A result of local exact controllability for 1-d compressible Navier-Stokes equations

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in collaboration with

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Outline

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Incompressible case

Extensive study of the problem since the beginning of 90's and questions asked by J.-L.Lions on the subject.
Several results depending on boundary conditions and regularity by Fursikov-Imanuvilov, Coron, Fursikov-Coron, Imanuvilov, Fernandez Cara-Guerrero-Imanuvilov-Puel, Gonzales Burgos-Guerrero-Puel, Imanuvilov-Puel-Yamamoto.
Important notion of Exact Controllability to Trajectories.

Exact controllability to trajectories

Ω : bounded domain, ω : “small" subdomain.

$$\frac{\partial y}{\partial t} + (y.\nabla)y - \Delta y + \nabla p = f + h.\mathbf{1}_\omega \text{ in } \Omega \times (0,T),$$
$$\operatorname{div} y = 0 \text{ in } \Omega \times (0,T),$$
Boundary conditions,
$$y(0) = y^0 \text{ in } \Omega.$$

Here : distributed control h in order to simplify. Possibility of having boundary control.

“Ideal" trajectory of the same operator

$$\frac{\partial \bar{y}}{\partial t} + (\bar{y}.\nabla)\bar{y} - \Delta \bar{y} + \nabla \bar{p} = f \text{ in } \Omega \times (0,T),$$
$$\operatorname{div} \bar{y} = 0 \text{ in } \Omega \times (0,T),$$
Boundary conditions,
$$\bar{y}(0) = \bar{y}^0 \text{ in } \Omega.$$

Possibly, $\bar{y}$: stationary solution even unstable...

Global Exact Controllability to Trajectories : Given any y^0, can we find h such that

$$y(T) = \bar{y}(T) \ ?$$

Local Exact Controllability to Trajectory : Does there exist $\eta > 0$ such that for $||y^0 - \bar{y}^0|| \leq \eta$, we can find h such that

$$y(T) = \bar{y}(T) \ ?$$

Last result

Fernandez Cara-Guerrero-Imanuvilov-Puel , Imanuvilov-Puel-Yamamoto.

Theorem

Dimension $N = 2$ or 3.
Dirichlet (no-slip) boundary conditions.
Assume $\bar{y} \in L^\infty(\Omega \times (0, T))$. There exists $\eta > 0$ such that if $||y^0 - \bar{y}^0||_{L^4(\Omega)} \leq \eta$, there exists a control $h \in L^2(0, T; L^2(\omega))$ and there exists a solution y of the controlled system such that

$$y(T) = \bar{y}(T).$$

Some open problems

• What about global exact controllability to trajectories (when y^0 is not necessary close to $\bar{y}^0$) ?
• Is it possible to relax the hypothesis $\bar{y} \in L^\infty$?
• How to compute the corresponding control in an efficient way?
• .....

Compressible Navier-Stokes equations

Up to now, almost no study of controllability for viscous compressible fluids.
Example of problem with control on the whole boundary.

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho y) = 0 \text{ in } \Omega \times (0, T),$$

$$\rho\Big(\frac{\partial y}{\partial t} + y.\nabla y\Big) - \nu \Delta y + \nabla p(\rho) = 0 \text{ in } \Omega \times (0, T),$$

$$y(0) = y^0,\ \rho(0) = \rho^0 \text{ in } \Omega,$$

$$p(\rho) = C\rho^\gamma.$$

Can we find a solution of this system (the boundary conditions are not prescribed and play the role of controls) such that we have at time T

$$(y(T), \rho(T)) = (\bar{y}, \bar{\rho})$$

where (for example) $\bar{y}$ and $\bar{\rho}$ are constant ?
In dimension $N \geq 2$ the problem is open.

The $1-d$ compressible Navier-Stokes equations

$$y \longleftrightarrow u$$

$$\partial_t \rho + \partial_x(\rho u) = 0, \quad \text{in } (0, L) \times (0, T),$$
$$\rho(\partial_t u + u\partial_x u) - \nu\partial_{xx} u + \partial_x p(\rho) = 0, \quad \text{in } (0, L) \times (0, T).$$

- ρ is the density;
- u is the velocity;
- $p(\rho)$ is the pressure, assumed to satisfy $p'(\rho) > 0$, typically $p(\rho) = C\rho^\gamma$, $C > 0$, $\gamma \geq 1$;
- $\nu > 0$ is the viscosity (constant).

