Approximation par relaxation de systèmes hyperboliques

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Outline of the talk

- **The relaxation phenomenon**
  - Principles
  - Bibliography
- **Theoretical relaxation approximation**
  - The case of the Burgers equation
  - Gas dynamics equations
  - Hyperbolic fluid systems
- **Numerical schemes using the relaxation approximation**
  - Finite volume schemes and Godunov-type schemes
  - Numerical relaxation
  - Properties of relaxation schemes
- **Conclusion**

The relaxation phenomenon

General relaxation

- Friction, chemical reactions, external forces...

  ε characteristic time

- Convergence when $t \to +\infty$ to the equilibrium

  $$\nabla p = \rho g \mathbf{z} \, , \quad \mathbf{u}_1 = \mathbf{u}_2 \quad ...$$

- Convergence when $\varepsilon \to 0$ to the equilibrium

  $$c = C_{eq}(\rho) \, , \quad \mathbf{u}_1 = \mathbf{u}_2 \quad ...$$

- Equilibrium = algebraic relations

  $\Longrightarrow$ reduction of the number of PDE's

The relaxation phenomenon

Relaxation in hyperbolic systems

Relaxation system
$$\begin{cases} \partial_t U + \partial_x F(U,V) = 0 \\ \partial_t V + \partial_x G(U,V) = R(U,V)/\varepsilon \end{cases} \qquad (\mathscr{R})$$

Equilibrum: $R(U,V) = 0$

$$R(U,V) = 0 \iff V = V_{eq}(U)$$

Equilibrium system
$$\partial_t U + \partial_x F(U, V_{eq}(U)) = 0 \qquad (\mathscr{E})$$

(only formal compatibility)

The relaxation approximation

Principles:

- **Theoritical and numerical approximation** of hyperbolic systems
- Proposition of a **relaxation system** $(\mathscr{R})$ for a given hyperbolic system $(\mathscr{E})$
- $(\mathscr{R})$ must be **simpler to solve** than $(\mathscr{E})$ $\longrightarrow$ approximate Riemann solver
- Convergence: $(\mathscr{R}) \underset{\varepsilon \to 0}{\longrightarrow} (\mathscr{E})$?...

Some existing works:

- Whitham, Liu, Chen-Levermore-Liu, Natalini, Hanouzet-Natalini, Yong...
  General formalism: Relaxation $\approx$ Dissipation
- Jin-Xin, Serre, Bianchini
  Global linearisation: $(\mathscr{R})$ is a linear hyperbolic system
- Suliciu, Coquel-Perthame, Coquel *et al*, Bouchut...
  Linearize only the nonlinear terms

Here: Relaxation approximation $(\mathscr{R})$ for a large class of physical systems of $(\mathscr{E})$

The Jin-Xin approximation for a conservation law

Scalar conservation law:

$$(\mathscr{E}) \quad \partial_t u + \partial_x f(u) = 0$$

Nonlinear PDE

Approximation by the relaxation system:

$$(\mathscr{R}) \quad \begin{array}{rcl} \partial_t u + \partial_x v & = & 0 \\ \partial_t v + a^2\, \partial_x u & = & \dfrac{1}{\varepsilon}(f(u) - v) \end{array}$$

Linear system of PDE's + source term (a positive constant).

The Jin-Xin approximation for a conservation law

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Approximation by the relaxation system:

$$(\mathscr{R}) \quad \begin{aligned} \partial_t u + \partial_x v &= 0 \\ \partial_t v + a^2 \, \partial_x u &= \frac{1}{\varepsilon}(f(u) - v) \end{aligned}$$

Linear wave equation, with a the soundspeed

The Jin-Xin approximation for a conservation law

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Nonlinear PDE

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$$(\mathscr{R}) \quad \begin{array}{rcl} \partial_t u + \partial_x v & = & 0 \\ \partial_t v + a^2 \partial_x u & = & \dfrac{1}{\varepsilon}(f(u) - v) \end{array}$$

Relaxation source term:

When $\varepsilon \to 0$, we formally obtain $v = f(u)$ and the first PDE of $(\mathscr{R})$ becomes

$$\partial_t u + \partial_x f(u) = 0$$

Hilbert expansion for the Jin-Xin approximation

Solution near the equilibrium of the relaxation system $(\mathscr{R})$:

$$\boxed{v^\varepsilon = f(u^\varepsilon) + \varepsilon\, v_1^\varepsilon + \varepsilon^2\, v_2^\varepsilon + \dots}$$

$$\begin{aligned} \partial_t u^\varepsilon + \partial_x v^\varepsilon &= 0, \\ \partial_t v^\varepsilon + a^2\, \partial_x u^\varepsilon &= \frac{1}{\varepsilon}(f(u^\varepsilon) - v^\varepsilon). \end{aligned}$$

