A-priori and a-posteriori error estimates for a family of Reissner-Mindlin plate elements

Joint work with

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Contents

The Reissner-Mindlin plate

The appropriately scaled equations have the form:
Find $(w, \boldsymbol{\beta}) \in W \times \boldsymbol{V}$ such that

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) + t^{-2}(\nabla w - \boldsymbol{\beta}, \nabla v - \boldsymbol{\eta}) = (g, v)$$

for all admissible deflections and rotations $(v, \boldsymbol{\eta}) \in W \times \boldsymbol{V}$.

Here $\boldsymbol{\varepsilon}$ is the small strain operator, t is the thickness of the plate,

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) \cong (\boldsymbol{\varepsilon}(\boldsymbol{\beta}), \boldsymbol{\varepsilon}(\boldsymbol{\eta})) \cong (\nabla \boldsymbol{\beta}, \nabla \boldsymbol{\eta})$$

is the bending energy and

$$t^{-2}(\nabla w - \boldsymbol{\beta}, \nabla v - \boldsymbol{\eta})$$

is the shear energy.

RM continued

The shear force is given by

$$\boldsymbol{q} = t^{-2}(\nabla w - \boldsymbol{\beta}).$$

Suppose that we have clamped and free boundary conditions.

On Γ_C:

$$\boldsymbol{\beta} = \mathbf{0}, \quad w = 0.$$

On Γ_F:

$$\boldsymbol{\varepsilon}(\boldsymbol{\beta})\boldsymbol{n} = \mathbf{0}, \quad \boldsymbol{q} \cdot \boldsymbol{n} = \mathbf{0}.$$

The continuous spaces

$$\boldsymbol{V} = \{ \ \boldsymbol{\eta} \in [H^1(\Omega)]^2 \mid \boldsymbol{\eta}_{|\Gamma_C} = \mathbf{0} \ \},$$

$$W = \{ \ v \in H^1(\Omega) \mid v_{|\Gamma_C} = 0 \ \},$$

The Kirchhoff limit and locking

When we go to the limit $t \to 0$, we obtain the Kirchhoff constraint

$$\nabla w - \boldsymbol{\beta} = \mathbf{0}.$$

In addition, we loose one boundary condition.

For a straightforward FEM:
Find $(w_h, \boldsymbol{\beta}_h) \in W_h \times \boldsymbol{V}_h \subset W \times \boldsymbol{V}$ such that

$$a(\boldsymbol{\beta}_h, \boldsymbol{\eta}) + t^{-2}(\nabla w_h - \boldsymbol{\beta}_h, \nabla v - \boldsymbol{\eta}) = (g, v) \quad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h,$$

the K-constraint is inherited:

$$\nabla w_h - \boldsymbol{\beta}_h = \mathbf{0}.$$

For low-order elements this leads to the "locking":

$$w_h = 0, \qquad \boldsymbol{\beta}_h = \mathbf{0}.$$

"Quasi-optimal" error estimates

$$\|w - w_h\|_1 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 \leq t^{-1} C \Big\{ \inf_{v \in W_h} \|w - v\|_1 + \inf_{\boldsymbol{\eta} \in \boldsymbol{V}_h} \|\boldsymbol{\beta} - \boldsymbol{\eta}\|_1 \Big\}.$$

This "explodes" as $t \to 0$.

The engineering remedy is to loosen up the constraint:
Find $(w_h, \boldsymbol{\beta}_h) \in W_h \times \boldsymbol{V}_h \subset W \times \boldsymbol{V}$ such that

$$a(\boldsymbol{\beta}_h, \boldsymbol{\eta}) + t^{-2}(\mathbf{\Pi}_h(\nabla w_h - \boldsymbol{\beta}_h), \mathbf{\Pi}_h(\nabla v - \boldsymbol{\eta})) = (g, v).$$

There exists $\mathcal{O}(10^2) - \mathcal{O}(10^3)$ of papers on this.

But only $\mathcal{O}(10)$ with a mathematical analysis.

The Falk-Tu family

$\mathbf{\Pi}_h$ is the L^2-projection onto a discrete shear space $\boldsymbol{Q}_h$.

This is equivalent to a mixed method.