It is clear that if $\bar{\rho}$ and $\bar{u}$ are constant, $(\bar{\rho}, \bar{u})$ is a (constant) stationary solution of the system.
The problem is then : given initial conditions (ρ^0, u^0), can we find a solution of the system such that we have at time T

$$(\rho(T), u(T)) = (\bar{\rho}, \bar{u}).$$

The boundary conditions are not prescribed. They will be the controls, e.g.

$$u(t,0) = v_0(t), \quad u(t,L) = v_L(t),$$

and

$$\rho(t,0) = w_0(t) \text{ if } u(t,0) > 0, \quad \rho(t,L) = w_L(t) \text{ if } u(t,L) < 0,$$

These functions (v_0, v_L, w_0, w_L) can be chosen freely.

Outline

Our result : S.Ervedoza, O.Glass, S.Guerrero, J.-P.P.

To appear in Arch. for Rat. Mech. and Analysis.

Theorem

Let $(\bar{\rho}, \bar{u}) \in \mathbb{R}_+^* \times \mathbb{R}^*$ *and* $T > L/|\bar{u}|$. *Then there exists* $\varepsilon > 0$ *such that for all* $(\rho^0, u^0) \in H^3(0, L)^2$ *satisfying*

$$||(\rho_0, u_0) - (\bar{\rho}, \bar{u})||_{(H^3)^2} \leq \varepsilon,$$

there exists a controlled trajectory (ρ, u)

$$\rho \in H^1((0, T) \times (0, L))$$
$$u \in H^1((0, T); L^2(0, L)) \cap L^2((0, T); H^2(0, L))$$

satisfying

$$(\rho(T), u(T)) = (\bar{\rho}, \bar{u}).$$

Comments

- We need $\bar{u} \neq 0$. Indeed, it can be shown that the linearized problem around $(\bar{\rho}, 0)$ is not controllable. This has also been noticed by Rosier-Rouchon (2007). In that case, it can be shown that there exist density waves which are essentially not moving. This does not say that the nonlinear problem is not controllable near $\bar{u} = 0$!
- We have a condition on time T which is $T > L/|\bar{u}|$. As the velocities u will stay close to $\bar{u}$, due to the hyperbolic character of the density equation, this condition will appear to be natural as the density waves will travel at velocity u and will have to cover the whole interval $(0, T)$.
- Our method requires regularity on both ρ and u and we will use a result on the Cauchy problem (on the whole real line) by Matsumura-Nishida (1980). The regularity needed here might be only technical. Other results on the Cauchy problem can be found in P.-L.Lions' book for example.
- There is a recent result of local controllability for the $1-d$ compressible Navier-Stokes equations by Amosova (2011) using Lagrangian coordinates when the initial density is already on the “targeted" trajectory.

Outline

We extend ρ^0 and u^0 to $\mathbb{R}$ so that

$$\rho^0 - \bar{\rho} \text{ and } u^0 - \bar{u} \text{ have compact support in } \mathbb{R},$$

$$||\rho_0 - \bar{\rho}||_{H^3(\mathbb{R})} \leq C||\rho_0 - \bar{\rho}||_{H^3(0,L)}, \quad ||u_0 - \bar{u}||_{H^3(\mathbb{R})} \leq C||u_0 - \bar{u}||_{H^3(0,L)}.$$

We then introduce (for simplicity we take $\bar{u} > 0$)

- (ρ_{in}, u_{in}) solution of the uncontrolled problem on the whole of $\mathbb{R}$ following Matsumura-Nishida.
- η a smooth cut-off function of time taking values 1 near $t = 0$ and 0 near $t = T$.
- $(\hat{\rho}, \hat{u})$ such that

$$\rho = \bar{\rho} + \eta(t)(\rho_{in} - \bar{\rho}) + \hat{\rho}, \quad u = \bar{u} + \eta(t)(u_{in} - \bar{u}) + \hat{u}$$

