = \partial\mathcal{D}$ or $\mathcal{C} = \partial\mathcal{U}_i$ for some i. If $\mathcal{C} = \partial\mathcal{D}$ then sliding a hyperplane with normal N from $x \cdot N = +\infty$ until it touches $\mathcal{C}$ for the first time at some P, we find a good direction at P belonging to $\{z \in \mathbb{R}^m : z \cdot N < 0\}$, a contradiction. Similarly, if $\mathcal{C} = \partial\mathcal{U}_i$ for some i, then sliding such a hyperplane from $x \cdot N \to -\infty$ until it touches $\mathcal{C}$ for the first time, and recalling that by Lemma 6.2 $\mathcal{U}_i$ is of class C^0, gives a similar contradiction.

(ii) We assume for contradiction that $\mathbb{G}(\mathcal{C})$ is not connected. Then $\mathbb{G}(\mathcal{C})$ can be decomposed as $\mathbb{G}(\mathcal{C}) = A \cup B$ where A, B are nonempty sets and $A \cap \bar{B} = \bar{A} \cap B = \emptyset$. Let $P \in \mathcal{C}$. We claim that either $\mathbb{G}(P) \subset A$ or $\mathbb{G}(P) \subset B$. Indeed if $n_1 \in A, n_2 \in B$ where $n_1, n_2 \in \mathbb{G}(P)$ then since $\mathbb{G}(P)$ is convex

$$n(\lambda) \stackrel{\text{def}}{=} \frac{\lambda n_1 + (1-\lambda)n_2}{|\lambda n_1 + (1-\lambda)n_2|} \in \mathbb{G}(P) \text{ for all } \lambda \in [0,1].$$

But the sets $\{\lambda \in [0,1] : n(\lambda) \in A\}$ and $\{\lambda \in [0,1] : n(\lambda) \in B\}$ are relatively open and their union is $[0,1]$, contradicting the connectedness of $[0,1]$.

Now consider the sets $\mathcal{C}_A = \{P \in \mathcal{C} : \mathbb{G}(P) \subset A\}$ and $\mathcal{C}_B = \{P \in \mathcal{C} : \mathbb{G}(P) \subset B\}$, whose disjoint union is $\mathcal{C}$. Since $\mathcal{C}$ is connected, one of the sets $\mathcal{C}_A \cap \bar{\mathcal{C}}_B$, $\mathcal{C}_B \cap \bar{\mathcal{C}}_A$ is nonempty. Suppose, for example, that $P \in \mathcal{C}_A \cap \bar{\mathcal{C}}_B$. Then there exists a sequence $P_j \to P$ with $\mathbb{G}(P_j) \subset B$. Let $n \in \mathbb{G}(P)$. Then $n \in A$ but also $n \in \mathbb{G}(P_j)$ for sufficiently large j, and hence $n \in B$, a contradiction. □

7 Partial regularity of bounded C^0 domains

In this section we show that if $\Omega \subset \mathbb{R}^m$ is a *bounded* domain of class C^0 then[1] Ω has a Lipschitz boundary portion.

Lemma 7.1. *Let $\Omega \subset \mathbb{R}^m$, $m \geq 2$, be a bounded domain of class C^0. If the set of good directions (pseudonormals) at a point $P \in \partial\Omega$ contains m linearly independent directions then $\partial\Omega$ is Lipschitz in a neighbourhood of P, that is for some $\delta > 0$ and a suitable orthonormal coordinate system $Y \stackrel{\text{def}}{=} (y', y_m)$ with origin at P*

$$\Omega \cap B(P,\delta) = \{y \in \mathbb{R}^m : y_m > f(y'),\ |y| < \delta\} \tag{7.1}$$

where $f : \mathbb{R}^{m-1} \to \mathbb{R}$ is Lipschitz.

