The equations of elasticity and the problem of dynamic cavitation

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Outline

The equations of elasticity, requirements from mechanics

Extensions of polyconvex elasticity

Radial solutions for elastic isotropic materials

The equations of elasticity

$$\frac{\partial^2 y}{\partial t^2} = \mathrm{div}\, \frac{\partial W}{\partial F}(\nabla y)$$

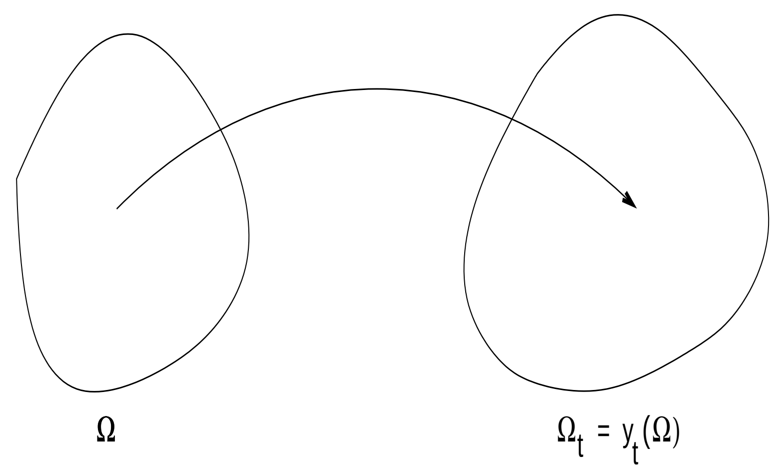
$$\begin{cases} \partial_t F_{i\alpha} = \partial_\alpha v_i \\ \partial_t v_i = \partial_\alpha \dfrac{\partial W}{\partial F_{i\alpha}}(F) \\ \partial_\alpha F_{i\beta} - \partial_\beta F_{i\alpha} = 0 \end{cases}$$

motion $y(x,t)$

velocity $v = \dfrac{\partial y}{\partial t}$

deformation gradient $F = \nabla y$

$W(F)$ stored energy $S = \dfrac{\partial W}{\partial F}$



Requirements from mechanics

- **Material frame indifference**

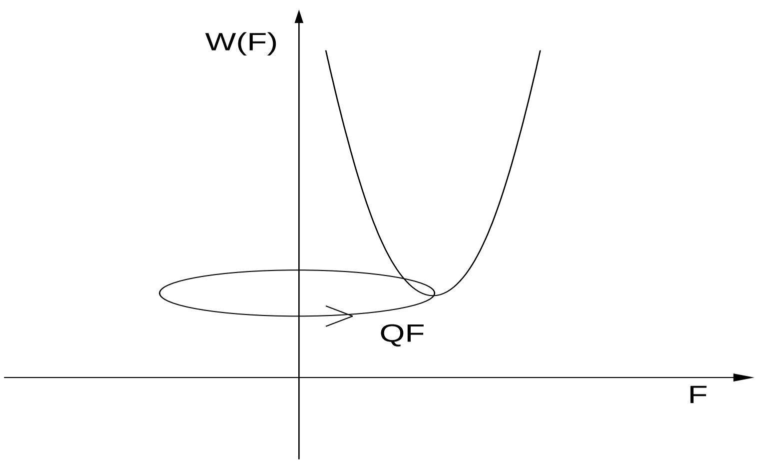
$$W(QF) = W(F) \quad \forall\, Q \in \mathcal{O}^3$$

- **Realizability of mechanical motions**

avoid interpenetration of matter

at least positivity of the Jacobian

$$\det F > 0$$



$$W(F) \to \infty \quad \text{as} \quad \det F \to 0$$

It is too restrictive to take $W(F)$ convex

$$\text{Hyperbolicity} \quad \Longleftrightarrow \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}}(F)\, \xi_i \xi_j \nu_\alpha \nu_\beta > 0 \quad \forall \xi \neq 0\,,\, \nu \in \mathcal{S}^2$$

$$\Longleftrightarrow \quad W(F) \quad \text{is } \text{rank-1 convex}$$

Energy identity

$$\partial_t \left(\frac{1}{2} |v|^2 + W(F) \right) + \partial_\alpha \left(v_i \frac{\partial W}{\partial F_{i\alpha}} \right) = 0$$

Question

Conservation law theory is intricately connected to convexity of the energy.
How to handle the elastodynamics system ?

