Existence of minimizers in nonlinear elasticity

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Plan

1. The minimization problem in nonlinear elasticity
2. Existence of minimizers by IFT
3. Existence of minimizers by SWLSC

The minimization problem in nonlinear elasticity

$\Omega \subset \mathbb{R}^3$	undeformed body
$\boldsymbol{u}(\Omega) \subset \mathbb{R}^3$	deformed body
$\boldsymbol{u} : \Omega \to \mathbb{R}^3$	deformation of the body

Total energy of the body, assumed to be made of a hyperelastic material and subjected to dead body forces:

$$\boxed{J(\boldsymbol{u}) = \int_\Omega W(x, \nabla \boldsymbol{u}(x))\, dx - \int_\Omega \mathbf{f}(x) \cdot \boldsymbol{u}(x)\, dx.}$$

$W : \overline{\Omega} \times \mathbb{R}^{3\times 3} \to \mathbb{R}$	characterizes the hyperelastic material
$\mathbf{f} : \Omega \to \mathbb{R}^3$	is the density of the applied forces

Frame-indifference:

$$J(\boldsymbol{u}) = \int_\Omega W(x, \mathbf{E}(x))\ dx - \int_\Omega \mathbf{f}(x) \cdot \boldsymbol{u}(x)\ dx$$

where

$$\mathbf{E}(x) := \frac{1}{2}\Big\{\nabla \boldsymbol{u}^T(x) \nabla \boldsymbol{u}(x) - I_3\Big\}$$

Example. Homogeneity + Isotropy + Ω unconstrained:

$$W(x, E) = \frac{\lambda}{2}(\operatorname{tr} E)^2 + \mu |E|^2 + o(|E|^2).$$

$\lambda > 0$ and $\mu > 0$ are Lamé's constants

Toy model:

$$\boxed{J(\boldsymbol{u}) = \int_\Omega |\mathbf{E}(x)|^2\ dx - \int_\Omega \mathbf{f}(x) \cdot \boldsymbol{u}(x)\ dx}$$

Minimization problem:

Find a minimizer of the total energy

$$J(\boldsymbol{u}) = \int_\Omega W(x, \mathbf{E}(x))\ dx - \int_\Omega \mathbf{f}(x) \cdot \boldsymbol{u}(x)\ dx,$$

where

$$\mathbf{E}(x) := \frac{1}{2}\Big\{\nabla \boldsymbol{u}^T(x) \nabla \boldsymbol{u}(x) - I_3\Big\},$$

in a given set $\mathcal{M} := \{\boldsymbol{u} \in\ ?\}$ of admissible deformations:

- regularity

- preservation of orientation: $\det \nabla \boldsymbol{u}(x) > 0$ for all $x \in \Omega$

- noninterpenetration of matter: $\boldsymbol{u}$ injective on Ω.

- boundary conditions

Existence of a minimizer by IFT

Zhang [1991]:

$$W(x,F) = a|F|^p + b|\mathrm{Cof}\,F|^q + G(F, \mathrm{Cof}\,F, \det F)$$

$a > 0$, $b > 0$, $p \geq 2$, $q > \frac{p}{p-1}$,
G convex, bounded below, C^4, $G \to \infty$ if $\det F \to 0^+$.

Then

$$J(\boldsymbol{u}) = \int_\Omega W(x, \nabla \boldsymbol{u}(x))\; dx - \int_\Omega \mathbf{f}(x) \cdot \boldsymbol{u}(x)\; dx$$

has a minimizer in the set

$$\mathcal{M} := \{\boldsymbol{u} \in W^{1,p};\ \mathrm{Cof}\nabla\boldsymbol{u} \in L^q,\ \det\nabla\boldsymbol{u} > 0, \boldsymbol{u}|_{\partial\Omega} = \mathbf{id}\}$$

if $\mathbf{f}$ is small enough in L^r-norm ($r > 3$).