Proof. Let $\{n_1, \ldots, n_m\}$ be a set of linearly independent good directions at P and let $\tilde{n}$ be an interior point of the geodesically convex hull $\mathrm{co_g}\{\mathrm{n_1}, \ldots, \mathrm{n_m}\}$ of $\{n_1, \ldots, n_m\}$, which by Lemma 2.2 is also an interior point of the set of good directions at P. For example we can take

$$\tilde{n} = \frac{\sum_{i=1}^m n_i}{|\sum_{i=1}^m n_i|}.$$

Choosing an orthonormal coordinate system $Y = (y', y_m)$ with origin at P and $\tilde{n} = e_m$ we have by Lemma 2.2 that the representation (7.1) holds for some $\delta > 0$ with $f : \mathbb{R}^{m-1} \to \mathbb{R}$ continuous. We

[1] if $m \geq 4$ under the possibly unnecessary assumption that the Euler characteristic of Ω is nonzero

claim that f is Lipschitz in a neighbourhood of 0. Then extending f outside this neighbourhood to a Lipschitz map on $\mathbb{R}^{m-1}$, and choosing δ smaller if necessary, gives the result.

If the claim were false there would exist sequences $S_j \to 0$, $T_j \to 0$ in $\mathbb{R}^{m-1}$ such that $S_j \neq T_j$ and

$$\frac{f(S_j) - f(T_j)}{|S_j - T_j|} \to \infty \text{ as } j \to \infty.$$

But for large enough j the unit vector

$$N_j \stackrel{\text{def}}{=} \frac{(S_j, f(S_j)) - (T_j, f(T_j))}{\sqrt{|S_j - T_j|^2 + |f(S_j) - f(T_j)|^2}}$$

is not a good direction, since the line $t \to (T_j, f(T_j)) + tN_j$ meets $\partial\Omega$ twice in $B(P, \delta)$ at $(T_j, f(T_j))$ and $(S_j, f(S_j))$. But $\lim_{j\to\infty} N_j = e_m$, contradicting that $\tilde{n} = e_m$ is an interior point of the set of good directions at P. □

We continue by showing that if part of the boundary of a 2D domain is the graph of a nowhere differentiable function then there is exactly one good direction for each point on that part of the boundary.

Lemma 7.2. *Let $f : (a, b) \to \mathbb{R}$ be a continuous nowhere differentiable function. Let*

$$\mathcal{G} \stackrel{\text{def}}{=} \{(x, f(x)) \in \mathbb{R}^2 : x \in (a, b)\}$$

be a subset of the boundary $\partial\Omega$ of a bounded domain $\Omega \subset \mathbb{R}^2$ of class C^0.Then at any point $P \in \mathcal{G}$ there exists a unique good direction namely $\pm(0, 1)$ (the sign being independent of $P \in \mathcal{G}$).

Proof. We can assume without loss of generality that $(0, 1)$ is a good direction for all $P \in \mathcal{G}$, i.e. that Ω lies locally above $\mathcal{G}$. Let us assume for contradiction that there exists a point $P \in \mathcal{G}$ with another good direction $n \in \mathbb{S}^1$ with $n \neq (0, 1)$. Then there exists a whole neighbourhood of P in $\mathcal{G}$ with two good directions $(0, 1)$ and n. Restricting the interval (a, b) if necessary we may thus assume that for any $P \in \mathcal{G}$ there are two good directions $(0, 1)$ and n.

Then for any $m = (m_1, m_2) \in \mathbb{S}^1$ in the interior of the geodesically convex hull of $(0, 1)$ and n we have by Lemma 7.1 that $\mathcal{G}$ is the graph, along m, of a Lipschitz function, say g. Then g is differentiable almost everywhere, so that for almost all points in $\mathcal{G}$ there exists a tangent to $\mathcal{G}$. Let

$$\mathcal{G}^* \stackrel{\text{def}}{=} \{P \in \mathcal{G} : \text{ the tangent to } \mathcal{G} \text{ at P exists and is parallel to } (0, 1)\}.$$

We claim now that if the function g is differentiable at the point $P \in \mathcal{G}\backslash\mathcal{G}^*$ then so is f. More precisely let us denote $P = (\bar{x}^*, \bar{y}^*)$ in the system of coordinates with axis $O\bar{y} = m$ and $O\bar{x} = m^\perp = (m_2, -m_1)$ respectively $P = (x^*, y^*)$ in the system of coordinates with axis $Oy = (0, 1)$ and $Ox = (1, 0)$. That g is differentiable means that

$$\frac{g(\bar{x}) - g(\bar{x}^*)}{\bar{x} - \bar{x}^*} \to g'(\bar{x}^*), \text{ as } \bar{x} \to \bar{x}^*. \tag{7.2}$$

Let θ be the angle between the $\bar{x}$ axis and the x axis (see Figure 5). Letting $R(\theta) \stackrel{\text{def}}{=} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