Then, the relaxation system becomes

$$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = -\varepsilon\, \partial_x v_1^\varepsilon + \mathcal{O}(\varepsilon^2)$$

$$\partial_t f(u^\varepsilon) + a^2\, \partial_x u^\varepsilon = -v_1^\varepsilon + \mathcal{O}(\varepsilon)$$

Hilbert expansion for the Jin-Xin approximation

Solution near the equilibrium of the relaxation system $(\mathscr{R})$:

$$\boxed{v^\varepsilon = f(u^\varepsilon) + \varepsilon\, v_1^\varepsilon + \varepsilon^2\, v_2^\varepsilon + \ldots}$$

$$\begin{aligned} \partial_t u^\varepsilon + \partial_x v^\varepsilon &= 0, \\ \partial_t v^\varepsilon + a^2\, \partial_x u^\varepsilon &= \frac{1}{\varepsilon}(f(u^\varepsilon) - v^\varepsilon). \end{aligned}$$

Then, the relaxation system becomes

$$\begin{aligned} f'(u^\varepsilon) \times \qquad \partial_t u^\varepsilon + \partial_x f(u^\varepsilon) &= -\varepsilon\, \partial_x v_1^\varepsilon + \mathcal{O}(\varepsilon^2) \\ \Longrightarrow \qquad \partial_t f(u^\varepsilon) + (f'(u^\varepsilon))^2\, \partial_x u^\varepsilon &= \mathcal{O}(\varepsilon) \end{aligned}$$

$$\partial_t f(u^\varepsilon) + a^2\, \partial_x u^\varepsilon = -v_1^\varepsilon + \mathcal{O}(\varepsilon)$$

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Then, the relaxation system becomes

$$\begin{aligned} \partial_t u^\varepsilon + \partial_x f(u^\varepsilon) &= -\varepsilon\, \partial_x v_1^\varepsilon + \mathcal{O}(\varepsilon^2) \\ (1) \qquad \partial_t f(u^\varepsilon) + (f'(u^\varepsilon))^2\, \partial_x u^\varepsilon &= \mathcal{O}(\varepsilon) \end{aligned}$$

$$(2) \qquad \partial_t f(u^\varepsilon) + a^2\, \partial_x u^\varepsilon = -v_1^\varepsilon + \mathcal{O}(\varepsilon)$$

$$(1) - (2) \qquad v_1^\varepsilon = \left[(f'(u^\varepsilon))^2 - a^2\right] \partial_x u^\varepsilon + \mathcal{O}(\varepsilon)$$

Hilbert expansion for the Jin-Xin approximation

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Then, the relaxation system becomes

$$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \varepsilon\, \partial_x \left(\left[a^2 - (f'(u^\varepsilon))^2\right] \partial_x u^\varepsilon \right) + \mathcal{O}(\varepsilon^2).$$

Hilbert expansion for the Jin-Xin approximation

Solution near the equilibrium of the relaxation system $(\mathscr{R})$:

$$\boxed{v^\varepsilon = f(u^\varepsilon) + \varepsilon\, v_1^\varepsilon + \varepsilon^2\, v_2^\varepsilon + \ldots}$$

$$\begin{aligned} \partial_t u^\varepsilon + \partial_x v^\varepsilon &= 0, \\ \partial_t v^\varepsilon + a^2\, \partial_x u^\varepsilon &= \frac{1}{\varepsilon}\big(f(u^\varepsilon) - v^\varepsilon\big). \end{aligned}$$

Then, the relaxation system becomes

$$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \varepsilon\, \partial_x \Big(\big[a^2 - (f'(u^\varepsilon))^2\big]\, \partial_x u^\varepsilon \Big) + \mathcal{O}(\varepsilon^2).$$

Equation $(\mathscr{E})$ + diffusion if

$$a > \sup_u |f'(u)| \qquad \text{(Whitham stability condition)}$$

Relaxation approximation for the gas dynamics equations

Euler equations in the barotropic case:

$$\partial_t \rho + \partial_x(\rho u) = 0,$$

$$\partial_t(\rho u) + \partial_x(\rho u^2 + p(v)) = 0,$$

where $v = 1/\rho$.

Decoupling of the **linear** and **nonlinear** parts:

- ▷ Switch to Lagrangian coordinates: $\mathrm{D}_t := \partial_t + u\partial_x$ and $\partial_y = v\,\partial_x$.
- ▷ Relaxation approximation of the nonlinear part.
- ▷ Switch to Eulerian coordinates.