M. [2011]:

W is C^3, $\dfrac{\partial W}{\partial E}(x,0) = 0$, $\exists\ \alpha, \varepsilon > 0$ such that $\forall\ x$, $|E| < \varepsilon$, H,

$$W(x, E+H) \geq W(x,E) + \frac{\partial W}{\partial E}(x,E) : H + \alpha |H|^2.$$

Then

$$J(\boldsymbol{u}) = \int_\Omega W(x, \mathbf{E}(x))\ dx - \int_\Omega \mathbf{f}(x) \cdot \boldsymbol{u}(x)\ dx$$

$$\mathbf{E}(x) := \frac{1}{2}\Big\{\nabla \boldsymbol{u}^T(x) \nabla \boldsymbol{u}(x) - I_3\Big\},$$

has a minimizer in the set

$$\mathcal{M} := \{\boldsymbol{u} \in W^{1,4};\ \det \nabla \boldsymbol{u} > 0, \boldsymbol{u}|_{\partial\Omega} = \mathbf{id}\}$$

if $\mathbf{f}$ is small enough in L^r-norm ($r > 3$).

Proof.

1) Existence of a critical point of J:

$$-\mathbf{div}\left(\nabla\boldsymbol{u}\frac{\partial W}{\partial E}(\cdot,\mathbf{E}(\boldsymbol{u}))\right)=\mathbf{f} \quad \text{in } \Omega$$
$$\boldsymbol{u}=\mathbf{id} \quad \text{on } \partial\Omega$$

Linearization around $\boldsymbol{u}=\mathbf{id}$:

$$-\mathbf{div}\left(\frac{\partial^2 W}{\partial^2 E}(\cdot,0)(\nabla_s\boldsymbol{d})\right)=\mathbf{f} \quad \text{in } \Omega$$
$$\boldsymbol{d}=\mathbf{0} \quad \text{on } \partial\Omega$$

where, for all $H\in\mathbb{S}^3$,

$$\frac{\partial^2 W}{\partial E^2}(x,0)(H):=\lim_{t\to 0}\frac{1}{t}\left(\frac{\partial W}{\partial E}(x,tH)-\frac{\partial W}{\partial E}(x,0)\right).$$

The linearized problem is elliptic since:

$$\frac{\partial^2 W}{\partial E^2}(x, 0 + H) \geq 2\alpha |H|^2 \ \forall x \in \overline{\Omega},\ \forall H \in \mathbb{S}^3$$

Linear Korn inequality.

IFT: $\|\mathbf{f}\|_{L^r} < \delta \Rightarrow \exists\ \boldsymbol{u}^* \in W^{2,r}$, $\|\boldsymbol{u}^* - \mathbf{id}\|_{W^{2,r}} < \varepsilon_1$, such that

$$J'(\boldsymbol{u}^*) = 0$$

$\boldsymbol{u}^*$ belongs to the minimization set $\mathcal{M}$ if ε_1 is small enough since:

$$\begin{aligned} \varepsilon_1 << 1,\ r > 3 &\Rightarrow \ \|\nabla \boldsymbol{u}^* - I_3\|_{L^\infty} < 1, \\ &\Rightarrow \det \nabla \boldsymbol{u}^* > 0 \\ &\Rightarrow \boldsymbol{u}^* \in \mathcal{M} \end{aligned}$$

2) The critical point $\boldsymbol{u}^*$ is a global minimizer for J:

For all $\boldsymbol{u} \in \mathcal{M}$,

$$J(\boldsymbol{u}) - J(\boldsymbol{u}^*) \geq J'(\boldsymbol{u}^*)(\boldsymbol{u} - \boldsymbol{u}^*) + \alpha \int_\Omega |\mathbf{E}(\boldsymbol{u}) - \mathbf{E}(\boldsymbol{u}^*)|^2 + \int_\Omega \frac{\partial W}{\partial E}(\cdot, \mathbf{E}(\boldsymbol{u}^*)) : (\nabla \boldsymbol{u} - \nabla \boldsymbol{u}^*)^T (\nabla \boldsymbol{u} - \nabla \boldsymbol{u}^*)$$

$$\geq \frac{\alpha}{4} \|\nabla \boldsymbol{u}^T \nabla \boldsymbol{u} - (\nabla \boldsymbol{u}^*)^T \nabla \boldsymbol{u}^*\|^2_{L^2(\Omega)} - \underbrace{\frac{1}{2} \left\| \frac{\partial W}{\partial E}(\cdot, \mathbf{E}(\boldsymbol{u}^*)) \right\|_{L^\infty(\Omega)}}_{\text{small}} \|\nabla \boldsymbol{u} - \nabla \boldsymbol{u}^*\|^2_{L^2(\Omega)}$$

