A Fast and Accurate FFT-Based Method for Pricing Early-Exercise Options

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Overview

- Risk-neutral option valuation context
- FFT pricing of options with early-exercise under Lévy processes
- Numerical results for Bermudan and American options
- The extension to multi-asset options
- Discussion of sparse grid technique

Pricing Approach: Risk-neutral valuation

- By the risk-neutral valuation formula the price of any option without early exercise can be written as

$$V(t, S(t)) = e^{-r\tau}\mathbb{E}[V(T, S(T))], \quad \tau = T - t$$

- From this, one can apply several numerical techniques to calculate the option price:
  - Monte Carlo simulation,
  - Numerical solution of the partial-(integro) differential equation (P(I)DE)
  - Numerical integration.
- The CONV method falls into the category of FFT-based pricing methods

Early Exercise Option Valuation; Bermudans

- Pricing Bermudan options
- Set of exercise dates $\mathcal{T} = \{t_1, \ldots, t_M\}$, $t_m < t_{m+1}, 0 = t_0 \leq t_1$;
- At $t \in \mathcal{T}$ we may exercise into $E(t, S(t))$.
- Setting $V(t_M, S(t_M)) = E(t_M, S(t_M))$ we find the Bermudan option price via backward induction in:

$$\begin{cases} V(t_M, S(t_M)) = E(t_M, S(t_M)) \\ C(t_m, S(t_m)) = e^{-r\Delta t}\mathbb{E}_{t_m}\left[V(t_{m+1}, S(t_{m+1}))\right] \\ V(t_m, S(t_m)) = \max\{C(t_m, S(t_m)), E(t_m, S(t_m))\}, \\ V(t_0, S(t_0)) = C(t_0, S(t_0)), \end{cases} m = M-1, \ldots, 1,$$

Discounted Expected Payoff

- Write, in the case of deterministic interest rates, as an integral:

$$C(t_m, S(t_m)) \; = \; e^{-r\Delta t} \int_{-\infty}^{\infty} V(t_{m+1}, y) f(y|S(t_m)) dy$$

- Here, $f(y|S(t_m))$ is the probability density describing the transition from $S(t_m)$ at t_m to y at t_{m+1}.

$\Rightarrow$ Basis for the QUAD Method by Andricopoulos, Widdicks, Duck and Newton (2003);

- O'Sullivan(2005): Generalization to exponential Lévy processes, as the density can be recovered via Fourier inversion.

$$f(y|x) = \frac{dF(y|x)}{dy} = \frac{1}{\pi} \int_0^{\infty} \text{Re } (e^{-ivy} \phi(v|x)) dv$$

- With the midpoint rule, f.e., the density can be approximated and resolved by the FFT. Overall complexity of $O(MN^2)$ for M-times exercisable Bermudan options.

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Motivation

- Our motivation: To derive a method which is
  - Computationally fast
  - Not restricted to Gaussian-based models
  - It should work as long as we have a characteristic function available,

$$\phi(v) = \int_{-\infty}^{\infty} e^{ivx} f(x) dx; \quad f(x) = \frac{1}{\pi} \int_0^{\infty} \text{Re}\, (\phi(v) e^{-ivx}) dv$$

The CONV method

- The main premise of the CONV method is that the conditional probability density $f(y|x)$ only depends on x and y via their difference,

$$f(y|x) = f(y-x).$$

- Assumption is clearly satisfied in exp. Lévy models, where x and y then represent log-asset prices. The assumption means that log-returns are independent.
- The CONV method departs from

$$\begin{aligned} C(t_m, x) &= e^{-r\Delta t} \int_{-\infty}^{\infty} V(t_{m+1}, y) f(y|x) dy \\ &= e^{-r\Delta t} \int_{-\infty}^{\infty} V(t_{m+1}, x+z) f(z) dz. \\ V(t_m, x) &= \max\left(E(t_m, x), C(t_m, x)\right) \end{aligned}$$

- The key insight is the notion that, apart from the discounting, the equation is a cross-correlation of V with the density function f.

