Problèmes elliptiques
partiellement et globalement surdéterminés

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Serrin's problem

A model in fluid mechanics (R. L. Fosdick).

$$\begin{cases} -\Delta u = 1 & \text{in} \quad \Omega, \\ \quad u = 0, \qquad \frac{\partial u}{\partial \nu} = const. & \text{on} \quad \partial\Omega \end{cases} \qquad (P)$$

is used to study a viscous incompressible fluid moving through a straight pipe with planar cross section $\Omega \subset \mathbb{R}^2$.

- u represents the *flow velocity of the fluid,*
- $\frac{\partial u}{\partial \nu}$ is the *tangential stress on the pipe,*
- the Dirichlet condition is the *adherence condition to the wall.*

Question : Assuming that the *adherence condition* holds on the entire pipe wall, and that the *tangential stress* is constant everywhere on the wall, is it true that the pipe has a circular cross section ? (i.e. Is true that Ω a disk ?)

To answer Fosdick's question, *J. Serrin* considered the overdetermined boundary value problem

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ \quad u > 0 & \text{in} \quad \Omega, \\ \quad u = 0, \qquad \frac{\partial u}{\partial \nu} = const. & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

and proved the following celebrated result

Theorem 1 (J. Serrin, 1971) *Let Ω be a connected bounded open set of class C^2. If the overdetermined boundary value problem* (1) *admits a $C^2(\overline{\Omega})$, then Ω must be a ball and u is radially symmetric about its center.*

Serrin's problem in unbounded domains

In 1997, *H. Berestycki, L. Caffarelli* and *L. Nirenberg*, motivated by questions on the regularity of some one-phase free boundary problems, are led to the study of semilinear problems of *bistable type* in *globally Lipschitz unbounded* domains.

• They considered the case of a *smooth, globally Lipschitz epigraph*, i.e. a domain Ω of the form :

$$\Omega := \{ (x^{'}, x_N) \in \mathbb{R}^N \, : \, \varphi(x^{'}) < x_N \},$$

where $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ is a globally Lipschitz smooth function.

- A typical ex. of *bistable nonlinearity* is given by

$$f(u) = u(1-u^2)$$

$$\Downarrow$$

$$-\Delta u = u - u^3 \qquad \text{(Allen-Cahn equation)}$$

Theorem 2 (Berestycki, Caffarelli, Nirenberg, 1997) *Let $f \in C^1$ be of bistable type and Ω a globally lipschitz smooth epigraph of $\mathbb{R}^N$, with $N \geq 2$, and satisfying the additional condition on the graph of φ :*

$$(H_\varphi) \qquad \forall \tau \in \mathbb{R}^{N-1}, \qquad \lim_{|x'| \to +\infty} (\varphi(x'+\tau) - \varphi(x')) = 0.$$

If the overdetermined boundary value problem

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ \quad u > 0 & \text{in} \quad \Omega, \\ \quad u = 0, \quad \frac{\partial u}{\partial \nu} = const. & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

admits a bounded $C^2(\overline{\Omega})$-solution, then Ω is a half-space.

Conjecture (Berestycki, Caffarelli, Nirenberg, 1997) *Assuming that* Ω *is a smooth domain with* Ω^c *connected and that there exists a* bounded *smooth solution of*

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ \quad u > 0 & \text{in} \quad \Omega, \\ \quad u = 0, \quad \frac{\partial u}{\partial \nu} = const. & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

for some Lipschitz function f, then Ω *is either a half-space, a ball, a circular-cylinder-type domain:* $\mathbb{R}^j \times B$*, with* B *a ball in* $\mathbb{R}^{N-j}$ *or the complement of one these regions.*

Theorem 3 (F.) *Let $f \in C^1$ be of bistable type and Ω a globally lipschitz smooth epigraph of $\mathbb{R}^N$, with $\underline{N = 2, 3}$. If the overdetermined boundary value problem*

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ \quad u > 0 & \text{in} \quad \Omega, \\ \quad u = 0, \quad \frac{\partial u}{\partial \nu} = const. & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

admits a bounded $C^2(\overline{\Omega})$-solution, then (up to isometry) Ω is the half-space $\mathbb{R}^N_+ := \mathbb{R}^{N-1} \times (0, +\infty)$ and u is one-dimensional and monotone (that is $u = u(x_N)$ and $\frac{\partial u}{\partial x_N} > 0$).

- No additional assumption (H_φ), but we require $N = 2, 3$.

Proof

The proof is in 3 steps.