Relaxation approximation for the gas dynamics equations

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Decoupling of the **linear** and **nonlinear** parts:

- ▷ Switch to Lagrangian coordinates: $\mathsf{D}_t := \partial_t + u\partial_x$ and $\partial_y = v\,\partial_x$

$$\mathsf{D}_t v - \partial_y u = 0,$$
$$\mathsf{D}_t u + \partial_y p(v) = 0.$$

$p(v)$: nonlinear part

Relaxation approximation for the gas dynamics equations

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$$\partial_t \rho + \partial_x(\rho u) = 0,$$
$$\partial_t(\rho u) + \partial_x(\rho u^2 + p(v)) = 0.$$

where $v = 1/\rho$.

Decoupling of the **linear** and **nonlinear** parts:

- ▷ Relaxation approximation of the nonlinear part:

$$\mathsf{D}_t v - \partial_y u = 0,$$
$$\mathsf{D}_t u + \partial_y \pi = 0,$$
$$\mathsf{D}_t \pi + a^2 \partial_y u = \frac{1}{\varepsilon}(p(v) - \pi).$$

Relaxation approximation for the gas dynamics equations

Euler equations in the barotropic case:

$$\partial_t \rho + \partial_x(\rho u) = 0,$$
$$\partial_t(\rho u) + \partial_x(\rho u^2 + p(v)) = 0.$$

where $v = 1/\rho$.

Decoupling of the **linear** and **nonlinear** parts:

- ▷ Switch to Eulerian coordinates:

$$\partial_t \rho + \partial_x(\rho u) = 0,$$
$$\partial_t(\rho u) + \partial_x(\rho u^2 + \pi) = 0,$$
$$\partial_t(\rho\pi) + \partial_x(\rho u \pi + a^2 u) = \frac{1}{\varepsilon}\rho\,(p(v) - \pi).$$

Relaxation approximation for the gas dynamics equations

Relaxation approximation of the Euler equations:

$$(\mathscr{E}) \qquad\qquad (\mathscr{R})$$

$$\boxed{\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0, \\ &\partial_t(\rho u) + \partial_x(\rho u^2 + p(v)) = 0. \end{aligned}} \quad \xleftarrow{\varepsilon \to 0} \quad \boxed{\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0, \\ &\partial_t(\rho u) + \partial_x(\rho u^2 + \pi) = 0, \\ &\partial_t(\rho \pi) + \partial_x(\rho u \pi + a^2 u) = \tfrac{1}{\varepsilon}\rho\,(p(v) - \pi). \end{aligned}}$$

- Stability condition (Whitham) :

$$a^2 > \sup_v |p'(v)|.$$

- The relaxation system is linearly degenerate (but not linear), all the waves are contact discontinuities $\Longrightarrow$ easy to solve.

Suliciu, Coquel *et al*, Bouchut, Chalons-Coulombel...

Rewriting the relaxation system

A new pressure law can be derived from the relaxation system $(\mathscr{R})$:

$$(\mathscr{R}) \qquad \begin{aligned} \mathsf{D}_t v - \partial_y u &= 0 \\ \mathsf{D}_t u + \partial_y \pi &= 0 \\ \mathsf{D}_t \pi + a^2 \partial_y u &= 0 \end{aligned}$$

- ▷ Introduce a new variable $\mathscr{T}$, such that

$$\mathsf{D}_t \mathscr{T} = 0$$

- ▷ Consider the new pressure law $\pi = \pi(\mathscr{T}, v)$. Then, the 3rd eq of $(\mathscr{R})$ gives

$$\begin{aligned} & \partial_v \pi(\mathscr{T}, v) = -a^2 \\ \Longrightarrow \quad & \pi(\mathscr{T}, v) = -a^2 v + f(\mathscr{T}) \end{aligned}$$

- ▷ We require $\pi(v, v) = p(v)$, then a simple choice is

$$\boxed{\pi(\mathscr{T}, v) = p(\mathscr{T}) + a^2(\mathscr{T} - v)}$$

Rewriting the relaxation system

The previous relaxation system

$$(\mathscr{R}) \qquad \begin{aligned} &\mathsf{D}_t v - \partial_y u = 0 \\ &\mathsf{D}_t u + \partial_y \pi = 0 \\ &\mathsf{D}_t \pi + a^2 \partial_y u = \tfrac{1}{\varepsilon}\left(p(v) - \pi\right) \end{aligned}$$