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Early Exercise Option Valuation

- Premultiplying by $\exp(\alpha x)$ and taking its Fourier transform, gives:

$$
\begin{aligned}
e^{r\Delta t}\mathcal{F}\{e^{\alpha x}C(t_m,x)\} &= e^{r\Delta t}\int_{-\infty}^{\infty} e^{iux}e^{\alpha x}C(t_m,x)dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iux}e^{\alpha x}V(t_{m+1},x+z)f(z)dzdx \\
&= \int_{-\infty}^{\infty} e^{i(u-i\alpha)y}V(t_{m+1},y)dy\int_{-\infty}^{\infty} e^{-i(u-i\alpha)z}f(z)dz \\
&= \widetilde{V}(t_{m+1},u-i\alpha)\phi\left(-(u-i\alpha)\right).
\end{aligned}
$$

- A computation for resolving the (conditional) density function is avoided, only the characteristic function ϕ is involved.

- The option price is recovered by the inverse Fourier transform:

$$
e^{r\Delta t}C(t_m,x) = e^{-\alpha x}\mathcal{F}^{-1}\{\widetilde{V}(t_{m+1},u-i\alpha)\phi\left(-(u-i\alpha)\right)\}.
$$

Some Details

- The extended characteristic function

$$\phi(x+yi) = \int_{-\infty}^{\infty} e^{i(x+yi)z} f(z)dz,$$

is well-defined when $\phi(yi) < \infty$, as $|\phi(x+yi)| \leq |\phi(yi)|$.

⇒ This puts a restriction on the damping coefficient α, because $\phi(\alpha i)$ must be finite.

- The damping factor is necessary when considering e.g. a Bermudan put, as then $V(t_{m+1}, x)$ tends to a constant when $x \to -\infty$, and as such is not L^1-integrable.

- The difference with the Carr-Madan approach is that we take a transform with respect to the log-spot price instead of the log-strike price.

⇒ The idea for GBM is already present in a presentation by Eric Reiner (2000)

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Expressions for hedge parameters

- The CONV formulae for two hedge parameters, Δ and Γ, defined as,

$$\Delta = \frac{\partial V}{\partial S} = \frac{1}{S}\frac{\partial V}{\partial x}, \quad \Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{1}{S^2}\left(-\frac{\partial V}{\partial x} + \frac{\partial^2 V}{\partial x^2}\right). \tag{1}$$

- Define, $\mathcal{F}\{e^{\alpha x}V(t_0, x)\} = e^{-r\Delta t}A(u)$,
  where $A(u) = \mathcal{F}\{e^{\alpha y}V(t_1, y)\} \cdot \phi(-u + i\alpha)$.

- CONV formula for Δ and Γ,

$$\Delta = \frac{e^{-\alpha x}e^{-r\Delta t}}{S}\Big[\mathcal{F}^{-1}\{-iuA(u)\} - \alpha\mathcal{F}^{-1}\{A(u)\}\Big],$$

$$\Gamma = \frac{e^{-\alpha x}e^{-r\Delta t}}{S^2}\Big[\mathcal{F}^{-1}\{(-iu)^2A(u)\} - (1+2\alpha)\mathcal{F}^{-1}\{-iuA(u)\} + \alpha(\alpha+1)\mathcal{F}^{-1}\{A(u)\}\Big].$$

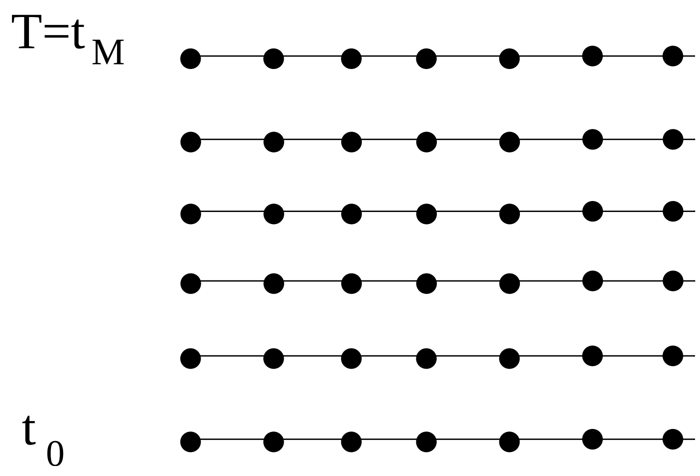
- The only additional calculations occur at the final step of the CONV algorithm, where we calculate the value of the option given the continuation and exercise values at t_1.