Null-Lagrangians and transport identities

$W(F)$ is polyconvex

$$W(F) = g(F, \text{cof}\, F, \det F) = g \circ \Phi(F) \qquad \text{with } g(\Xi) \text{ convex}$$

$\Phi(F)$ is a null-Lagrangean iff

$$\int_\Omega \Phi(\nabla y + \nabla \phi)\, dx = \int_\Omega \Phi(\nabla y) dx \quad \forall\, y \in W^{1,p},\ \phi \in C_c^\infty$$

$$\iff \quad \partial_\alpha \left(\frac{\partial \Phi}{\partial F_{i\alpha}}(\nabla y) \right) = 0 \quad \text{in } \mathcal{D}'$$

$$\iff \quad \Phi(F) = \alpha\, F + \beta\, \text{cof}\, F + c \det F$$

null-Lagrangeans $\Phi(\nabla y)$ are weakly continuous in $W^{1,p}$.

J. Ball 77, J. Ericksen 62

Transport identities

$$\partial_t F_{i\alpha} = \partial_\alpha v_i$$

$$\partial_t \Phi^A(F) = \partial_\alpha \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}} v_i \right) \quad A = 1, ..., 19$$

identity follows from property $\partial_\alpha \left(\frac{\partial \Phi}{\partial F_{i\alpha}} (\nabla y) \right) = 0$ in $\mathcal{D}'$

$$\frac{\partial}{\partial t} \det F = \frac{\partial}{\partial x^\alpha} \big((\operatorname{cof} F)_{i\alpha} v_i \big)$$

$$\frac{\partial}{\partial t} (\operatorname{cof} F)_{k\gamma} = \frac{\partial}{\partial x^\alpha} (\epsilon_{ijk} \epsilon_{\alpha\beta\gamma} F_{j\beta} v_i)$$

T. Qin 98

The extended elasticity system

Elasticity with transport identities; variables (v, F)

$$\partial_t v_i = \partial_\alpha \left(\frac{\partial g}{\partial \Xi^A}(\Phi(F)) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \right)$$
$$\partial_t \Phi(F)^A = \partial_\alpha \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\, v_i \right).$$

EXTENDED ELASTICITY SYSTEM; variables (v, Ξ)

$$\partial_t v_i = \partial_\alpha \left(\frac{\partial g}{\partial \Xi^A}(\Xi) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \right)$$
$$\partial_t \Xi^A = \partial_\alpha \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\, v_i \right).$$

both subject to the propagating constraint - involutions

$$\partial_\alpha F_{i\beta} - \partial_\beta F_{i\alpha} = 0$$

Properties of extension

(a) Elastodynamics is viewed as a constrained evolution:

$$\Xi(\cdot,0) = \Phi(F(\cdot,0)) \quad \Longrightarrow \quad \Xi(\cdot,t) = \Phi(F(\cdot,t)) \ \ \forall t$$

(b) The extended system is symmetrizable

$$\partial_t \left(\frac{1}{2}|v|^2 + g(\Xi) \right) - \partial_\alpha \left(v_i \frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} \right) = 0$$

(c) Relative entropy. Let (v,F) entropy weak solution, $(\bar{v},\bar{F})$ Lipschitz (conservative) solution

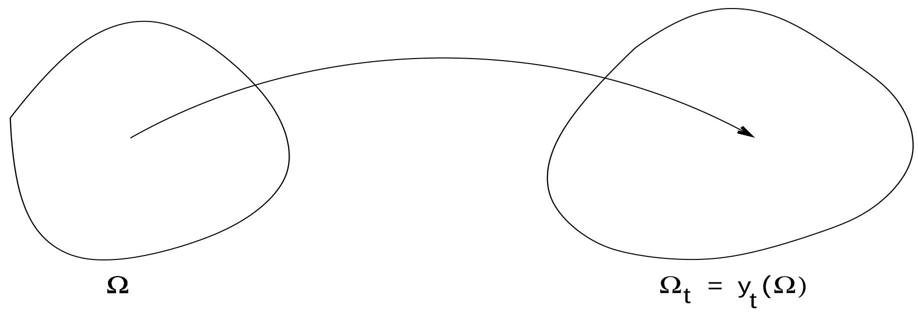
relative entropy $\quad \eta((v,F)|(\bar{v},\bar{F})) = \frac{1}{2}|v-\bar{v}|^2 + g\left(\Phi(F)|\Phi(\bar{F})\right)$

relative flux $\quad q_\alpha((v,F)|(\bar{v},\bar{F})) = \left(\frac{\partial g}{\partial \Xi^A}(\Phi(F)) - \frac{\partial g}{\partial \Xi^A}(\Phi(\bar{F})) \right) (v_i - \bar{v}_i) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)$

$$\partial_t \eta_{rel} + \operatorname{div} q_{rel} \le O\left(\|\nabla \bar{F}\|_\infty \right) |\Phi(F) - \Phi(\bar{F})|^2$$