Step 1 - Monotonicity of u - If u solves (2) then u is monotone with respect to the x_N-variable, that is

$$\frac{\partial u}{\partial x_N} > 0 \qquad \text{in} \quad \Omega.$$

[*Berestycki, Caffarelli, Nirenberg* (1997)].

Step 2 - A geometric Poincaré-type formula - By step 1 and the implicit function theorem, any connected component of each level set $\{ u = \alpha \}$ is a $N-1$-dimensional smooth manifold, hence we can introduce the principal curvatures $\kappa_1, \ldots, \kappa_{N-1}$ at any point of such manifold. We set

$$\mathcal{K}^2 := \kappa_1^2 + \ldots + \kappa_{N-1}^2$$

∇_T = tangential gradient along level sets
(*the orthogonal projection of the gradient on the tangent space to level sets*).

Theorem 4 *Under the assumptions of Theorem 3 we have*

$$\int_\Omega \left(|\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2\right)\varphi^2 \le \int_\Omega |\nabla u|^2 |\nabla \varphi|^2 \tag{3}$$

for any $\varphi \in C_c^{0,1}(\mathbb{R}^N)$.

- Inequality (3) holds for any f (*not necessarily of bistable type*).
- Theorem 4 holds in any dimension $N \ge 2$.

Step 3 - End of proof of Theorem 3 for $N = 2$

- $\forall\, R > 1, \qquad \varphi_R(x) := \chi_{B_{\sqrt{R}}}(x) + \frac{2\ln(R/|x|)}{\ln R}\chi_{B_R \setminus B_{\sqrt{R}}}(x).$

- Plug φ_R inside (3) to obtain

$$\int_{\Omega \cap B_{\sqrt{R}}} \Big(|\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2 \Big) \leq \frac{C}{(\ln R)^2} \int_{\Omega \cap (B_R \setminus B_{\sqrt{R}})} \frac{1}{|x|^2} \leq \frac{C'}{\log R}$$

for appropriate C, $C' > 0$, since $|\nabla u|$ is bounded.

- $R \longrightarrow +\infty \qquad \Longrightarrow \qquad \mathcal{K} = |\nabla_T |\nabla u|| \equiv 0 \quad \textit{on} \quad \Omega$

$$\Downarrow$$

$$\Omega = \textit{half-plane}.$$

Question (*J.L. Vázquez*, Stockholm 2008)

" *What happens for the overdetermined boundary value problem* (2) *when* Ω *is a cone ?* "

For $N \geq 2$, given $\alpha \in (0, +\infty)$, we write the cone as

$$\mathcal{C} := \Big\{ (x^{'}, x_N) \in \mathbb{R}^N \, : \, x_N > \alpha |x^{'}| \Big\}.$$

Theorem 6 (F.) *Let* $f \in C^1$ *and* $N = 2, 3$. *There exists no solution* $u \in C^2(\overline{\mathcal{C}} \setminus \{0\}) \cap W^{1,\infty}(\mathcal{C})$ *of*

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \mathcal{C}, \\ u > 0 & \text{in} \quad \mathcal{C}, \\ u = 0 & \text{on} \quad \partial\mathcal{C}, \\ \frac{\partial u}{\partial \nu} = c & \text{on} \quad \partial\mathcal{C} \setminus \{0\}. \end{cases} \tag{6}$$

- f is any (*not necessarily of bistable type*).

Partially overdetermined problems

The above results can be seen as an example of *partially overdetermined problem* (and for a *non-smooth domain*).

A partially overdetermined problem is a *bvp whose overdetermined prescription occurs only on a portion of the boundary.*

Partially overdetermined problems naturally appear in the analysis of some questions in shape optimization:

• *the minimization problem of the first Dirichlet eigenvalue* $\lambda_1(\Omega)$ *of* $-\Delta$ *among all open sets constrained to lie in a given box* B *and also with a given volume.*

• *the minimization problem of the second Dirichlet eigenvalue* $\lambda_2(\Omega)$ *of* $-\Delta$ *among all planar convex domains of given area.*

⋄ *A. Henrot, Extremum problems for eigenvalues of elliptic operators, 2006.*

We want to develop a *general approach* to partially overdetermined bvp of the form

$$\begin{cases} -\Delta u = f(x, u, \nabla u) & \text{in} \quad \Omega, \\ u = \phi & \text{on} \quad \partial\Omega, \\ \frac{\partial u}{\partial \nu} = \psi & \text{on} \quad \Gamma, \end{cases} \tag{8}$$

where

- $\Omega \subset \mathbb{R}^N$ is an open domain (possibly *unbounded*, *non-smooth*, with *unbounded* and *non-connected* boundary).
- Γ is a C^1-portion of the boundary $\partial\Omega$,
- $f \in \mathrm{Lip}(\Omega \times \mathbb{R} \times \mathbb{R}^N)$,
- ϕ and ψ are given smooth functions.