CONV Method, FFT

- Implementation using the FFT, so we require uniform grids for u, x and y and $j = 0, \ldots, N-1$:

$$u_j = u_0 + j\Delta u, \; x_j = x_0 + j\Delta x, \; y_j = y_0 + j\Delta x$$

- x represent the log-asset price at t_m, y at t_{m+1}.
- Further, the Nyquist relation must be satisfied: $\Delta u \cdot \Delta x = 2\pi/N$.



CONV Method, FFT

- Step 1 - The payoff transform

$$\mathcal{F}\{e^{\alpha y}V(t_{m+1},y)\}(u) \;=\; \int_{-\infty}^{\infty} e^{iuy}e^{\alpha y}V(t_{m+1},y)dy \approx \Delta y \sum_{n=0}^{N-1} w_n e^{iu_j y_n} e^{\alpha y_n} V(t_{m+1},y_n)$$

  Can be evaluated using the FFT. We use the Trapezoidal rule as this yields the most stable results, though higher order Newton-Côtes can in principle be used.

- Step 2 - Convolution

  Calculate $\mathcal{F}\{e^{\alpha y}V(t_{m+1},y)\}(u)\cdot\phi(-u_j+i\alpha),\quad$ for $j=0,\ldots,N-1$.

- Step 3 - Inverting the Fourier Transform: Can again be evaluated using the FFT.

  The left-rectangle rule is used here. The decay of the CF dictates the convergence.

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Error analysis of the CONV method

- Rederive discretized CONV formula by a Fourier series expansion of continuation value.
- This reveals that
  - Only moment restriction on α is necessary (L^1 integrability is replaced by L^1-summability);
  - If ϕ decays faster than a polynomial, the discretized CONV formula converges as $O(1/N^2)$ for continuous payoff functions;
  - If ϕ decays as x^β, the order is $O(1/N^{\min\{1+\beta,2\}})$ for continuous payoff functions.

$\Rightarrow$ Details in our paper.

- The method is $O(MN \ln N)$, lower than the QUAD method, but higher than Broadie and Yamamoto's algorithm.
- However, the CONV method is generally applicable.

Lévy Processes

- Each Lévy process can be characterised by a triplet (μ, σ, ν) with $\mu \in \mathbb{R}, \sigma \geq 0$ and ν a measure satisfying $\nu(0) = 0$ and

$$\int_{\mathbb{R}} \min(1, |x|^2)\nu(dx) < \infty.$$

- In terms of this triplet the characteristic function of the Lévy process equals:

$$\begin{aligned} \phi(u) &= \mathbb{E}[\exp(iuL(t))] \\ &= \exp(t(i\mu u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbf{1}_{[|x|<1]}\nu(dx))), \end{aligned}$$

the celebrated Lévy-Khinchine formula.

Characteristic function

- The particular model we will consider is the extended CGMY model.
- The characteristic function of the log-asset price can be found in closed-form as:

$$\phi(u) = S(0)^{iu} \exp\left(iu\mu t - \frac{1}{2}u^2\sigma^2 t + tC\Gamma(-Y)[(M-iu)^Y - M^Y + (G+iu)^Y - G^Y] \right),$$

where $\Gamma(x)$ is the gamma function.

- When $\sigma = 0$ and $Y = 0$ we obtain the Variance Gamma (VG) model, which is often parameterised slightly differently with parameters σ, θ and ν, related to C, G and M:

$$C = \frac{1}{\nu},\ G = \frac{1}{\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu - \frac{1}{2}\theta\nu}},\ M = \frac{1}{\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu + \frac{1}{2}\theta\nu}}.$$