Lattanzio-T. 06, Dafermos 06

Elasticity in Eulerian coordinates



$$D_t\varphi = \partial_\alpha \psi_\alpha + p \quad \longleftrightarrow \quad \partial_t \frac{\varphi}{\det F} + \partial_j\big(u_j \frac{\varphi}{\det F}\big) = \partial_j\big(\psi_\alpha \frac{F_{j\alpha}}{\det F}\big) + \frac{p}{\det F}$$

Consider for simplicity $W(F) = G \circ (\Phi(F)) = g(F) + h(\det F)$

$$S_{i\alpha} = \frac{\partial g}{\partial F_{i\alpha}}(F) + \frac{\partial h}{\partial w}(\det F)\frac{\partial \det F}{\partial F_{i\alpha}}$$

$$T_{il} = S_{i\alpha}\frac{1}{\det F}F_{l\alpha} = \frac{\partial g}{\partial F_{i\alpha}}(F)(\operatorname{cof} F^{-1})_{\alpha l} + \frac{\partial h}{\partial w}(\det F)\delta_{il}$$

$$= \frac{\partial g}{\partial F_{i\alpha}}\Big(\frac{(\operatorname{cof} H)^T}{\rho}\Big)(\operatorname{cof} H)_{\alpha l} + \frac{\partial h}{\partial w}\Big(\frac{1}{\rho}\Big)\delta_{il}$$

where we used $H = F^{-1}$, $\rho = \frac{1}{\det F}$, $F^{-1} = \frac{1}{\det F}(\operatorname{cof} F)^T$

Extended Eulerian elasticity system

variables are (u, ρ, H, A) where $H = F^{-1}$, $A \sim \text{cof}\, H$, $\rho \sim \det H$

$$\begin{aligned}
\partial_t \rho + \partial_j(\rho u_j) &= 0 \\
\partial_t(\rho u_i) + \partial_j(\rho u_i u_j) &= \partial_l \Big(\frac{\partial g}{\partial F_{i\alpha}}(\frac{A^T}{\rho}) A_{\alpha l} + \frac{\partial h}{\partial w}(\frac{1}{\rho}) \delta_{il} \Big) \\
\partial_t A_{\alpha i} + \partial_j(u_j A_{\alpha i}) &= \partial_l(u_i A_{\alpha l}))
\end{aligned}$$

subject to the constraint $\partial_l A_{\alpha l} = 0$ which propagates from the initial data.

System has an "entropy"

$$\partial_t \rho \Big(\frac{1}{2}|u|^2 + g(\frac{A^T}{\rho}) + h(\frac{1}{\rho}) \Big) + \partial_j u_j \rho \Big(\frac{1}{2}|u|^2 + g(\frac{A^T}{\rho}) + h(\frac{1}{\rho}) \Big) = \partial_l (T^{ext}_{il} u_i)$$

Note that $E = \rho \Big(\frac{1}{2} |\frac{m}{\rho}|^2 + g(\frac{A^T}{\rho}) + h(\frac{1}{\rho}) \Big)$ is convex in (m, ρ, A) Wagner 09

For gas dynamics the original and the extended system coincide.

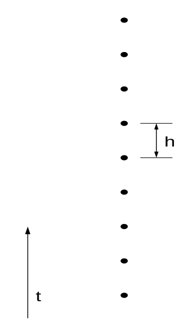
Variational approximation 3-d

Extended elasticity system:

$$\partial_t v_i = \partial_\alpha \left(\frac{\partial g}{\partial \Xi^A}(\Xi) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \right)$$
$$\partial_t \Xi^A = \partial_\alpha \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\, v_i \right).$$

suggests the implicit-explicit iterative scheme : time step h

$$\frac{v_i^J - v_i^{J-1}}{h} = \frac{\partial}{\partial x^\alpha} \left(\frac{\partial g}{\partial \Xi^A}(\Xi^J)\, \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F^{J-1}) \right)$$
$$\frac{(\Xi^J - \Xi^{J-1})^A}{h} = \frac{\partial}{\partial x^\alpha} \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F^{J-1}) v_i^J \right).$$