It remains to prove:

$$\|\nabla \boldsymbol{u}^T \nabla \boldsymbol{u}^* - (\nabla \boldsymbol{u}^*)^T \nabla \boldsymbol{u}^*\|_{L^2(\Omega)} \geq C \|\nabla \boldsymbol{u} - \nabla \boldsymbol{u}^*\|_{L^2(\Omega)}$$

3) Nonlinear Korn inequality: $\boldsymbol{u} \in W^{1,4}(\Omega;\mathbb{R}^3)$, $\det \nabla \boldsymbol{u} > 0$ a.e. in Ω, $\boldsymbol{u}|_{\partial\Omega} = \mathbf{id}|_{\partial\Omega} \Rightarrow$

$$\begin{aligned} \|\nabla \boldsymbol{u} - \nabla \boldsymbol{u}^*\|_{L^2(\Omega)} &\leq C(\boldsymbol{u}^*)\ \|\nabla \boldsymbol{u}^T \nabla \boldsymbol{u} - (\nabla \boldsymbol{u}^*)^T \nabla \boldsymbol{u}^*\|_{L^2(\Omega)} \\ &\leq C\ \|\nabla \boldsymbol{u}^T \nabla \boldsymbol{u} - (\nabla \boldsymbol{u}^*)^T \nabla \boldsymbol{u}^*\|_{L^2(\Omega)} \end{aligned}$$

Proof:

Extend $\boldsymbol{u}^*$ in an open ball $B \supset \overline{\Omega}$ such that $\boldsymbol{u}^* = \mathbf{id}$ on ∂B, then extend $\boldsymbol{u}$ by $\boldsymbol{u}^*$ in B. Then

$$\left.\begin{array}{r} \det \nabla \boldsymbol{u}^* > 0 \text{ in } \overline{B} \\ \boldsymbol{u}^*(x) = x,\ \ x \in \partial B \end{array}\right\} \Rightarrow \boldsymbol{u}^* : \overline{B} \to \overline{B} \text{ is a } C^1\text{–diffeomorphism}$$

Friesecke, James & Müller [1992] for $\boldsymbol{\varphi} := \boldsymbol{u} \circ (\boldsymbol{u}^*)^{-1}$:

$$\inf_{R \in SO(3)} \int_B |\nabla \boldsymbol{\varphi}(x) - R|^2 dx \leq C_0 \int_B \inf_{R \in SO(3)} |\nabla \boldsymbol{\varphi}(x) - R|^2\, dx$$

Polar decomposition of the 3×3 matrix $\nabla \boldsymbol{\varphi}(x)$:

$$\inf_{R \in SO(3)} |\nabla \boldsymbol{\varphi}(x) - R|^2 \leq |\nabla \boldsymbol{\varphi}(x)^T \nabla \boldsymbol{\varphi}(x) - I_3|^2$$

so

$$\inf_{R \in SO(3)} \int_B |\nabla \boldsymbol{\varphi}(x) - R|^2 dx \leq C_0 \int_B \ |\nabla \boldsymbol{\varphi}^T \nabla \boldsymbol{\varphi} - I_3|^2 \ dx$$

Since $\boldsymbol{\varphi} = \mathbf{id}$ on $B \setminus \Omega$,

$$I_3 = \frac{1}{|B \setminus \Omega|} \int_{B \setminus \Omega} \nabla \boldsymbol{\varphi}(x) dx$$

so, $\forall\ R \in SO(3)$,

$$\begin{aligned}
\int_B |\nabla \boldsymbol{\varphi} - I_3|^2 dx &\leq 2 \int_B |\nabla \boldsymbol{\varphi}(x) - R|^2 dx + 2|B||R - I_3|^2 \\
&\leq 2 \int_B |\nabla \boldsymbol{\varphi}(x) - R|^2 dx + \frac{2|B|}{|B \setminus \Omega|^2} \Big(\int_{B \setminus \Omega} |R - \nabla \boldsymbol{\varphi}(x)|\ dx \Big)^2 \\
&\leq 2 \Big(1 + \frac{|B|}{|B \setminus \Omega|} \Big) \int_B |\nabla \boldsymbol{\varphi}(x) - R|^2 dx.
\end{aligned}$$