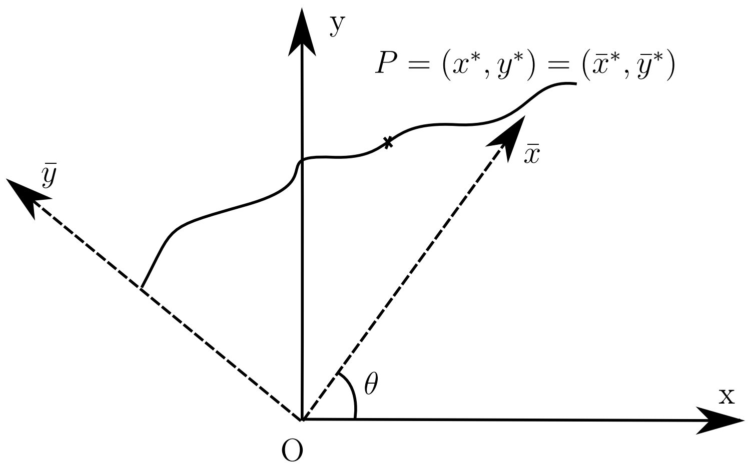


Figure 5: Changing the coordinate system

we have that $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = R(\theta) \begin{pmatrix} x \\ y \end{pmatrix}$, which implies that

$$\begin{aligned} \bar{x} - \bar{x}^* &= \cos\theta\,(x - x^*) + \sin\theta\,[f(x) - f(x^*)] \\ g(\bar{x}) - g(\bar{x}^*) &= -\sin\theta\,(x - x^*) + \cos\theta\,[f(x) - f(x^*)] \end{aligned} \tag{7.3}$$

Hence

$$\frac{f(x) - f(x^*)}{x - x^*} = \frac{\sin\theta\ + \cos\theta\ \left(\frac{g(\bar{x}) - g(\bar{x}^*)}{\bar{x} - \bar{x}^*}\right)}{\cos\theta\ - \sin\theta\ \left(\frac{g(\bar{x}) - g(\bar{x}^*)}{\bar{x} - \bar{x}^*}\right)}. \tag{7.4}$$

Note now that (7.3) and the continuity of f imply that $\bar{x} \to \bar{x}^*$ when $x \to x^*$, which together with (7.2) and (7.4) implies

$$\frac{f(x) - f(x^*)}{x - x^*} \to \frac{\sin\theta + \cos\theta\, g'(\bar{x}^*)}{\cos\theta - \sin\theta\, g'(\bar{x}^*)} \text{ as } x \to x^*$$

(note that $\cos\theta - \sin\theta g'(\bar{x}^*) \neq 0$ by our assumption that $P = (\bar{x}^*, g(\bar{x}^*)) \notin \mathcal{G}^*$). Hence f is differentiable at P as claimed.

There are now two cases:

Case I: The measure of $\mathcal{G}^*$ is strictly smaller than the measure of $\mathcal{G}$. Then at all points $P = (x, f(x)) \in \mathcal{G} \setminus \mathcal{G}^*$ we have that f is differentiable at x. Thus we obtain a contradiction with the assumption that f is nowhere differentiable.

Case II: The measure of $\mathcal{G}^*$ is the same as that $\mathcal{G}$, hence g' is almost everywhere $\cot\theta$. Then $\mathcal{G}$ is contained in a straight line in the direction $(0, 1)$, and hence is not a graph in the direction $(0, 1)$. Thus we again obtain a contradiction. □

We now show that under a topological condition bounded domains with continuous boundary necessarily have Lipschitz boundary portions.

Theorem 7.1. *Let Ω be a bounded domain of class C^0 in $\mathbb{R}^m$, $m \geq 3$. If Ω has non-zero Euler characteristic then there exists a point $P \in \partial\Omega$ in the neighbourhood of which $\partial\Omega$ is Lipschitz.*