Iterates (v, Ξ) constructed by solving the constrained variational problem
Given v^0, $\Xi^0 = (F^0, Z^0, w^0)$,

$$\min \int_{\mathbb{T}^3} \left(\frac{1}{2}|v - v^0|^2 + g(F, Z, w) \right) dx$$

over the affine subspace

$$\mathcal{C} := \Big\{ (v, F, Z, w) : \mathbb{T}^3 \to \mathbb{R}^{22} \text{ subject to the constraints}$$
$$\frac{1}{h}(F_{i\alpha} - F^0_{i\alpha}) = \partial_\alpha v_i \,,$$
$$\frac{1}{h}(Z_{k\gamma} - Z^0_{k\gamma}) = \partial_\alpha(\epsilon_{ijk}\epsilon_{\alpha\beta\gamma} F^0_{j\beta} v_i) \,,$$
$$\frac{1}{h}(w - w^0) = \partial_\alpha((\mathrm{cof}\, F^0)_{i\alpha} v_i) \qquad \Big\}.$$

Iterates decrease the mechanical energy, obey bounds

$$\sup_j \int_{\mathbb{T}} \frac{1}{2}|v^j|^2 + g(\Xi^j)\, dx + \sum_j |v^j - v^{j-1}|^2_{L^2_x} + |\Xi^j - \Xi^{j-1}|^2_{L^2_x} \le E_0$$

Existence of dissipative mv-solutions

Under coercivity for g and bounds for g and $\frac{\partial g}{\partial \Xi}$ we obtain a Young measure ν and a nonnegative concentration measure $\gamma(dxdt)$ such that

$$v^h \rightharpoonup v \quad \text{wk in } L^2$$

$$(F^h, Z^h, w^h) \rightharpoonup (F, \operatorname{cof} F, \det F) \quad \text{wk in } L^p \times L^q \times L^r$$

where $F = \langle \nu, \lambda_F \rangle$, $v = \langle \nu, \lambda_v \rangle$ satisfy

$$\partial_t v_i - \partial_\alpha \Big\langle \nu, \frac{\partial G}{\partial \Xi^A}(\lambda_\Xi) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\lambda_F) \Big\rangle = 0$$

$$\partial_t \Phi^A(F) - \partial_\alpha \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) v_i \Big) = 0$$

and

$$\iint \frac{d\theta}{dt} \Big(\langle \nu, \eta \rangle + \gamma \Big) dxdt + \int \theta(0) \eta_0(x) dx \geq 0 \,,$$

for test functions $\theta(t) \geq 0$

i.e, (v, F) is a dissipative measure-valued solution of elasticity, which satisfies the weak from of the geometric transport identities

Demoulini-Stuart-AT 01

Let ν, γ, v, F be a dissipative measure valued solution and $\hat{v}$, $\hat{F} \in W^{1,\infty}$ a Lipschitz solution. Then

$$\int \langle \nu, \eta\big((\lambda_v, \lambda_F)|(\bar{v}, \bar{F})\big)\rangle \, dx \le c_1 \Big(\int \eta((v_0, F_0)|(\bar{v}_0, \bar{F}_0)) \, dx \Big) \, e^{c_2 t}$$

$$\text{where} \quad \eta((v, F)|(\bar{v}, \bar{F})) = \frac{1}{2}|v - \bar{v}|^2 + g\left(\Phi(F)|\Phi(\bar{F})\right)$$

Based on an averaged relative entropy calculation and using the weak form of the transport identities and the null-Lagrangean property.

Uniqueness of classical solutions for polyconvex elasticity within the class of mv-solutions:

If $v_0 = \hat{v}_0$ and $F_0 = \hat{F}_0$ then

$$(v, F) = (\hat{v}, \hat{F}) \,, \quad \nu = \delta_{\hat{v}(x,t), \hat{F}(x,t)} \,, \quad \gamma = 0 \text{ on } Q_T$$

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Radial isotropic elasticity

$$\frac{\partial^2 y}{\partial t^2} = \text{div}\,\frac{\partial W}{\partial F}(\nabla y)$$

- Radial motions $y(x,t) = w(R,t)\frac{x}{R}$, $R = |x|$, $x \in \mathbb{R}^3$

$$F = w_R \hat{x} \otimes \hat{x} + \frac{w}{R}\hat{x}^\perp \otimes \hat{x}^\perp$$