The problem is reduced to find $(\hat\rho, \hat u)$ such that (after some calculations)

$$\partial_t\hat\rho + (\bar u + \eta(t)(u_{in} - \bar u) + \hat u)\partial_x\hat\rho + \bar\rho\partial_x\hat u + \frac{\bar\rho}{\nu}p'(\bar\rho)\hat\rho = f(\hat\rho, \hat u),$$

$$(\bar\rho + \eta(t)(\rho_{in} - \bar\rho))\,(\partial_t\hat u + \bar u\partial_x\hat u) - \nu\partial_{xx}\hat u + p'(\bar\rho)\partial_x\hat\rho = g(\hat\rho, \hat u).$$

$$(\hat\rho, \hat u)(t = 0) = (0, 0), \quad (\hat\rho, \hat u)(t = T) = (0, 0),$$

where f and g are small when $(\hat\rho, \hat u)$ are small, for example

$$f(\hat\rho, \hat u) = -\eta'\rho_{in} + (\eta - \eta^2)\partial_x(\rho_{in}u_{in}) - \eta\partial_x(\rho_{in}\hat u)$$

$$-\eta\hat\rho\partial_x u_{in} - \hat\rho\partial_x\hat u + \frac{\bar\rho}{\nu}p'(\bar\rho)\hat\rho.$$

- From now on we omit the notation $\hat{}$.
- The left hand side is not linear due to the term $u\partial_x\rho$ in the equation for ρ.
- When the right hand sides are given, the problem appears to be decoupled in u and then ρ.

Fixed point mapping

Following the last remark we want to define a mapping $F\ :\ (\hat\rho,\hat u))\to(\rho,u)$ such that

$$\partial_t\rho+(\bar u+\eta(t)(u_{in}-\bar u)+u)\partial_x\rho+\bar\rho\partial_x u+\frac{\bar\rho}{\nu}p'(\bar\rho)\rho=f(\hat\rho,\hat u),$$
$$(\bar\rho+\eta(t)(\rho_{in}-\bar\rho))\,(\partial_t u+\bar u\partial_x u)-\nu\partial_{xx}u+p'(\bar\rho)\partial_x\hat\rho=g(\hat\rho,\hat u).$$
$$(\rho,u)(t=0)=(0,0),\quad(\rho,u)(t=T)=(0,0).$$

This corresponds to a (linear) parabolic controllability problem for u and then to a (nonlinear) hyperbolic controllability problem for ρ.
We have to

- find a good functional class,
- show that this mapping is well defined,
- obtain estimates in order to prove existence of a fixed point.

Outline

Construction of u

We consider the control problem for the velocity u

$$(\bar{\rho} + \eta(t)(\rho_{in} - \bar{\rho}))\,(\partial_t u + \bar{u}\partial_x u) - \nu\partial_{xx} u + p'(\bar{\rho})\partial_x\hat{\rho} = g(\hat{\rho}, \hat{u}).$$
$$u(0,x) = 0, \qquad u(T,x) = 0.$$

We use Carleman estimates for the adjoint equation to obtain an observability inequality but

- we extend the domain for $x \in (\tilde{L}, 0)$ (every data extended by zero) and truncate afterwards on $(0, L)$. We then put a distributed control located in a subdomain ω.
- the control region ω is inside the part $(\tilde{L}, 0)$.
- Carleman estimates requires $\rho_{in} \in W^{1,\infty}((0,T) \times (\tilde{L}, L))$.
- Because of the needed estimates, the weight function will have to travel along the characteristics !

Instead of the usual weight functions, we take

Weight functions

Let $\tilde{L} < -3\bar{u}T$. Let $\psi = \psi(x)$, $\psi : [\tilde{L} - 2\bar{u}T, L] \mapsto [3, 4]$, with

$$\psi'(x) < 0 \text{ near } x = L \quad \text{and } \psi'(x) > 0 \text{ near } \ x = -3\bar{u}T$$
$$\inf_{(\tilde{L},L)\setminus(-3\bar{u}T,-2\bar{u}T)} |\psi'| > 0.$$