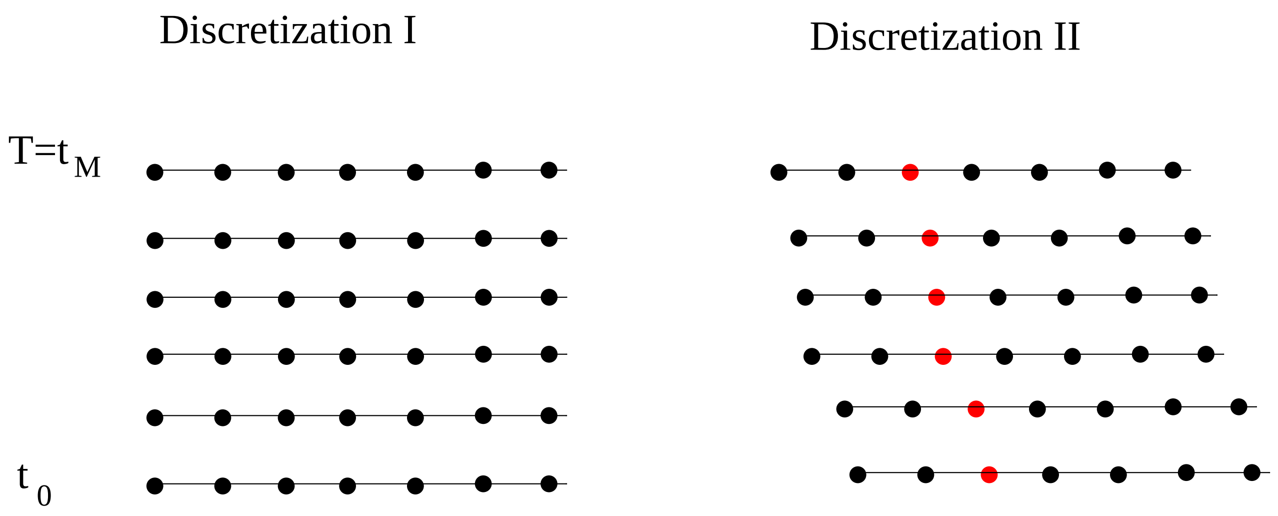
- Finally, when $C = 0$ the model collapses to the Black-Scholes model.

Numerical results

- Variety of results in Black-Scholes, VG and CGMY
- Reference values from literature or from CONV method with 2^{20} grid points
- Univariate results in C^{++} on an Intel Xeon CPU 5160, 3 GHz with 2 GB RAM
- Multivariate results in C on an Intel Core 2 CPU 6700, 2.66Ghz with 8 GB RAM

Dealing with discontinuities

Especially for Bermudan Options



- We consider two discretizations:
  - Discretization I: $x = y$ throughout, and $\ln S(0)$ lies on the grid;
  - Discretization II: At each time, t_m, we place d_m on the x-grid.

1. Estimate d_m in $C(t_m, d_m) = E(t_m, d_m)$;
2. Place d_m on the x-grid and recalculate $C(t_m)$;
3. Re-evaluate exercise decision and continue.