Proof. We consider the canonical field of good directions obtained in the proof of Proposition 2.1, namely

$$G(P) \stackrel{\text{def}}{=} \frac{\Sigma_{i=1}^{k}\alpha_i(P)n_{P_i}}{|\Sigma_{i=1}^{k}\alpha_i(P)n_{P_i}|} \tag{7.5}$$

with n_{P_i} a good direction at P_i with respect to the ball $B(P_i,\delta_i)$ and thus at each point $R \in B(P_i, \frac{1}{2}\delta_{P_i})$, $i = 1,\dots,k$, and $\alpha_i, i = 1,\dots,k$, a partition of unity subordinate to the covering $B(P_i, \frac{1}{2}\delta_{P_i})$, $i = 1,\dots,k$. Then $G|_{\partial\Omega}$ is a continuous pseudonormal field and hence Theorem 6.1, part (i), provides that $G : \partial\Omega \to \mathbb{S}^{m-1}$ is surjective. We claim now that there exist m linearly independent good directions $\bar{n}_1,\dots,\bar{n}_m \in \{n_{P_i},\ i = 1,\dots,k\}$ such that $\partial\Omega$ and the m corresponding balls, on which they are good directions, have a non-empty intersection. Lemma 7.1 then shows that $\partial\Omega$ is Lipschitz in a neighbourhood of any point P in their intersection.

In order to prove the claim we assume, for contradiction, that at each point $Q \in \partial\Omega$ the subset of $\{n_{P_i} \in \mathbb{S}^{m-1}, i = 1,\dots,k\}$ that are good directions at Q is contained in a hyperplane. For each $i = 1,\dots,k$, let $E_i = \{P \in \partial\Omega : \alpha_i(P) > 0\}$. Then each ∂E_i is a closed nowhere dense subset of $\partial\Omega$, and so by the Baire Category theorem $\cup_{i=1}^{k}\partial E_i$ is a closed nowhere dense subset of $\partial\Omega$. Consider any nonempty subset of the form $A_{i_1,\dots,i_r} = \cap_{j=1}^{r} E_{i_j}$, where $1 \le r \le k$ and $1 \le i_1 < \cdots < i_r \le k$. For $P \in A_{i_1,\dots,i_r}$ we have by our assumption that the vectors $n_{P_{i_j}}$, $1 \le j \le r$, lie in a hyperplane, and thus from (7.5) we have that $G(A_{i_1,\dots,i_r})$ is contained in this hyperplane. Hence $G(\partial\Omega \setminus \cup_{i=1}^{k}\partial E_i)$ is contained in a closed nowhere dense subset A of $\mathbb{S}^{m-1}$ (the union of the intersection with $\mathbb{S}^{m-1}$ of a finite number of hyperplanes). Since G is continuous and $\cup_{i=1}^{k}\partial E_i$ is nowhere dense, it follows that $G(\partial\Omega) \subset A$, contradicting the surjectivity of G. □

We continue by studying the set of all pseudonormals in the general case of C^0 domains with no topological restrictions imposed on $\partial\Omega$.

Lemma 7.3. *Let Ω be a bounded domain of class C^0 in $\mathbb{R}^m$, $m > 1$. For each connected component $\mathcal{C}$ of $\partial\Omega$ there exists a point $P \in \mathcal{C}$ at which the set of good directions at P is not a singleton.*

Proof. Assume for contradiction that for any $P \in \mathcal{C}$ there exists only one good direction $n(P) \in \mathbb{S}^{m-1}$. Note that if N is a good direction at some point $x \in \partial\Omega$ then N is also a good direction at points $z \in \partial\Omega$ sufficiently close to x. Pick $P_1 \in \mathcal{C}$ and define $E = \{P \in \mathcal{C} : n(P) = n(P_1)\}$. The preceding property implies that E is both open and closed in $\mathcal{C}$. Since $\mathcal{C}$ is connected it follows that $E = \mathcal{C}$, in contradiction to Proposition 6.2(i). □

The last lemma, combined with the characterization of Lipschitz regularity of the boundary in Lemma 7.1 immediately imply:

Theorem 7.2. *Let Ω be a bounded domain of class C^0 in $\mathbb{R}^2$. For each connected component $\mathcal{C}$ of $\partial\Omega$ there exists a point $P \in \mathcal{C}$ in the neighbourhood of which $\partial\Omega$ is Lipschitz.*

Remark 7.1. Note that Theorem 7.2 is not in general true for *unbounded* domains in $\mathbb{R}^2$ of class C^0. Indeed, by Lemma 7.2, if $f : \mathbb{R} \to \mathbb{R}$ is nowhere differentiable then the domain $\Omega = \{(x_1, x_2) : x_2 > f(x_1)\}$ has no Lipschitz boundary portions.