For $s, \lambda \geq 1$, define

$$\varphi(t,x) = \frac{1}{\theta(t)}\left(e^{5\lambda} - e^{\lambda\psi(x-\bar{u}t)}\right), \quad \xi(t,x) = \frac{e^{\lambda\psi(x-\bar{u}t)}}{\theta(t)}$$

with θ such that, for T_0 small,

$$\theta(t) = \begin{cases} t \text{ in } (0, T_0), \\ 1 \text{ in } (2T_0, T - 2T_0) \\ T - t \text{ in } (T - T_0, T) \end{cases} \qquad \begin{cases} \theta' > 0 \text{ in } (T_0, 2T_0), \\ \theta' < 0 \text{ in } (T - 2T_0, T - T_0). \end{cases}$$

We obtain the following estimate.

Carleman estimates

There exist $s_0, \lambda_0 > 1$ such that for all $s \geq s_0$ $\lambda \geq \lambda_0$, any smooth z such that $z(t, \tilde{L}) = z(t, L) = 0$ satisfies

$$
\begin{aligned}
s^3\lambda^4 \int_0^T \int_{\tilde{L}}^L \xi^3 e^{-2s\varphi} |z|^2 + s\lambda^2 \int_0^T \int_{\tilde{L}}^L \xi e^{-2s\varphi} |\partial_x z|^2 \\
+ \frac{1}{s} \int_0^T \int_{\tilde{L}}^L \frac{1}{\xi} e^{-2s\varphi} \left(|\partial_{xx} z|^2 + |\partial_t z|^2 \right) \\
\leq C \int_0^T \int_{\tilde{L}}^L e^{-2s\varphi} |-\partial_t z + \partial_{xx} z|^2 \\
+ Cs^3\lambda^4 \int_0^T \int_{(-3\bar{u}T, -\bar{u}T)} \xi^3 e^{-2s\varphi} |z|^2
\end{aligned}
$$

Now using classical arguments, we can obtain an observability inequality for the adjoint equation and find a controlled trajectory u which satisfies our controllability problem when restricted to $(0, L)$, the controls appearing now on the boundary condition at $x = 0$.
Moreover we can obtain an estimate on this trajectory, namely

$$s^3\lambda^4 \int_0^T \int_0^L |u|^2 e^{2s\varphi} + s\lambda^2 \int_0^T \int_0^L \frac{e^{2s\varphi}}{\xi^2} |\partial_x u|^2 + \frac{1}{s} \int_0^T \int_0^L \frac{e^{2s\varphi}}{\xi^4} \left(|\partial_t u|^2 + |\partial_{xx} u|^2\right) + \lambda \int_0^T \frac{e^{2s\varphi}}{\xi^3} |\partial_x u(t,0)|^2 \leq C \int_0^T \int_0^L \frac{e^{2s\varphi}}{\xi^3} |\hat{g}|^2 + C \int_0^T \int_0^L \frac{e^{2s\varphi}}{\xi^3} |\partial_x \hat{\rho}|^2.$$

To get estimates on

$$\int_0^T \int_0^L \frac{e^{2s\varphi}}{\xi^3} |\partial_x \rho|^2$$

in order to run the fixed point argument will be a real difficulty.

Outline

1. Introduction
   - Controllability for incompressible Navier-Stokes equations
   - Case of viscous compressible fluids
2. Our result
   - Result
   - Comments and references
3. Structure of the proof
   - Preparation for the initial and final datas
   - Fixed point mapping
4. Controlling the parabolic part. Construction of the velocity u
5. Controlling the hyperbolic part. Construction of the density ρ
6. Estimates. Main difficulties.

Construction of ρ

Let us recall the control problem for the density ρ which is now (as u is already known) a linear control problem.

$$\partial_t \rho + (\bar{u} + \eta(t)(u_{in} - \bar{u}) + u)\partial_x \rho + \bar{\rho}\partial_x u + \frac{\bar{\rho}}{\nu} p'(\bar{\rho})\rho = f(\hat{\rho}, \hat{u}),$$
$$\rho(0, x) = 0, \quad \rho(T, x) = 0 \quad \text{on } (0, L).$$

$\rightsquigarrow$ Transport equation at a velocity $\simeq \bar{u}$.