- $W(F)$ is frame indifferent and isotropic

$$W(F) = \Phi(v_1, v_2, v_3)$$

with Φ symmetric function of eigenvalues v_1, v_2, v_3 of $\sqrt{F^T F}$ and polyconvex, *e.g.*

$$\Phi = \frac{1}{2}(v_1^2 + v_2^2 + v_3^2) + h(v_1 v_2 v_3) \quad \text{with } h(\delta) \to +\infty \text{ as } \delta \to 0+$$

- Piola-Kirchhoff stress

$$S = \frac{\partial W}{\partial F} = \frac{\partial \Phi}{\partial v_1}\hat{x} \otimes \hat{x} + \frac{\partial \Phi}{\partial v_2}\hat{x}^{\perp} \otimes \hat{x}^{\perp}$$

- To represent a physically realizable motion: $\det F > 0$ with $F = \nabla y$.

$$\det F = w_R (w/R)^2 > 0$$

Also sufficient condition for avoiding interpenetration of matter.

$$w_{tt} = \frac{1}{R^2}\,\partial_R\left(R^2 \frac{\partial \Phi}{\partial v_1}\big(w_R, \frac{w}{R}, \frac{w}{R}\big)\right) - \frac{1}{R}\Big(\frac{\partial \Phi}{\partial v_2} + \frac{\partial \Phi}{\partial v_3}\Big)\big(w_R, \frac{w}{R}, \frac{w}{R}\big)$$

$$w_{tt} = \frac{1}{R^2}\,\partial_R\left(R^2\frac{\partial\Phi}{\partial v_1}\big(w_R, \frac{w}{R}, \frac{w}{R}\big)\right) - \frac{1}{R}\Big(\frac{\partial\Phi}{\partial v_2} + \frac{\partial\Phi}{\partial v_3}\Big)\big(w_R, \frac{w}{R}, \frac{w}{R}\big)$$

$$w_{tt} = \frac{1}{R^2}\,\partial_R\left(R^2\frac{\partial\Phi}{\partial v_1}\big(w_R, \frac{w}{R}, \frac{w}{R}\big)\right) - \frac{1}{R}\Big(\frac{\partial\Phi}{\partial v_2} + \frac{\partial\Phi}{\partial v_3}\Big)\big(w_R, \frac{w}{R}, \frac{w}{R}\big)$$

Set

$$v = w_t \quad a = w_R \quad b = \frac{w}{R}$$

Write as a first order system

$$\begin{aligned}
\partial_t v &= \frac{1}{R^2}\,\partial_R\left(R^2\frac{\partial\Phi}{\partial v_1}\big(a, b, b\big)\right) - \frac{1}{R}\Big(\frac{\partial\Phi}{\partial v_2} + \frac{\partial\Phi}{\partial v_3}\Big)\big(a, b, b\big)\\
\partial_t a &= \partial_R v\\
\partial_t b &= \frac{1}{R} v
\end{aligned}$$

Null-Lagrangeans: Potential energies $\Psi(v_1, v_2, v_3; R)$ for which the functional

$$I[w] = \int_0^1 \Psi\big(\big(w_R, \frac{w}{R}, \frac{w}{R}\big); R\big)\, dR$$

has variational derivative zero: $\Psi = v_1,\ v_1 v_2 R,\ v_1 v_3 R,\ v_1 v_2 v_3 R^2$