We also have a similar result in 3D, but the proof is considerably more intricate.

Theorem 7.3. *Let Ω be a bounded domain of class C^0 in $\mathbb{R}^3$. For each connected component of $\partial\Omega$ there exists a point $P \in \partial\Omega$ in the neighbourhood of which $\partial\Omega$ is Lipschitz.*

Proof. We consider again the canonical field of good directions obtained in the proof of Proposition 2.1, namely

$$G(P) \stackrel{\text{def}}{=} \frac{\sum_{i=1}^{k} \alpha_i(P) n_{P_i}}{|\sum_{i=1}^{k} \alpha_i(P) n_{P_i}|} \tag{7.6}$$

with n_{P_i} a good direction at P_i with respect to the ball $B(P_i, \delta_i)$ and thus at each point $R \in B(P_i, \frac{1}{2}\delta_{P_i})$, $i = 1, \ldots, k$, and $\alpha_i, i = 1, \ldots, k$, a partition of unity subordinate to the covering $B(P_i, \frac{1}{2}\delta_{P_i})$, $i = 1, \ldots, k$.

We continue by analyzing the image of G when restricted to an arbitrary connected component $\mathcal{S} = \mathcal{S}_\varepsilon$ of $\partial\Omega_\varepsilon$ for the approximating Ω_ε as in Theorem 5.1, with $\varepsilon > 0$ sufficiently small. We will show that $G(\mathcal{S})$ has non-empty interior and that this allows one to infer that there exist three linearly independent directions at some point $Q_\varepsilon \in \partial\Omega_\varepsilon$ for all $\varepsilon > 0$. Then, thanks to the special structure of the canonical field G, it will be shown that the same can be claimed at some point $Q \in \partial\Omega$.

We start by noting that if the connected component $\mathcal{S}$ of $\partial\Omega_\varepsilon$ has non-zero Euler characteristic then Theorem 6.1 and Remark 6.1 ensure that $G(\mathcal{S}) = \mathbb{S}^2$ hence $G(\mathcal{S})$ has non-empty interior.

The case when $\mathcal{S}$ has zero Euler characteristic is more delicate as in this situation the field $G|_{\mathcal{S}}$ is not necessarily surjective, as was shown in Proposition 6.1. The proof continues in two steps. In the first we show that we can assume, without loss of generality, that $\mathcal{S} = \mathbb{T}^2$ is the standard torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ embedded in $\mathbb{R}^3$, for which we know by Proposition 6.1 that any smooth field of good directions $G : \mathcal{S} \to \mathbb{S}^2$ has an image $G(\mathcal{S})$ with non-empty interior. Then, in the second step, we show that $G(\mathcal{S})$ having non-empty interior implies (irrespective of the Euler characteristic of $\mathcal{S}$ being zero or not) that there exist three linearly independent directions at some point $Q_\varepsilon \in \partial\Omega_\varepsilon$ for all $\varepsilon > 0$, and that the same holds also at some point $Q \in \partial\Omega$.

Step 1 (reduction to the standard torus and consequences): We note that $\mathcal{S}$ is a smooth 2-dimensional compact, connected and orientable manifold without boundary, that has zero Euler characteristic. Let

$$\mathbb{T}^2 \stackrel{\text{def}}{=} \{(\cos\theta\,(2+\cos\varphi), \sin\theta\,(2+\cos\varphi), \sin\varphi) : \theta, \varphi \in [0, 2\pi]\} \tag{7.7}$$

denote a standard torus embedded in $\mathbb{R}^3$. Then $\mathbb{T}^2$ is a 2-dimensional compact, connected and orientable manifold without boundary of zero Euler characteristic and by the theorem of classification of 2-dimensional compact manifolds [15, Chapter 9] there exists a diffeomorphism $D : \mathcal{S} \to \mathbb{T}^2$.

We use the diffeomorphism D to transport the field of good directions, in a bijective manner, from $\mathcal{S}$ onto $\mathbb{T}^2$, while preserving the angle between the good direction and the normal.