Hence

$$\int_B |\nabla \boldsymbol{\varphi}(x) - I_3|^2 dx \leq C_1 \int_B \ \ |\nabla \boldsymbol{\varphi}^T \nabla \boldsymbol{\varphi} - I_3|^2\ dx$$

$$\int_B |\nabla \boldsymbol{\varphi}(x) - I_3|^2 dx \le C_1 \int_B \ |\nabla \boldsymbol{\varphi}^T \nabla \boldsymbol{\varphi} - I_3|^2 \ dx.$$

But $\boldsymbol{\varphi} = \boldsymbol{u} \circ (\boldsymbol{u}^*)^{-1}$. Change of variables $x = \boldsymbol{u}^*(y)$:

$$\int_B |\nabla \boldsymbol{u} - \nabla \boldsymbol{u}^*|^2 \ \frac{\det(\nabla \boldsymbol{u}^*)}{|\nabla \boldsymbol{u}^*|^2} \ dy$$
$$\le C_1 \int_B |\nabla \boldsymbol{u}^T \nabla \boldsymbol{u} - (\nabla \boldsymbol{u}^*)^T \nabla \boldsymbol{u}^*|^2 \ |(\nabla \boldsymbol{u}^*)^{-1}|^4 \det(\nabla \boldsymbol{u}^*) \ dy$$

So

$$\begin{aligned} \|\nabla \boldsymbol{u} - \nabla \boldsymbol{u}^*\|_{L^2(\Omega)} &\le C(\boldsymbol{u}^*) \ \|\nabla \boldsymbol{u}^T \nabla \boldsymbol{u} - (\nabla \boldsymbol{u}^*)^T \nabla \boldsymbol{u}^*\|_{L^2(\Omega)} \\ &\le \frac{C}{(1-\lambda)^{7/2}} \ \|\nabla \boldsymbol{u}^T \nabla \boldsymbol{u} - (\nabla \boldsymbol{u}^*)^T \nabla \boldsymbol{u}^*\|_{L^2(\Omega)} \end{aligned}$$

if $\|\nabla \boldsymbol{u}^* - I_3\|_{L^\infty(\Omega)} \le \lambda$.

Existence of a minimizer by SWLSC

Minimization problem in 3D nonlinear elasticity: Find a minimizer of the total energy

$$J(\boldsymbol{u}) = \int_\Omega W(x, \nabla \boldsymbol{u}(x))\, dx - \int_\Omega \mathbf{f}(x) \cdot \boldsymbol{u}(x)\, dx$$

in a given set $\mathcal{M} := \{\boldsymbol{u} : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3;\ ??\}$ of admissible deformations:

- regularity

- preservation of orientation: $\det \nabla \boldsymbol{u}(x) > 0$ for all $x \in \Omega$

- noninterpenetration of matter: $\boldsymbol{u}$ injective on Ω.

- boundary conditions

Ball [1977]:

$$W(x,F) = G(x,F,\mathrm{Cof}\,F,\det F)$$

with $G(x,\cdot)$ convex, $G(\cdot,F,H,\delta)$ measurable.

$$W(x,F) \geq \alpha(|F|^p + |\mathrm{Cof}\,F|^q + |\det F|^r) - \beta$$

with $p \geq 2$, $q \geq \frac{p}{p-1}$, $r > 1$, $\alpha > 0$, $\beta \in \mathbb{R}$.

$$W(x,F) \to +\infty \;\; \text{if} \;\; \det F \to 0.$$

Then, $\forall \mathbf{f} \in L^{6/5}(\Omega;\mathbb{R}^3)$,

$$J(\boldsymbol{u}) = \int_\Omega W(x,\nabla\boldsymbol{u}(x))\;dx - \int_\Omega \mathbf{f}(x)\cdot\boldsymbol{u}(x)\;dx$$

has a minimizer in the set

$$\mathcal{M} := \{\boldsymbol{u} \in W^{1,p};\; \mathrm{Cof}\nabla\boldsymbol{u} \in L^q,\; \det\nabla\boldsymbol{u} \in L^r,\; \det\nabla\boldsymbol{u} > 0, \boldsymbol{u}|_{\Gamma_0} = \mathbf{id}|_{\Gamma_0}\}$$

Minimization problem in nonlinear shell theory:

$\omega \subset \mathbb{R}^2$, $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$ and $\boldsymbol{\varphi} : \omega \to \mathbb{R}^3$ embeddings,

$S := \boldsymbol{\theta}(\omega) \subset \mathbb{R}^3$ middle surface of the undeformed shell

$\tilde{S} := \boldsymbol{\varphi}(\omega) \subset \mathbb{R}^3$ middle surface of the deformed shell

$\boldsymbol{n} : \omega \to \mathbb{R}^3$ normal to S, $\boldsymbol{\zeta} : \omega \to \mathbb{R}^3$ transversal to $\tilde{S}$,

$\{\boldsymbol{\theta}(y) + z\boldsymbol{n}(y);\ y \in \omega, -1 < z < 1\}$ undeformed shell

$\{\boldsymbol{\varphi}(y) + z\boldsymbol{\zeta}(y);\ y \in \omega, -1 < z < 1\}$ deformed shell

Total energy of the deformation $\boldsymbol{v} := (\boldsymbol{\varphi}, \boldsymbol{\zeta})$:

$$\boxed{I(\boldsymbol{v}) = \int_\omega W(y, \boldsymbol{v}(y), \nabla \boldsymbol{v}(y))\, dy - L(\boldsymbol{v})}$$

$W : \overline{\omega} \times \mathbb{R}^6 \times \mathbb{R}^{6\times 2} \to \mathbb{R}$ characterizes the material constituting the shell

$L : W^{1,r}(\omega; \mathbb{R}^6) \to \mathbb{R}$ linear, continuous: characterizes the applied forces

Minimization set:

$$\left.\begin{aligned}\mathcal{M} := \{\boldsymbol{v} = (\boldsymbol{\varphi}, \boldsymbol{\zeta}) \in ?? \ ; \ & \partial_1\boldsymbol{\varphi} \wedge \partial_2\boldsymbol{\varphi} \neq \mathbf{0} \ \text{ in } \overline{\omega}, \\ & |\boldsymbol{\zeta}| = 1 \text{ and } (\partial_1\boldsymbol{\varphi} \wedge \partial_2\boldsymbol{\varphi}) \cdot \boldsymbol{\zeta} > 0 \ \text{ in } \overline{\omega}, \\ & \boldsymbol{\varphi} = \boldsymbol{\theta} \text{ and } \boldsymbol{\zeta} = \boldsymbol{n} \ \text{ on } \gamma_0 \subset \partial\omega \end{aligned}\right\}$$

(i.e., immersion, orientation preserving, boundary conditions)

The functional space depends on the properties of W (which characterizes the elastic material constituting the shell): coerciveness, orientation preserving, polyconvexity.

Polyconvexity of W:

We wish to find a function $\mathbb{W}$ such that (Ball, Currie, Olver [1981]):

$$W(y, \boldsymbol{v}(y), \nabla \boldsymbol{v}(y)) = \mathbb{W}(y, \boldsymbol{v}(y), \mathrm{Jac}(\nabla \boldsymbol{v}(y))$$

$\mathbb{W}$ convex w.r.t. last variable

But this contradicts orientation preserving:

$$W(y, \boldsymbol{v}(y), \nabla \boldsymbol{v}(y)) \to \infty \ \text{ if } (\partial_1 \boldsymbol{\varphi}(y) \wedge \partial_2 \boldsymbol{\varphi}(y)) \cdot \boldsymbol{\zeta}(y) \to 0^+.$$

Solution: Weaken the convexity assumption by restricting the domain of definition of $\mathbb{W}$.

$\mathbb{A}^6$ denotes the space of all 6×6 antisymmetric matrices

For each non-zero vector $\boldsymbol{q} = (q_i) \in \mathbb{R}^3$, the set

$$\mathbb{G}_+(\boldsymbol{q}) := \{G = (G_{k\ell}) \in \mathbb{A}^6;\ q_1 G_{23} + q_2 G_{31} + q_3 G_{12} > 0\}$$

is a *convex* subset of $\mathbb{A}^6$.