Euler-Lagrange identities

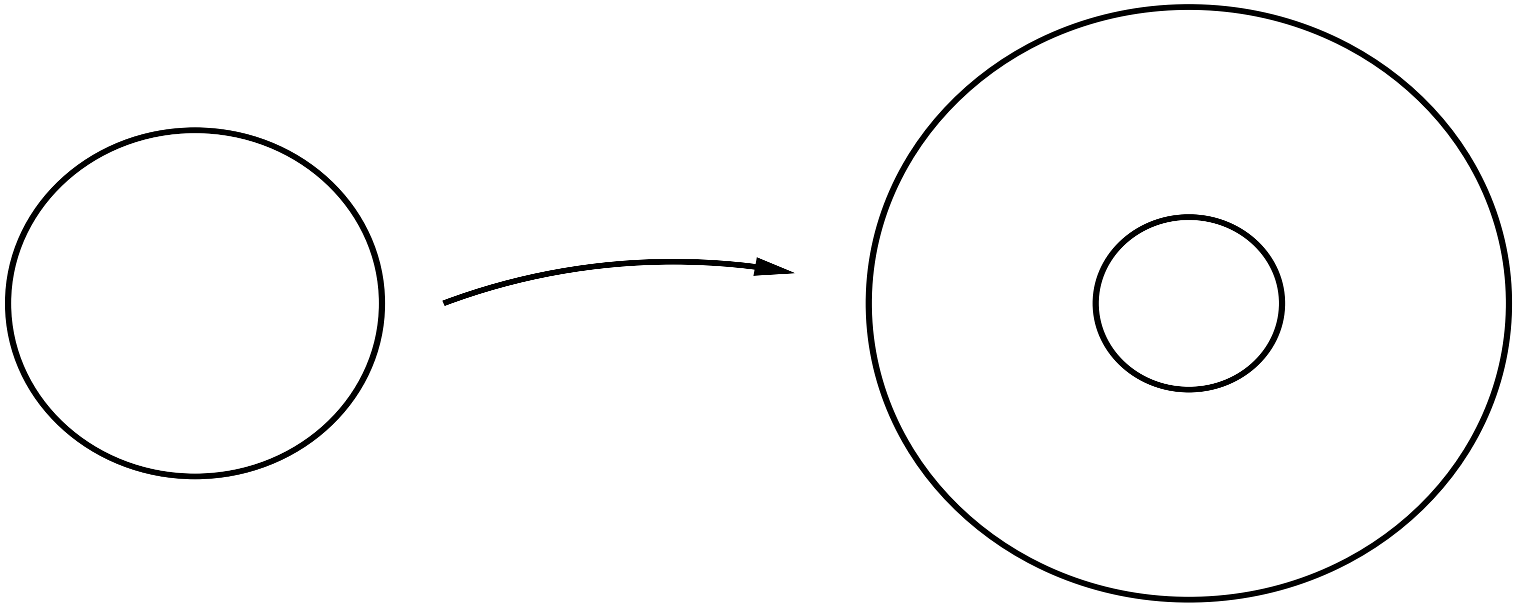
$$-\partial_R \Psi_{,1} + R^{-1}\left(\Psi_{,2} + \Psi_{,3}\right) = 0 \quad \forall w\,.$$

transport identities

$$\partial_t \Psi = \partial_R \left(\Psi_{,1}\, v\right)$$

It is possible to write (more than one) extended systems for polyconvex radial elasticity endowed with a convex entropy

Radial elasticity - Cavitation



$$y(x,t) = w(R,t)\frac{x}{R} = t\varphi\left(\frac{R}{t}\right)\frac{x}{R}$$

Thm (Pericak-Spector and Spector '88 , 95)
Under certain assumptions (main hyp $h'' > 0$, $h''' < 0$)

$$\Phi(v_1, v_2, v_3) = \frac{1}{2}(v_1^2 + v_2^2 + v_3^2) + h(v_1 v_2 v_3)$$

there exist cavitating solutions for dimension $d \geq 3$

$$R^2 w_{tt} = \partial_R \left(R^2 \frac{\partial \Phi}{\partial v_1} (w_R, \frac{w}{R}, \frac{w}{R}) \right) - R \Big(\frac{\partial \Phi}{\partial v_2} + \frac{\partial \Phi}{\partial v_3} \Big) (w_R, \frac{w}{R}, \frac{w}{R}) \ , \ 0 < R < 1 \, , \ t > 0$$

$$w(1,t) = \lambda$$

- One solution: uniformly deformed state $w(R,t) = \lambda R$
- Ansatz: self-similar solution

$$w(R,t) = t\varphi(\frac{R}{t}) \qquad s = \frac{R}{t}$$

$$w(0,t) = t\varphi(0)$$

$$(s^2 - \Phi_{11})\ddot{\varphi} = \frac{2}{s}(\dot{\varphi} - \frac{\varphi}{s}) \underbrace{\left[\Phi_{12} + \frac{\Phi_1 - \Phi_2}{\dot{\varphi} - \frac{\varphi}{s}} \right]}_{P(\dot{\varphi}, \frac{\varphi}{s}, \frac{\varphi}{s}) \geq 1}$$

Write as $a = \dot{\varphi}$, $b = \frac{\varphi}{s}$

$$\big(s^2 - \Phi_{11}(a,b,b)\big)\dot{a} = \frac{2}{s}(a-b)P(a,b,b)$$
$$\dot{b} = \frac{1}{s}(a-b)$$