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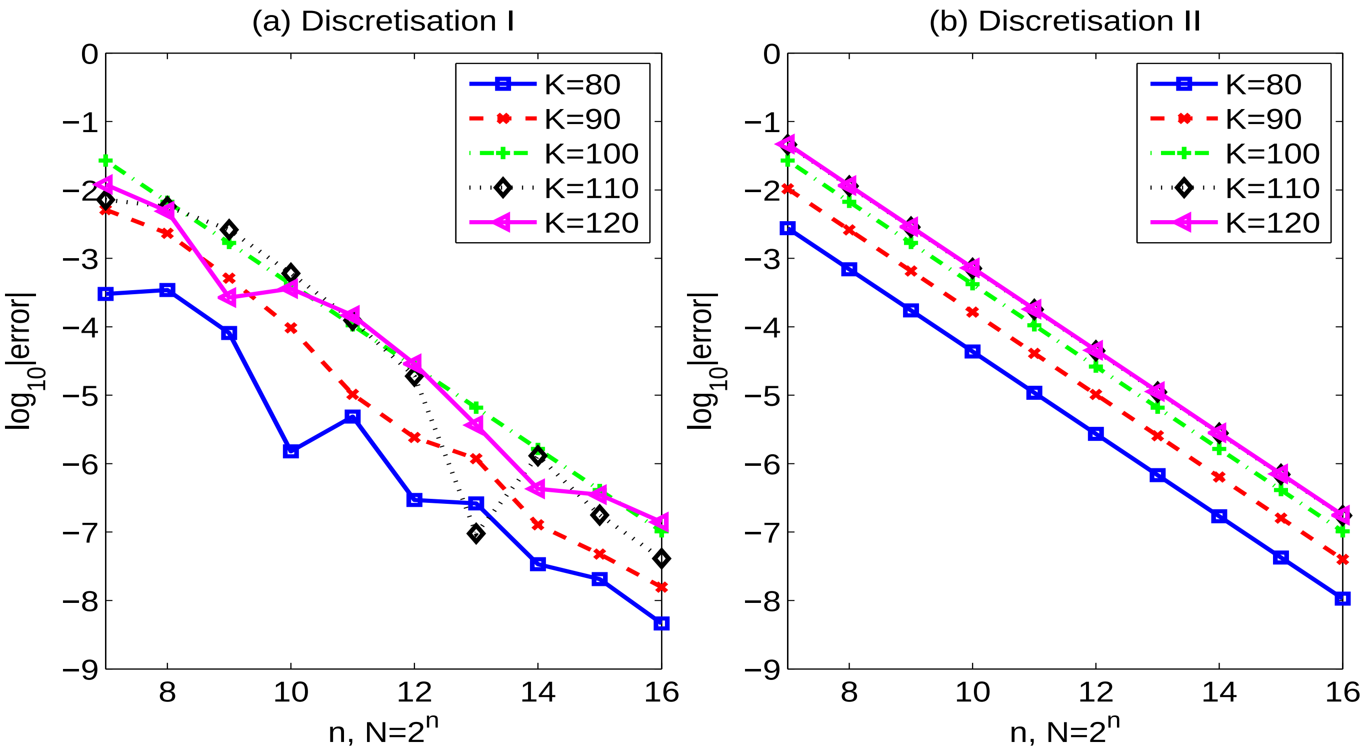
Discretization I vs. II

European Option

- Placing the discontinuity (here the strike) on the grid ensures smooth convergence:



Bermudan option under GBM and VG

Discretization II

- Pricing 10-times exercisable Bermudan put under GBM and VG
- $S_0 = 100, K = 110, T = 1, r = 0.1, q = 0;$
- For GBM: $\sigma = 0.25$, reference$= 11.1352431;$
- For VG: $\sigma = 0.12, \theta = -0.14, \nu = 0.2$, reference$= 9.040646114;$

$(N = 2^n)$	GBM			VG		
n	time(msec)	abs. error	conv.	time(msec)	abs. error	conv.
7	0.23	-2.7-02	–	0.28	-9.6e-02	–
8	0.46	-7.4-03	3.7	0.55	-1.1e-02	9.0
9	0.90	-2.0e-03	3.7	1.09	-2.3e-03	4.7
10	2.00	-5.2e-04	3.8	2.15	-6.1e-04	3.8
11	3.85	-1.3e-04	4.0	4.38	-1.6e-04	3.8
12	7.84	-3.3e-05	4.0	9.29	-4.1e-05	3.9

Approximation of American option

- The value of an American option can be approximated
  - either by a Bermudan with many exercise dates,
  - or, by Richardson extrapolation on a series of Bermudan options with an increasingly number of exercise dates (Geske,Johnson, 1984)
- We choose the repeated Richardson extrapolation proposed by Chang, Chung and Stapleton [2001], and compare the two approaches

American option under GBM

- $S_0 = 100, K = 110, T = 1, \sigma = 0.25, r = 0.1, q = 0;$
- Reference value: $V_{ref}(0, S(0)) = 12.169417$ (Black-Scholes)
- Richardson extrapolation with 128, 64 and 32 exercise opportunities

$(N = 2^n)$	P(N/2)			Richardson		
n	time(msec)	error	conv.	time(msec)	error	conv.
7	0.97	-5.9e-02	–	3.3	-3.1e-02	–
8	3.7	-2.2e-03	2.6	6.6	-7.8e-03	3.9
9	14.8	-9.3e-03	2.4	14.0	-2.1e-03	3.8
10	60.0	-4.16e-03	2.2	28.4	-5.2e-04	4.0
11	251.7	-2.0e-03	2.1	66.4	-1.2e-04	4.3
12	1108.1	-9.4e-04	2.1	151.9	-2.1e-05	5.8