becomes

$$(\mathscr{R}) \qquad \begin{aligned} &\mathsf{D}_t v - \partial_y u = 0 \\ &\mathsf{D}_t u + \partial_y \pi(\mathscr{T}, v) = 0 \\ &\mathsf{D}_t \mathscr{T} \simeq \tfrac{1}{\varepsilon}\left(v - \mathscr{T}\right) \end{aligned}$$

with $\pi(\mathscr{T}, v) = p(\mathscr{T}) + a^2(\mathscr{T} - v)$. In Eulerian coordinates

$$(\mathscr{R}) \qquad \begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0 \\ &\partial_t(\rho u) + \partial_x(\rho u^2 + \pi) = 0 \\ &\partial_t(\rho \mathscr{T}) + \partial_x(\rho u \mathscr{T}) = \tfrac{1}{\varepsilon}\left(1 - \rho \mathscr{T}\right) \end{aligned}$$

Fluid systems

Class of systems of conservation laws introduced by Després

If a hyperbolic system of conservation laws satisfies

- The entropy s verifies $\mathsf{D}_t s = 0$ in Lagrangian coordinates
- The unknowns are $(\mathbf{v}, \mathbf{u}, e) \in \mathbb{R}^{n-1-d} \times \mathbb{R}^d \times \mathbb{R}$ and $e = |u|^2/2 + \epsilon(\mathbf{v}, s)$
- Invariance by Galilean tranformations
- Reversibility for smooth solutions

then, it can be written in Lagrangian coordinates under the form

$$\begin{cases} \mathsf{D}_t \mathbf{v} - N\partial_y \mathbf{u} = 0 \\ \mathsf{D}_t \mathbf{u} - N^T \partial_y [\partial_{\mathbf{v}} e] = 0 \\ \mathsf{D}_t s = 0 \end{cases}$$

where N is a rectangular $(n-1-d) \times d$ constant matrix.

Relaxation approximation of fluid systems (isentropic case)

In the isentropic case, $\partial_{\mathbf{v}} e(\mathbf{v}, \mathbf{u}) = \epsilon'(\mathbf{v})$. The fluid system becomes

$$\mathsf{D}_t \mathbf{v} - N \partial_y \mathbf{u} = 0$$
$$\mathsf{D}_t \mathbf{u} - N^T \partial_y [\epsilon'(\mathbf{v})] = 0$$

$[\epsilon'(\mathbf{v})]$: nonlinear term

Same structure as the gas dynamics equations !

Relaxation approximation of fluid systems (isentropic case)

A isentropic fluid system writes

$$
\begin{aligned}
&\mathsf{D}_t \mathbf{v} - N \partial_y \mathbf{u} = 0 \\
&\mathsf{D}_t \mathbf{u} - N^T \partial_y [\epsilon'(\mathbf{v})] = 0
\end{aligned}
$$

the relaxation approximation of this system writes

$$
\begin{aligned}
&\mathsf{D}_t \mathbf{v} - N \partial_y \mathbf{u} = 0 \\
&\mathsf{D}_t \mathbf{u} - N^T \partial_y [\boldsymbol{\pi}(\mathscr{T}, \mathbf{v})] = 0 \\
&\mathsf{D}_t \mathscr{T} = \frac{1}{\varepsilon} (\mathbf{v} - \mathscr{T})
\end{aligned}
$$

with the pressure law

$$
\boldsymbol{\pi}(\mathscr{T}, \mathbf{v}) = \epsilon'(\mathscr{T}) + \boldsymbol{\theta}'(\mathbf{v} - \mathscr{T})
$$

where $\boldsymbol{\theta} \colon \mathbb{R}^{n-1-d} \mapsto \mathbb{R}$ is a quadratic function ($\boldsymbol{\theta}''$ is a constant matrix).

Relaxation approximation of fluid systems (isentropic case)

The relaxation approximation of isentropic fluid systems writes

$$\begin{aligned} &\mathsf{D}_t \mathbf{v} - N \partial_y \mathbf{u} = 0 \\ &\mathsf{D}_t \mathbf{u} - N^T \partial_y \boldsymbol{\pi}(\mathscr{T}, \mathbf{v}) = 0 \\ &\mathsf{D}_t \mathscr{T} = \frac{1}{\varepsilon} \left(\mathbf{v} - \mathscr{T} \right) \end{aligned}$$

with the pressure law

$$\boldsymbol{\pi}(\mathscr{T}, \mathbf{v}) = \epsilon'(\mathscr{T}) + \boldsymbol{\theta}'(\mathbf{v} - \mathscr{T})$$

The stability condition (Whitham) is now

the matrix $\boldsymbol{\theta}'' - \epsilon''(\mathbf{v})$ is positive definite

(for Euler equations: $a^2 > \sup_v |p'(v)|$)

Relaxation approximation of fluid systems

Under the Whitham stability condition, we have:

Theorem.