We will introduce, in a rather natural way, two functions ρ_f and ρ_b (for forward and backward) which are solutions of the following equations.

$$\partial_t \rho_f + (\bar{u} + \eta(t)(u_{in} - \bar{u}) + u)\partial_x \rho_f + \bar{\rho}\partial_x u + \frac{\bar{\rho}}{\nu} p'(\bar{\rho})\rho_f = f(\hat{\rho}, \hat{u}),$$
$$\rho_f(0, x) = 0, \quad \rho_f(t, 0) = 0,$$

and

$$\partial_t \rho_b + (\bar{u} + \eta(t)(u_{in} - \bar{u}) + u)\partial_x \rho_b + \bar{\rho}\partial_x u + \frac{\bar{\rho}}{\nu} p'(\bar{\rho})\rho_b = f(\hat{\rho}, \hat{u}),$$
$$\rho_b(T, x) = 0, \quad \rho_b(t, L) = 0.$$

- if the time T is large enough, one can glue ρ_f and ρ_b to obtain a solution of the control problem.

Let X be the flow associated to $\tilde{u} = (\bar{u} + \eta(t)(u_{in} - \bar{u}) + u)$, i.e.

$$\frac{dX}{dt}(t,s,x) = (\bar{u} + \eta(t)(u_{in} - \bar{u}) + u)\,(t, X(t,s,x)), \quad X(s,s,x) = 0.$$

Let a and b such that $a < b < 0$.
Let $\tilde{\theta}$ (regular) such that $\tilde{\theta} = 1$ for $x > b$ and $\tilde{\theta} = 0$ for $x < a$ (with $0 \leq \tilde{\theta} \leq 1$), then

$$\rho(t,x) = \rho_f(t,x)\tilde{\theta}(X(0,t,x)) + \rho_b(t,x)(1 - \tilde{\theta}(X(0,t,x))$$

solves the control problem.

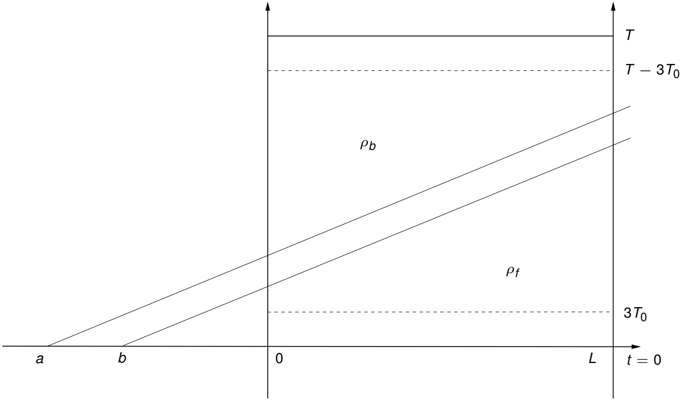

Figure: Geometric setting on a, b. The straight lines represent the lines $t \mapsto (t, a + \overline{u}t)$ and $t \mapsto (t, b + \overline{u}t)$, which approximate the flow X.

Outline

Estimating $\partial_x \rho$

How to get estimates on $\partial_x \rho$ in weighted norms !
Actually, one should rather study the equation of $\partial_x \rho$:

$$\partial_t \partial_x \rho + \partial_x \left(\eta(t)(u_{in} - \bar{u}) + u\right) \partial_x \rho$$
$$+ \left(\bar{u} + \eta(t)(u_{in} - \bar{u}) + u\right) \partial_{xx} \rho + \bar{\rho} \partial_{xx} u + \frac{\bar{\rho}}{\nu} p'(\bar{\rho}) \partial_x \rho = \partial_x f(\hat{\rho}, \hat{u}),$$
$$\partial_x \rho(0, x) = 0, \quad \partial_x \rho(T, x) = 0 \quad \text{on } (0, L).$$

$\rightsquigarrow$ $\partial_x \rho$ is similar to $\partial_{xx} u$!!! $\Rightarrow$ direct estimates do not allow to conclude !

New functions μ_f and μ_b

But we can introduce μ_f and μ_b

$$\mu_f = u + \frac{\nu}{\bar\rho^2}\partial_x \rho_f, \ \mu_b = u + \frac{\nu}{\bar\rho^2}\partial_x \rho_b$$

for which the equations do not make use of $\partial_{xx} u$ and are of pure transport !