Lemma 7.4. *Let $\nu_S : \mathcal{S} \to \mathbb{S}^2$, $\nu_{\mathbb{T}^2} : \mathbb{T}^2 \to \mathbb{S}^2$ denote the interior normals to $\mathcal{S}$, respectively $\mathbb{T}^2$. There exist smooth functions $e, \hat{e} : \mathcal{S} \to \mathbb{S}^2$, $f, \hat{f} : \mathbb{T}^2 \to \mathbb{S}^2$ such that $e, \hat{e}$ and $\nu_{\mathcal{S}}$, respectively $f, \hat{f}$ and $\nu_{\mathbb{T}^2}$ are pairwise orthogonal at each point.*

Proof. The proof is essentially an easy consequence of the fact that $\mathcal{S}$ and $\mathbb{T}^2$ are parallelizable manifolds embedded in $\mathbb{R}^3$. We consider the torus $\mathbb{T}^2$ as in (7.7). Then $\nu_{\mathbb{T}^2}(\theta, \varphi) = -(\cos\theta\cos\varphi, \sin\theta\cos\varphi, \sin\varphi$ for any $\theta, \varphi \in [0, 2\pi] \times [0, 2\pi]$. We take

$$f(\theta, \varphi) \stackrel{\text{def}}{=} (-\sin\theta, \cos\theta, 0), \quad \hat{f}(\theta, \varphi) \stackrel{\text{def}}{=} (-\cos\theta\sin\varphi, -\sin\theta\sin\varphi, \cos\varphi).$$

We consider the derivative $TD : T\mathcal{S} \to \mathbb{T}^2$ of the diffeomorphism $D : \mathcal{S} \to \mathbb{T}^2$, acting between the tangent bundles $T\mathcal{S}$ and $T\mathbb{T}^2$. Then for any point $P \in \mathcal{S}$ we have that the linear map $T_P D : T_P\mathcal{S} \to T_{D(P)}\mathbb{T}^2$ is an invertible linear function, and as such $T_P D^{-1}\big(f(D(P))\big)$ and $T_P D^{-1}\big(\hat{f}(D(P))\big)$ define a basis in $T_P\mathcal{S}$ which varies smoothly in P. However this basis need not be an orthogonal basis and in order to obtain an orthogonal, smoothly varying, basis, we take:

$$e(P) \stackrel{\text{def}}{=} \frac{T_P D^{-1}\big(f(D(P))\big)}{|T_P D^{-1}\big(f(D(P))\big)|}, \quad \hat{e}(P) = e(P) \times \nu_{\mathcal{S}}(P).$$

□

Continuing the proof of Theorem 7.3, let $G : \mathcal{S} \to \mathbb{S}^2$ be a smooth field of good directions on $\mathcal{S}$ such that $G(P) \cdot \nu_{\mathcal{S}}(P) > 0$ for all $P \in \mathcal{S}$. We define

$$\tilde{G}(D(P)) \stackrel{\text{def}}{=} \big(G(P) \cdot \nu_{\mathcal{S}}(P)\big)\nu_{\mathbb{T}^2}(D(P)) + \big(G(P) \cdot e(P)\big)f(D(P)) + \big(G(P) \cdot \hat{e}(P)\big)\hat{f}(D(P))$$

We have then that $\tilde{G}(Q) \cdot \nu_{\mathbb{T}^2}(Q) = G(D^{-1}(Q)) \cdot \nu_{\mathcal{S}}(D^{-1}(Q)) > 0$ for all $Q \in \mathbb{T}^2$ and thus $\tilde{G}$ is a smooth field of good directions on $\mathbb{T}^2$, the transported version to $\mathbb{T}^2$ of the field G.

We claim now that $G(\mathcal{S})$ has nonempty interior if and only if the transported version $\tilde{G}(\mathbb{T}^2)$ has nonempty interior.

To this end let us define the continuous function $\mathcal{H} : \mathcal{S} \times \mathbb{S}^2 \to \mathbb{T}^2 \times \mathbb{S}^2$ by

$$\mathcal{H}(P, n) \stackrel{\text{def}}{=} \Big(D, \big(n \cdot \nu_{\mathcal{S}}\big)\nu_{\mathbb{T}^2}(D) + \big(n \cdot e\big)f(D) + \big(n \cdot \hat{e}\big)\hat{f}(D)\Big)(P)$$