American option pricing under VG and CGMY

- VG: $S_0 = 100, K = 110, T = 1, \sigma = 0.12, \theta = -0.14, \nu = 0.2, r = 0.1, q = 0;$
- CGMY ($Y < 1$):$Y = 0.5, C = 1, G = M = 5, S_0 = 1, r = 0.1;$
- CGMY ($Y > 1$): $Y = 1.0102, C = 0.42, G = 4.37, M = 191.2, S_0 = 90, r = 0.06.$

$(N = 2^n)$	$K = 110, T = 1$ $V_{ref}(0, S(0)) = 10.0000$		$K = 1, T = 1$ $V_{ref}(0, S(0) = 0.112152$		$K = 98, T = 0.25$ $V_{ref}(0, S(0) = 9.225439$	
n	time(msec)	error	time(msec)	error	time(msec)	error
8	6.9	4.3e-2	7.6	9.5e-5	7.7	6.6e-3
9	14.3	1.3e-2	15.9	-1.0e-4	15.8	-1.9e-3
10	29.0	-5.0e-3	32.2	-1.6e-5	33.4	-5.4e-6
11	61.7	-1.9e-2	68.2	-1.1e-5	68.6	-1.7e-4
12	135.1	1.3e-3	148.2	3.7e-6	148.0	-7.9e-5

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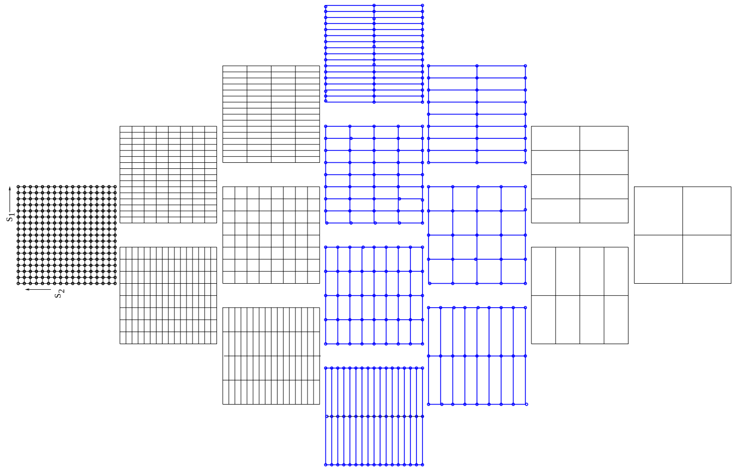
4-Asset Basket Options

- Price multi-asset basket put options with the multi-D version of CONV method.
- 64-bit machine with 8 GB Memory and 2.66 GHz Bus frequency
- $S_0 = 40, K = 40, T = 1, r = 0.06, q = 0.04, \sigma_i = 0.2, \rho_{ij} = 0.25.$
- A good accuracy on relatively coarse grids (in only a few seconds).

	European Call		10-times Bermudan Put	
N	result	time (sec)	result	time (sec)
16^4	1.6428	0.02	1.7721	0.15
32^4	1.6537	0.51	1.7390	3.12
64^4	1.6539	7	1.7394	61.6
128^4	1.6538	159	1.7393	1511

Sparse Grids



$$2D\ :\ u_h^c = \sum_{|I|=n+1} u_I - \sum_{|I|=n} u_I$$

$$dD\ :\ u_n^c = \sum_{k=0}^{d-1} (-1)^{k+1} \binom{d}{k} \sum_{|I|=n+k} u_I$$

Number of points: $\mathcal{O}(N(\log N)^{d-1})$ vs. $\mathcal{O}(N^d)$ ("classical"); 20K vs 33M (32^5)
Accuracy: $\mathcal{O}(N^{-2}(\log N)^{d-1})$ vs. $\mathcal{O}(N^{-2})$ (2nd order),
Sparse grid technique converges for solutions with bounded mixed derivatives.