$$\mathbb{F}_+ := \{(\boldsymbol{p}, \boldsymbol{q}, F, G);\ \boldsymbol{p} \in \mathbb{R}^3, \boldsymbol{q} \in \mathbb{R}^3 \setminus \{0\}, F \in \mathbb{R}^{6\times 2}, G \in \mathbb{G}_+(\boldsymbol{q})\}$$

$$\mathbb{E}_+ := \{(\boldsymbol{p}, \boldsymbol{q}, F);\ (\boldsymbol{p}, \boldsymbol{q}, F, J(F)) \in \mathbb{F}_+\},$$

where, for each $F = (F_{k\alpha}) \in \mathbb{R}^{6\times 2}$,

$$J(F) := (J_{k\ell}(F)) \in \mathbb{A}^6$$

is defined by

$$J_{k\ell}(F) := F_{k1} F_{\ell 2} - F_{k2} F_{\ell 1}$$

Assumptions on the elastic material: $W : \omega \times \mathbb{E}_+ \to \mathbb{R}$ is such that:

Polyconvexity: $\exists\ \mathbb{W} : \omega \times \mathbb{F}_+ \to \mathbb{R}$ such that

$$W(y, (\boldsymbol{p}, \boldsymbol{q}), F) = \mathbb{W}(y, (\boldsymbol{p}, \boldsymbol{q}), F, J(F)) \ \text{ for all } (y, (\boldsymbol{p}, \boldsymbol{q}, F)) \in \omega \times \mathbb{E}_+$$

$\mathbb{W}(y, (\boldsymbol{p}, \boldsymbol{q}), \cdot, \cdot) : \mathbb{F} \times \mathbb{G}_+(\boldsymbol{q}) \to \mathbb{R}$ is convex $\forall y, \boldsymbol{p}, \boldsymbol{q} \neq 0$

$\mathbb{W}(\cdot, (\boldsymbol{p}, \boldsymbol{q}, F, G)) : \omega \to \mathbb{R}$ is measurable $\forall (\boldsymbol{p}, \boldsymbol{q}, F, G) \in \mathbb{F}_+$

$\mathbb{W}(y, \cdot) : \mathbb{F}_+ \to \mathbb{R}$ is continuous a.e. $y \in \omega$

Orientation-preserving condition: For almost all $y \in \omega$,

$$\mathbb{W}(y, \boldsymbol{p}, \boldsymbol{q}, F, G) \to +\infty \ \text{ if } q_1 G_{23} + q_2 G_{31} + q_3 G_{12} \to 0^+.$$

Coerciveness: $\exists\ \alpha > 0$, $\beta \in \mathbb{R}$, $r > 4/3$, $s > 1$, $g \in L^1(\omega)$ such that

$$W(y, (\boldsymbol{p}, \boldsymbol{q}, F)) \geq \alpha(\|F\|^r + \|J(F)\|^s) + \beta(|\boldsymbol{p}| + |\boldsymbol{q}|) + g(y)$$

for almost all $y \in \omega$ and all $(\boldsymbol{p}, \boldsymbol{q}, F) \in \mathbb{E}_+$.

Ciarlet, Gogu, M. [2011]:

The total energy

$$I(\boldsymbol{v}) = \int_{\omega} W(y, \boldsymbol{v}(y), \nabla \boldsymbol{v}(y))\; dy - L(\boldsymbol{v})$$

possesses a minimizer in the set

$$\left.\begin{aligned}\mathcal{M} := \{\boldsymbol{v} = (\boldsymbol{\varphi}, \boldsymbol{\zeta}) \in W^{1,r}(\omega; \mathbb{R}^6);\; & J(\nabla \boldsymbol{v}) \in L^s(\omega; \mathbb{A}^6) \\ & \partial_1 \boldsymbol{\varphi} \wedge \partial_2 \boldsymbol{\varphi} \neq \mathbf{0} \;\text{ in } \overline{\omega}, \\ & |\boldsymbol{\zeta}| = 1 \text{ and } (\partial_1 \boldsymbol{\varphi} \wedge \partial_2 \boldsymbol{\varphi}) \cdot \boldsymbol{\zeta} > 0 \;\text{ in } \overline{\omega}, \\ & \boldsymbol{\varphi} = \boldsymbol{\theta} \text{ and } \boldsymbol{\zeta} = \boldsymbol{n} \;\text{ on } \gamma_0 \subset \partial\omega\end{aligned}\right\}$$

Proof.

$$\boldsymbol{v} \in \mathcal{M} \Rightarrow \int_\omega W(y, \boldsymbol{v}(y), \nabla \boldsymbol{v}(y))\,\mathrm{d}y \in \mathbb{R} \cup \{+\infty\}$$