- Sonic line $s^2 - \Phi_{11} = 0$
- One class of special solutions: uniform states : $a = b$
- Second class shocks

$$s^2[a] - [\Phi_1] = 0$$
$$[b] = 0$$

  $b_- = b_+$, Lax admissibility implies $a_- < a_+ = \lambda$
- Third class: continuous solutions of ode
- Monotonicity property, Resolution of singularity at $s = 0$
- Cauchy stress $T(0) = 0$ implies that $v(0) = \det F(0) = H$

The cavitating solution decreases the mechanical energy

$$E(y, B_\rho) = \int_{B_\rho} \frac{1}{2}|y_t|^2 + W(\nabla y)dx$$

$y_h = \lambda x$ is the homogeneous solution
y_c the solution with the cavity

$$\begin{aligned} E(y_c, B_\rho) - E(y_h, B_\rho) = (t\sigma)^3 c \Big[& \Phi(a_-, \lambda, \lambda) - \Phi(\lambda, \lambda, \lambda) \\ & + \frac{1}{2}\big(\Phi_1(a_-, \lambda, \lambda) + \Phi_1(\lambda, \lambda, \lambda)\big)(\lambda - a_-)\Big] \\ < 0 \qquad & \text{whenever} \quad a_- < \lambda \end{aligned}$$

Fracture in 1-d

In one-space dimension

$$y_{tt} = \partial_x \tau(y_x) \qquad \longleftrightarrow \qquad \begin{cases} u_t - v_x = 0 \\ v_t - \tau(u)_x = 0 \end{cases}$$

Homogeneous strain $\quad y_h(x,t) = \lambda x$, $y_h(\pm 1, t) = \pm\lambda$

The ansatz

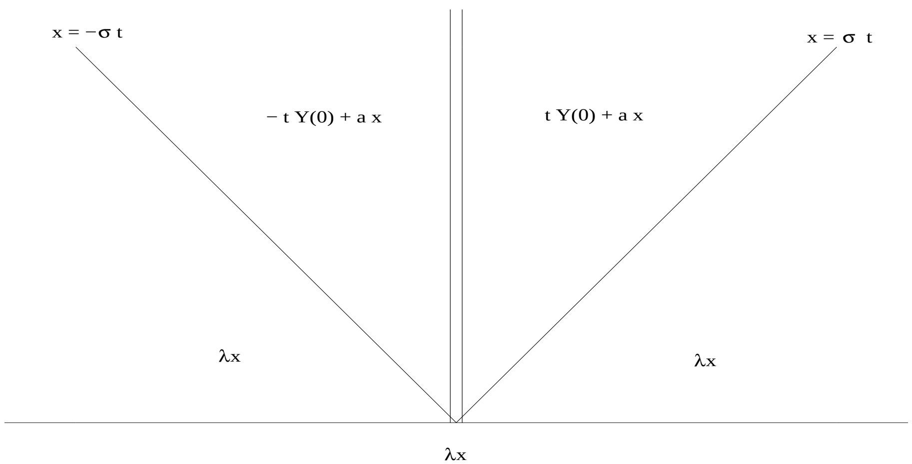
$$y(x,t) = t\varphi\Big(\frac{|x|}{t}\Big)\frac{x}{|x|} = tY(\frac{x}{t})$$

for $Y(0) > 0$ could conceivably provide a cavitating (in fact fracturing) solution, except that

$$\partial_x y = Y'(\xi) + 2Y(0)\delta_{\xi=0} \quad \text{and} \quad \tau(\partial_x y) = ??$$

Test the following candidate

$$y(x,t) = tY(\frac{x}{t}) \qquad Y(\xi) = \begin{cases} \lambda\xi & \xi < -\sigma \\ -Y(0) + \alpha\xi & -\sigma < \xi < 0 \\ Y(0) + \alpha\xi & 0 < \xi < \sigma \\ \lambda\xi & \sigma < \xi \end{cases}$$



$$Y(0) = \sigma(\lambda - \alpha)$$

$$\sigma^2 = \frac{\tau(\lambda) - \tau(\alpha)}{\lambda - \alpha}$$

$\tau''(u) < 0$ and $\alpha < \lambda \Longrightarrow$ both shocks admissible

slic solutions

Defn $y(x,t)$ is a singular limiting induced from continuum solution (slic solution) if $y_n = \phi_n \star y$ where ϕ is a mollifier satisfies

$$\partial_{tt} y_n - \partial_x \tau(\partial_x y_n) =: f_n \to 0 \quad \text{in } \mathcal{D}'$$