Theorem 7 (F.) *Let* Ω, $\tilde{\Omega} \subset \mathbb{R}^N$ *be open sets.*
Let Γ *be a* C^1*-subset of* $\partial\Omega$.
Let $\mathcal{C}$ *be the connected component of* $\partial\Omega$ *that contains* Γ.
Suppose that there exists a C^2*-hypersurface* $\mathcal{M}$, *with exterior normal* ν, *such that* $\Gamma \subseteq \mathcal{M}$ *and* $\Omega \cup (\mathcal{C} \cap \mathcal{M}) \subset \tilde{\Omega}$.

Let f be locally Lipschitz in $\Omega \times \mathbb{R} \times \mathbb{R}^N$.

Let $\phi : \mathbb{R}^N \to \mathbb{R}$ be C^1 in a neighborhood of $\mathcal{M}$ and $\psi \in C^0(\mathcal{M})$, with

$$\psi(x) - \partial_\nu \phi(x) \neq 0 \qquad \textit{for any} \qquad x \in \mathcal{M} \setminus \Gamma \tag{T}$$

Assume that for any $P \in \mathcal{M} \setminus \Gamma$ *there exist* $r(P) > 0$ *and*

$$u^{(P)} \in C^2(B_{r(P)}(P))$$

which solves

$$\begin{cases} -\Delta u^{(P)} = f(x, u^{(P)}, \nabla u^{(P)}) & \text{in} \quad B_{r(P)}(P), \\ \quad u^{(P)} = \phi & \text{on} \quad \mathcal{M} \cap B_{r(P)}(P), \\ \quad \frac{\partial u^{(P)}}{\partial \nu} = \psi & \text{on} \quad \mathcal{M} \cap B_{r(P)}(P). \end{cases} \tag{7}$$

Let $\epsilon_o > 0$ and $p_o \in \Gamma$.
Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega) \cap W^{2,2}(\Omega \cap B_{\epsilon_o}(p_o)) \cap C^1(\tilde{\Omega})$ be a solution of a problem that is partially overdetermined on $\partial\Omega$ by the following equations:

$$\begin{cases} -\Delta u = f(x, u, \nabla u) & \text{in} \quad \Omega, \\ \quad u = \phi & \text{on} \quad \partial\Omega, \\ \quad \frac{\partial u}{\partial \nu} = \psi & \text{on} \quad \Gamma. \end{cases} \tag{8}$$

Then,

$$\mathcal{C} = \mathcal{M}$$

and

$$u = \phi \qquad on \quad \mathcal{C}$$

$$\partial_\nu u = \psi \qquad on \quad \mathcal{C}$$

Its statement may be interpreted, roughly speaking, saying that

if the problem is partially overdetermined on some portion Γ *of* $\partial\Omega$ *and we can locally solve an overdetermined problem on some smooth manifold* $\mathcal{M}$ *that contains* Γ*, than the portion of the boundary on which the problem is overdetermined is the whole connected component* $\mathcal{C}$ *of* $\partial\Omega$ *that contains* Γ*.*

Domains whose boundary contains a piece of a hyperplane

Theorem 10 (F.) *Let $c \neq 0$. Let $\Omega \subset \mathbb{R}^N$ be connected and of class C^1. Let $\Gamma \subseteq \partial\Omega$ be nonempty and relatively open in $\partial\Omega$, with exterior normal ν. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solve*

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \frac{\partial u}{\partial \nu} = c & \text{on} \quad \Gamma. \end{cases} \tag{9}$$

with f Lipschitz.

Suppose that Γ agrees with a portion of hyperplane $\{ x_N = 0 \}$.

Then, one of the following four possibilities holds: $\Omega = \{x_N > 0\}$, $\Omega = \{x_N < 0\}$, $\Omega = \{0 < x_N < \kappa\}$, $\Omega = \{-\kappa < x_N < 0\}$, *for some $\kappa > 0$ and u is one-dimensional.*

Theorem 11 (F.) *Let c, Ω and Γ be as in Theorem 10. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solve*

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \frac{\partial u}{\partial \nu} = c & \text{on} \quad \Gamma. \end{cases} \tag{9}$$

with f Lipschitz.