One can check that $\widetilde{\mathcal{H}} : \mathbb{T}^2 \times \mathbb{S}^2 \to \mathcal{S} \times \mathbb{S}^2$ defined by

$$\widetilde{\mathcal{H}}(R, m) \stackrel{\text{def}}{=} \Big(D^{-1}, \big(m \cdot \nu_{\mathbb{T}^2}\big)\nu_{\mathbb{T}^2}(D^{-1}) + \big(m \cdot f\big)e(D^{-1}) + \big(m \cdot \hat{f}\big)\hat{e}(D^{-1})\Big)(R)$$

is the continuous inverse of $\mathcal{H}$ and thus $\mathcal{H}$ is a homeomorphism. Moreover we have:

$$\mathcal{H}(P, G(P)) = (D(P), \tilde{G}(D(P))). \tag{7.8}$$

Let us assume now that $G(\mathcal{S})$ has nonempty interior E. Then, since G is continuous, $G^{-1}(E)$ is nonempty and open. Since $\mathcal{H}$ is a homeomorphism, it takes nonempty open sets into nonempty open sets, so that, by (7.8), $\mathcal{H}(G^{-1}(E), E) = (D(G^{-1}(E), \tilde{G}(D(G^{-1}(E))))$. Thus $\tilde{G}(D(G^{-1})(E)) \subset \tilde{G}(\mathbb{T}^2)$ is a nonempty open set and therefore it has nonzero measure. One can argue in a similar way, using $\tilde{H}$ to show that if $\tilde{G}(\mathbb{T}^2)$ has nonempty interior then so does $G(\mathcal{S})$, thus proving our claim.

Proposition 6.1 now shows that $\tilde{G}(\mathbb{T}^2)$ has nonempty interior and therefore so does $G(\mathcal{S})$.

Step 2 (from the smooth approximating domains back to the rough one): Let Ω_ε be a sequence of smooth domains approximating Ω, as given in Theorem 5.1. Let $G : V \to \mathbb{S}^2$ be a canonical field of good directions, where V is an open set containing $\partial\Omega$ (hence there exists an $\varepsilon_0 > 0$ so that $\partial\Omega_\varepsilon \subset V$ for $0 < \varepsilon < \varepsilon_0$). The previous step and the remark before it show that for each connected component $\mathcal{S}_\varepsilon$ of $\partial\Omega_\varepsilon$ we have that the interior of $G(\mathcal{S}_\varepsilon)$ is nonempty.

Let us recall now the definition (7.6) of the canonical field G. Arguing as in the proof of Proposition 7.1 (and using the fact that $G(\mathcal{S}_\varepsilon)$ has nonempty interior, instead of surjectivity of G) we have that there exists a point $Q_\varepsilon \in \mathcal{S}_\varepsilon$ so that there exist three linearly independent good directions $n_{P_{i_1(\varepsilon)}}, n_{P_{i_2(\varepsilon)}}, n_{P_{i_3(\varepsilon)}}$ such that the three corresponding balls $B(P_{i_k(\varepsilon)}, \frac{1}{2}\delta_{i_k(\varepsilon)}), k = 1, 2, 3$, have a nonempty intersection containing Q_ε.

Since the number of balls in the cover is finite there exist three of them, say $B(\bar{P}_i, \frac{\bar{\delta}_i}{2}), i = 1,2,3$ such that there exist infinitely many points Q_{ε_j}, $\varepsilon_j \to 0+$, in their intersection. There exists then a $\bar{Q} \in \partial\Omega$, a limit point of the Q_{ε_j}, with $\bar{Q} \in \cap_{i=1}^3 \overline{B(\bar{P}_i, \frac{\bar{\delta}_i}{2})} \subset \cap_{i=1}^3 B(\bar{P}_i, \delta_i)$ and such that there are three linearly independent good directions at $\bar{Q}$. □

Remark 7.2. Despite some efforts we have been unable to decide whether Theorem 7.1 remains true for $m \geq 4$ when the Euler characteristic of Ω is zero.

Remark 7.3. Theorems 7.1-7.3 remain valid if the hypothesis that Ω is of class C^0 is replaced by the weaker hypothesis that Ω is the image under a C^1 diffeomorphism $\varphi : U \to \mathbb{R}^m$ of a bounded domain $\Omega' \subset \mathbb{R}^m$ of class C^0, where U is an open neighbourhood of Ω'. Indeed, by [16, Theorem 4.1] (in which the term strongly Lipschitz is used instead of our Lipschitz) if $\partial\Omega'$ is Lipschitz in a neighbourhood of P then $\partial\Omega$ is Lipschitz in a neighbourhood of $\varphi(P)$.

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