For the previous example

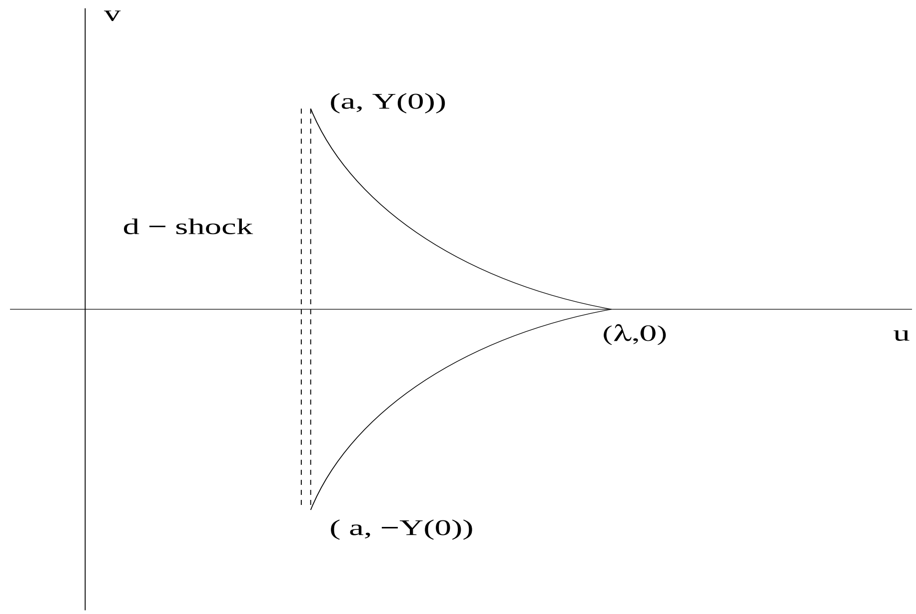
$$< \partial_{tt} y_n - \partial_x \tau(\partial_x y_n), \psi > = \int_0^\infty \int_{-\frac{1}{n}}^{\frac{1}{n}} \tau\big(\alpha + 2\phi_n(x) t Y(0)\big) \psi_x \, dx dt$$
$$\to 2tY(0)L \int_0^\infty \partial_x \psi(0,t) dt$$

where $L = \lim_{u\to\infty} \frac{\sigma(u)}{u}$

$L > 0$ then $y(x,t)$ not a solution

$L = 0$ then $y(x,t)$ is a slic-solution

Riemann diagram



$$\frac{d}{dt}\int_B \frac{1}{2}v_n^2 + W(\partial_x y_n)dx = \int_B f_n v_n dx$$

$$= Y(0)^2\sigma - 2\sigma(W(\alpha) - W(\lambda)) + 2\int_0^{\frac{1}{n}} \tau\big(\alpha + 2\phi_n(x)tY(0)\big)2Y(0)\phi_n(x)dx$$

$$\to \mu_{-\sigma} + \mu_\sigma + p_c =: P_{wf} > 0$$

If $\tau_\infty < \infty$ then $P_{wf} = +\infty$; if $\tau_\infty < \infty$ then $0 < P_{wf} < \infty$

Cavitation in 3-d

$$y(x,t) = \frac{r(s)}{s}x\,, \quad s = \frac{|x|}{t}$$

is the solution constructed by Spector - Spector.

We say a solution of the form $y(x,t) = \frac{r(s)}{s}x$, $s = \frac{x}{t}$ is a slic-solution if $y_n = \frac{r_n(s)}{s}x$, $r_n = \phi_n \star r$ satisfies

$$\frac{\partial^2 y_n}{\partial t^2} - \text{div}\,\frac{\partial W}{\partial F}(\nabla y_n) =: f_n \to 0 \quad \text{in } \mathcal{D}'$$

- If $\lim_{u\to\infty} \frac{h'(u^3)}{u} = 0$ then the weak solution constructed by Spector - Spector is a slic-solution.
- If $L = \lim_{u\to\infty} \frac{h(u)}{u}$ then

$$\lim_{n\to\infty} \int_B \frac{1}{2}v_n^2 + W(\nabla y_n) - W(\lambda)dx$$

$$= \underbrace{\int_B \Big[\frac{1}{2}v^2 + W(\nabla y) - W(\lambda)\Big]dx}_{<0} + t\underbrace{\frac{\omega_d}{d}r(0)^d L}_{>0} = P_{wf} > 0$$