7D Multi-Asset Results

work with C. Leentvaar

- 7D Put on maximum of assets $(K - \max_j S_j)^+$ and 7D put on minimum of assets
- Parallelization over the subproblems, and parallelization within the multi-D FFT
- $K = 100, T = 1, r = 0.045, 0.2 \leq \sigma_i \leq 0.35, 0.02 \leq q_i \leq 0.07, R_{i,j}$ corr. matrix

	7D Put on minimum		7D put on maximum	
n_s	price	error	price	error
7	26.15	1.2e-1	0.18	1.5e-2
8	26.22	6.3e-2	0.19	1.5e-2
9	26.19	2.7e-2	0.21	1.1e-2
10	26.20	1.3e-2	0.21	7.2e-3

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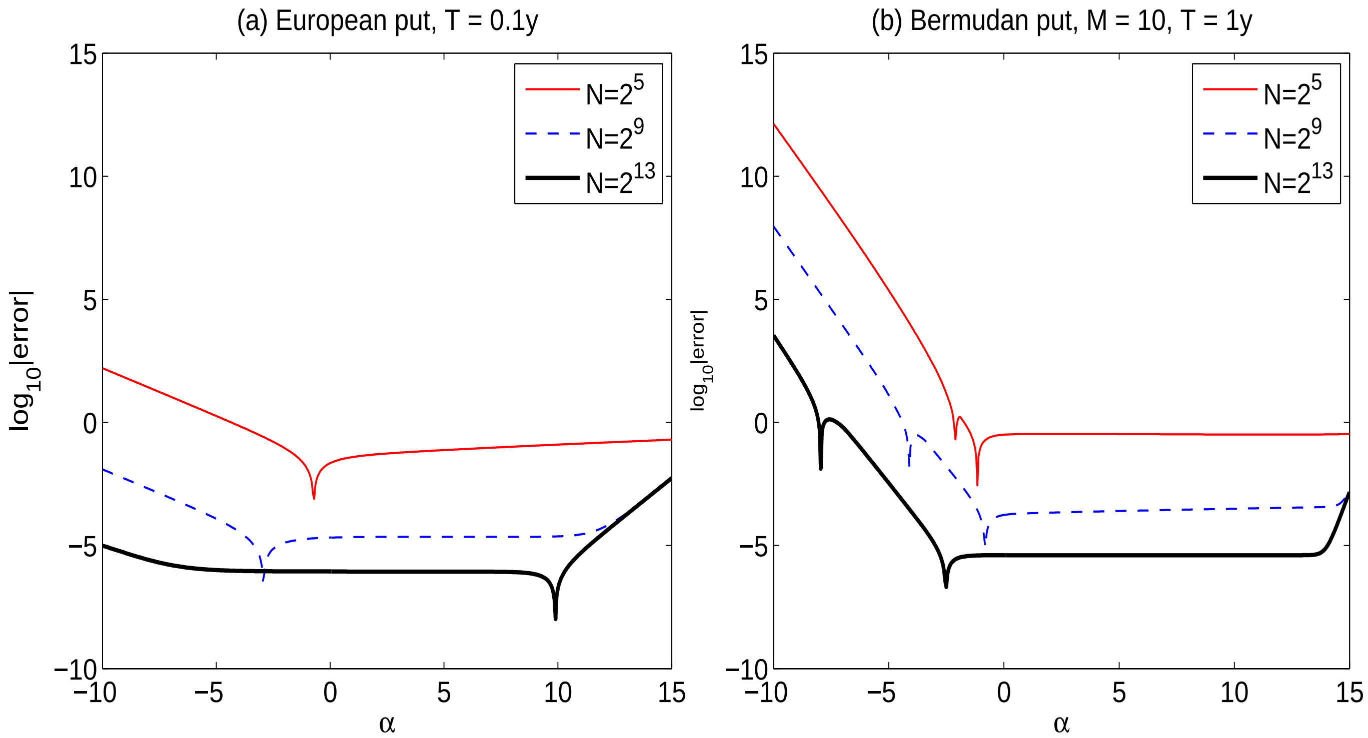
Conclusions

- We presented an FFT based method for early exercise options, the CONV method
- It is of $O(MN \log N)$ for M-times exercisable Bermudan options
- FFT based methods are quite flexible w.r.t. the choice of asset process, and the type of option contract
- The CONV method is highly efficient, for different single-asset early-exercise contracts
- It is also fast for some multi-asset options
- The CONV method appears to be a promising method for calibration purposes

Choice of damping coefficient

- Lord [2006] related the problem of α in the Carr-Madan framework to saddlepoint methods.
- We opt for $\alpha = 0$ as the payoff-transform is not generally known.
  Illustration for puts in VG



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