Suppose that Γ agrees with a portion of sphere ∂B_1.

Then, one of the following four possibilities holds:

$\Omega = B_1, \quad \Omega = \mathbb{R}^N \setminus B_1, \quad \Omega = B_1 \setminus B_{1-\kappa}, \quad \Omega = B_{1+\kappa} \setminus B_1,$

for some $\kappa > 0$ and u is radially symmetric.

Theorem 12 *The stadium (i.e. the convex hull of two identical tangent discs) does not realize the minimum of* λ_2 *among plane convex domains of given area.*

• The above result was originally obtained by *A. Henrot* and *E. Oudet* in 2003 and by different methods.

The analytic setting

Theorem 13 (F.) *Let $c \neq 0$. Let $\Omega \subset \mathbb{R}^N$ be open, connected and* <u>*bounded*</u> *with $\partial\Omega$* <u>*connected*</u>*. Let $\Gamma \subseteq \partial\Omega$, nonempty and relatively open in $\partial\Omega$. Assume there exists an open set Ω' with* <u>*analytic*</u> *boundary $\partial\Omega'$ such that $\Gamma \subset \partial\Omega'$. Also assume that f is an* <u>*analytic*</u> *function.*

If there exists a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of

$$\begin{cases} -\Delta u = f(u), \quad u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \frac{\partial u}{\partial \nu} = c & \text{on} \quad \Gamma, \end{cases} \tag{10}$$

then Ω is ball and u is radially symmetric.

The proof is *crucially* based on :

- the *analyticity of both* f and $\partial\Omega^{'}$ (Cauchy-Kowalewskya Th.),
- the *connectedness of both* $\partial\Omega$ and $\partial\Omega^{'}$,
- the *boundedness* of Ω.

- $\partial\Omega^{'}$, f *analytic* + *Cauchy-Kowalewskya Theorem*

$$\Downarrow$$

i) for any $P \in \partial\Omega^{'}$ there exist $r(P) > 0$ and $u^{(P)} \in C^2(B_{r(P)}(P))$ which solves

$$\begin{cases} -\Delta u^{(P)} = f(u^{(P)}) & \text{in} \quad B_{r(P)}(P), \\ \quad u^{(P)} = 0, \qquad \frac{\partial u^{(P)}}{\partial \nu} = const. & \text{on} \quad \mathcal{M} \cap B_{r(P)}(P). \end{cases} \tag{7}$$

ii)

$$\tilde{\Omega} := \Omega \bigcup_{P \in \partial\Omega'} B_{r(P)}(P)$$

iii)

$$\exists\ \tilde{u} \in C^2(\tilde{\Omega}) \quad : \quad \tilde{u}_{|\Omega} = u \quad \text{(by analytic continuation).}$$

• By <u>*Theorem* 7</u>, the original problem is *globally overdetermined*. That is u satisfies

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ \quad u > 0 & \text{in} \quad \Omega, \\ \quad u = 0 & \text{on} \quad \partial\Omega, \\ \quad \frac{\partial u}{\partial \nu} = const. & \text{on} \quad \underline{\partial\Omega}. \end{cases} \tag{10}$$

We conclude by applying *Serrin's result* !

An extension to unbounded domains

Theorem 14 (F.) *Let* Ω *be a globally lipschitz, analytic epigraph of* $\mathbb{R}^N$ *with* $N = 2, 3$. *Let* $\Gamma \subseteq \partial\Omega$, *nonempty and relatively open in* $\partial\Omega$. *If the* *partially overdetermined boundary value problem*

$$\begin{cases} -\Delta u = u - u^3 & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \\ \frac{\partial u}{\partial \nu} = const. & \text{on} \quad \Gamma, \end{cases} \tag{11}$$

admits a bounded $C^2(\overline{\Omega})$*-solution, then (up to isometry)*

$$\Omega = \mathbb{R}^{N-1} \times (0, +\infty), \qquad u(x) = \tanh\left(\frac{x_N}{\sqrt{2}}\right) \qquad \forall\, x \in \Omega$$

The talk is based on

[1] *On partially and globally overdetermined problems of elliptic type*, to appear in American Journal of Mathematics, 2012

[2] *Flattening results for elliptic PDEs in unbounded domains with applications to overdetermined problems*, Archive for Rational Mechanics and Analysis, 195 (2010), 1025 - 1058.