Equation satisfied by μ_f:

$$\partial_t \mu_f + (\bar u + \eta(u_{in} - \bar u) + u)\partial_x \mu_f + k\mu_f = \tilde f,$$
$$\mu_f(0, x) = 0 \quad \text{on } (0, L),$$

and the non-trivial boundary condition:

$$\mu_f(t, 0) = u(t, 0) + \frac{\nu}{\bar\rho^2}\left(\frac{1}{\bar u + \eta(t)(u_{in} - \bar u) + u(t, 0)}\right)(\hat f(t, 0) - \bar\rho \partial_x u(t, 0)).$$

A fundamental lemma

Lemma

If

$$\partial_t \mu_f + (\bar{u} + \tilde{u})\partial_x \mu_f + k\mu_f = \tilde{f},$$
$$\mu_f(0, x) = 0 \quad \text{on } (0, L),$$

where

- *k is bounded by 1 in $L^\infty(L^\infty)$*
- *$\tilde{u}$ is small enough in $L^\infty(L^\infty) \cap W^{1,1}(L^\infty)$-norm.*

we have

$$||\mu_f \xi^{-3/2} e^{s\varphi}||_{L^2(0,T-2T_0;L^2)} \leq C||\tilde{f}\xi^{-3/2} e^{s\varphi}||_{L^2(L^2)} + C||\mu_f(t,0)\xi^{-3/2} e^{s\varphi}||_{L^2(0,T-2T_0)}.$$

On the proof of the lemma

- Solve explicitly the transport equation.
- Requires estimates on the difference between the flow

$$X(t, s, x) \quad \text{Vs} \quad x + (t - s)\bar{u}.$$

- In this step, it is important that the weight function "follows the characteristics".

Comments:

- Also true for the backward problem.
- Yields suitable estimates for $\partial_x \rho_f$, $\partial_x \rho_b$ (use both s, λ).
- A large number of other estimates are also needed (not at all straightforward...)

Fixed point strategy

We take balls of the form

$$X_{s,\lambda,R_\rho} = \{\rho \text{ such that}$$
$$\left|\xi^{-1} e^{s\varphi}\rho\right|_{L^2((0,T)\times(0,L))} \leq R_\rho, \quad \left|\xi^{-3/2} e^{s\varphi}\partial_x\rho\right|_{L^2((0,T)\times(0,L))} \leq R_\rho,$$
$$\left|\partial_t\rho\right|_{L^2((0,T)\times(0,L))} \leq R_\rho, \quad \left|e^{s\check{\varphi}/2}\rho\right|_{L^\infty((0,T)\times(0,L))} \leq R_\rho,$$
$$\left|e^{s\check{\varphi}/2}\partial_x\rho\right|_{L^\infty((0,T);L^2(0,L))} \leq R_\rho, \quad \left|\lambda^{1/2}[\xi^{-3/2}e^{s\varphi}\rho](\cdot,0)\right|_{L^2(0,T)} \leq R_\rho,$$
$$\left|\lambda^{1/2}[\xi^{-3/2}e^{s\varphi}\rho](\cdot,L)\right|_{L^2(0,T)} \leq R_\rho\},$$

$$Y_{s,\lambda,R_u} = \{u \text{ such that } u(t,L) = 0,\ t \in (0,T),$$
$$\left|s^{3/2}\lambda^2 e^{s\varphi}u\right|_{L^2((0,T)\times(0,L))} \leq R_u, \quad \left|s^{1/2}\lambda\xi^{-1}e^{s\varphi}\partial_x u\right|_{L^2((0,T)\times(0,L))} \leq R_u,$$
$$\left|s^{-1/2}\xi^{-2}e^{s\varphi}\partial_{xx}u\right|_{L^2((0,T)\times(0,L))} \leq R_u, \quad \left|s^{-1/2}\xi^{-2}e^{s\varphi}\partial_t u\right|_{L^2((0,T)\times(0,L))} \leq R_u\}.$$

For s, λ large enough and R_ρ, R_u small enough, our map sends $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$ into itself and then by Schauder fixed point theorem, we get a controlled trajectory.

Thank you for your attention.