The LHS of the relaxation approximation of fluid systems is such that:

- ▷ The system is hyperbolic
- ▷ All the fields are linearly degenerate
  $\Longrightarrow$ Global well-posedness for BV smooth and nonsmooth solutions

Theorem. [Yong, Dressel-Yong]

The relaxation approximation of fluid systems satisfies:

- ▷ Global existence of smooth solutions for initial data near the equilibrium
- ▷ Convergence towards the equilibrium for smooth solutions
- ▷ Existence of traveling wave solutions

More results for nonsmooth solutions ? [Chen-Levermore-Liu, Serre...]

Numerical methods

Finite volume schemes for $\partial_t U + \partial_x F(U) = 0$:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}(U_i^n, U_{i+1}^n) - \mathcal{F}(U_{i-1}^n, U_i^n)\right)$$

where $\mathcal{F}(U_l, U_r)$ approximates the flux between U_l and U_r.

Godunov-type schemes by Harten-Lax-Van Leer:

$$U_i^{n+1} = \frac{1}{\Delta x}\left(\int_0^{\Delta x/2} \mathcal{U}(x/\Delta t, U_{i-1}^n, U_i^n)\, dx + \int_{-\Delta x/2}^{0} \mathcal{U}(x/\Delta t, U_i^n, U_{i+1}^n)\, dx\right)$$

where $\mathcal{U}(x/t; U_l, U_r)$ is an approximate Riemann solver.

$\longrightarrow$ Consistency, conservation and entropy properties.

Relaxation Riemann solver

Hyperbolic fluid systems

$$(\mathscr{E}) \quad \partial_t U + \partial_x F(U) = 0 \qquad\qquad (\mathscr{R}) \quad \partial_t W + \partial_x G(W) = \frac{1}{\varepsilon} R(W)$$

Define L and $M(.)$ such that

$$U = L\,W, \qquad L\,M(U) = U, \qquad L\,G(M(U)) = F(U).$$

Proposition.

Let $\mathcal{W}$ be the exact Riemann solver of $(\mathscr{R})$. Then under the Whitham condition,

$$\mathcal{U}(x/t, U_l, U_r) := L\,\mathcal{W}(x/t; M(U_l), M(U_r))$$

is an approximate Riemann solver for $(\mathscr{E})$ in the sense of Harten-Lax-Van Leer.

$\longrightarrow$ Consistent, conservative and entropy satisfying numerical schemes.

Relaxation scheme

Numerical flux of the relaxation scheme:

$$\mathcal{F}(U_l, U_r) = \frac{1}{2}\left(F(U_l) + F(U_r) - \sum_{k=1}^{l} |\lambda_k| (\mathcal{U}(\lambda_k^+; U_l, U_r) - \mathcal{U}(\lambda_k^-; U_l, U_r))\right)$$

Classical numerical flux for finite volume schemes.

In 1D:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}(U_i^n, U_{i+1}^n) - \mathcal{F}(U_{i-1}^n, U_i^n)\right)$$

In multiD:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{|M_i|} \sum_{M_k \in V(M_i)} |\sigma_{ik}| \mathcal{F}(U_i^n, U_k^n; \nu_{ik})$$

Properties of relaxation schemes

Relaxation schemes

- Application to hyperbolic fluid systems (Després)
  - Gas dynamics equations ($\equiv$ HLLC, cf Bouchut)
  - One-velocity one-pressure multifluid models
  - Multi-temperature models
  - Ideal MHD
  - ...
- Based on approximate Riemann solvers (Harten-Lax-Van Leer)
- Only Riemann problems with contact discontinuities to solve
- Consistent, conservative and entropy satisfying numerical schemes
- Positivity preserving schemes (domain invariance by relaxation)
- Finite volume interpretation for multiD applications

Numerical tests

- **Cavitation**
  - Isothermal model of phase transition
  - A box immersed in a liquid with high velocity
  - Apparition of bubbles of vapor
- **Shallow water with topography**
  - Topography dealt by the hydrostatic reconstruction [Audusse et al. 04]
  - Difficult phenomenon of transition between dry and wet area
  - Positivity of water height $\Longleftrightarrow$ conservation of mass

Conclusion

- **A new relaxation approximation**
  - Application to hyperbolic fluid systems
  - Linearly degenerate system
  - $\equiv$ diffusion for smooth solutions

- **New numerical schemes**
  - Application to hyperbolic fluid systems
  - Formalism of Harten-Lax-Van Leer
  - Riemann solvers with only linearly degenerate fields
  - Positive and entropy satisfying numerical schemes