The coerciveness of W implies that any infimizing sequence $(\boldsymbol{v}^k)$ satisfies

$$\|\boldsymbol{v}^k\|_{W^{1,r}} \leq C \quad \text{and} \quad \|J(\nabla \boldsymbol{v}^k))\|_{L^s} \leq C$$

Extract a subsequence:

$$\boldsymbol{v}^\ell \rightharpoonup \boldsymbol{u} \quad \text{in } W^{1,r}(\omega; \mathbb{R}^6$$

$$J(\nabla \boldsymbol{v}^\ell) \rightharpoonup \boldsymbol{H} \quad \text{in } L^s(\omega; \mathbb{A}^6)$$

Compensated compactness $\Rightarrow \boldsymbol{H} = J(\boldsymbol{u})$.

It remains to show that

$$\boldsymbol{u} \in \mathcal{M} \text{ and } I(\boldsymbol{u}) = \inf_{\boldsymbol{v} \in \mathcal{M}} I(\boldsymbol{v})$$

Lemma (Ball, Currie, Olver [1981], Dacorogna [2010]).

$$\int_\Omega f(x, u(x), v(x))dx \leq \liminf_{n\to\infty} \int_\Omega f(x, u_n(x), v_n(x))dx$$

if: $\Omega \subset \mathbb{R}^n$ is measurable with finite measure

$u_n \to u$ a.e. in Ω, $u_n, u : \Omega \to \mathbb{R}^p$ are measurable

$v_n \rightharpoonup v$ in $L^1(\Omega; \mathbb{R}^q)$-weak

$W : \Omega \times \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R} \cup \{+\infty\}$ satisfies:

$$W(\cdot, a, b) \text{ is measurable } \forall a \in \mathbb{R}^p,\ \forall b \in \mathbb{R}^p$$

$$W(x, \cdot, \cdot) \text{ is continuous for a.e. } x \in \Omega$$

$$W(x, a, \cdot) \text{ is convex for a.e. } x \in \Omega,\ \forall a \in \mathbb{R}^p$$

$\exists g \in L^1(\Omega)$ such that

$$W(x, u_n(x), v_n(x)) \geq g(x)$$

$$W(x, u_n(x), v_n(x)) \geq g(x).$$

Extend $\mathbb{W}$ by $+\infty$ outside $\omega \times \mathbb{F}_+$. This extension $\tilde{\mathbb{W}}$ satisfies the assumptions of the Lemma; so

$$\inf_{\boldsymbol{v}\in\mathcal{M}} I(\boldsymbol{v}) = \liminf_{\ell\to\infty} \Big\{ \int_\omega \tilde{\mathbb{W}}(y, \boldsymbol{v}^\ell(y), \nabla \boldsymbol{v}^\ell(y), J(\nabla \boldsymbol{v}^\ell(y)))\,\mathrm{d}y - L(\boldsymbol{v}^\ell) \Big\}$$
$$\geq \int_\omega \tilde{\mathbb{W}}(y, \boldsymbol{u}(y), \nabla \boldsymbol{u}(y), J(\nabla \boldsymbol{u}(y)))\,\mathrm{d}y - L(\boldsymbol{u}).$$

Finally,

$$\inf_{\boldsymbol{v}\in\mathcal{M}} I(\boldsymbol{v}) < \infty \Rightarrow \tilde{\mathbb{W}}(y, \boldsymbol{u}(y), \nabla \boldsymbol{u}(y), J(\nabla \boldsymbol{u}(y))) < \infty \text{ a.e. } y \in \omega$$

This implies that $\boldsymbol{v} \in \mathcal{M}$ and that

$$\inf_{\boldsymbol{v}\in\mathcal{M}} I(\boldsymbol{v}) \geq \int_\omega \mathbb{W}(y, \boldsymbol{u}(y), \nabla \boldsymbol{u}(y), J(\nabla \boldsymbol{u}(y)))\,\mathrm{d}y - L(\boldsymbol{u})$$
$$= \int_\omega W(y, \boldsymbol{u}(y), \nabla \boldsymbol{u}(y))\,\mathrm{d}y - L(\boldsymbol{u}) = I(\boldsymbol{u})$$