Find $(w_h, \boldsymbol{\beta}_h, \boldsymbol{q}_h) \in W_h \times \boldsymbol{V}_h \times \boldsymbol{Q}_h$ such that

$$\begin{aligned} a(\boldsymbol{\beta}_h, \boldsymbol{\eta}) + (\boldsymbol{q}_h, \nabla v - \boldsymbol{\eta}) &= (g, v) \quad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h, \\ -t^2(\boldsymbol{q}_h, \boldsymbol{r}) + (\nabla w_h - \boldsymbol{\beta}_h, \boldsymbol{r}) &= 0 \qquad \forall \boldsymbol{r} \in \boldsymbol{Q}_h. \end{aligned}$$

In the limit $t \to 0$ we obtain the familiar saddle point structure (for the Kirchhoff solution).

The key question is what norms have to be used.

For this we take a look at:

Regularity and shift theorems

Kirchhoff.

$$\|w\|_2 \le C\|g\|_{-2}.$$

and

$$\|w\|_3 \le C\|g\|_{-1}.$$

Reissner–Mindlin. All works prior to 1998 use the estimates

$$\|w\|_1 + \|\boldsymbol{\beta}\|_1 \le C\|g\|_{-1}$$

and

$$\|w\|_2 + \|\boldsymbol{\beta}\|_2 \le C\|g\|_0.$$

Where do we see RM $\rightarrow$ K ?

Regularity of the Reissner–Mindlin model

Split $w = w_0 + w_r$, where

$$w_0 = \lim_{t \to 0} w$$

is the limiting Kirchhoff solution.

Then it holds (Chapelle, Stenberg 88, Lyly, Niiranen, Stenberg 06)

$$\|w_0\|_2 + t^{-1}\|w_r\|_1 + \|\boldsymbol{q}\|_{-1} + t\|\boldsymbol{q}\|_0 \leq C\big(\|g\|_{-2} + t\|g\|_{-1}\big).$$

and for a clamped plate and convex domain

$$\|w_0\|_3 + t^{-1}\|w_r\|_2 + \|\boldsymbol{q}\|_0 + t\|\boldsymbol{q}\|_1 \leq C\big(\|g\|_{-1} + t\|g\|_0\big).$$

This gives the previous results both for $t = 1$ and $t \to 0$.

Finite element norms

since $w_h \notin H^2(\Omega)$ the continous norms cannot be used.

FE-norms (Babuska, Osborn, Pitkäranta 1980, Pitkäranta 1988).

$$|||(v,\boldsymbol{\eta})|||_h^2 = \|\boldsymbol{\eta}\|_1^2 + \sum_{K\in\mathcal{C}_h} (t^2+h_K^2)^{-1}\|\nabla v - \boldsymbol{\eta}\|_{0,K}^2,$$

which in the limit $t \to 0$ gives the *Kirchhoff* norm

$$\|\boldsymbol{\eta}\|_1 + \|v\|_{2,h},$$

with

$$\|v\|_{2,h}^2 = \sum_{K\in\mathcal{C}_h} |v|_{2,K}^2 + \sum_{E\in\mathcal{I}_h} h_K^{-1}\| \left[\!\left[\frac{\partial v}{\partial \boldsymbol{n}} \right]\!\right] \|_{0,E}^2 \,.$$

The dual norm for the shear is

$$\|\boldsymbol{r}\|_h^2 = \sum_{K\in\mathcal{C}_h} (h_K^2+t^2)\|\boldsymbol{r}\|_0^2.$$

Finite element residual norms

Recall. Kirchhoff:

$$\|w\|_2 \le C\|g\|_{-2}$$

and Reissner-Mindlin:

$$\|w_0\|_2 + t^{-1}\|w_r\|_1 + \|\boldsymbol{q}\|_{-1} + t\|\boldsymbol{q}\|_0 \le C\big(\|g\|_{-2} + t\|g\|_{-1}\big).$$

For the a-posteriori analysis this indicates (K-hhoff):

$$\|\cdot\|_{-2} \approx h^2\|\cdot\|_0$$

and (RM):

$$\|\cdot\|_{-2} + t\|\cdot\|_{-1} \approx h^2\|\cdot\|_0 + th\|\cdot\|_0.$$

The Falk-Tu spaces

For the degree $k \geq 1$, we define

$$
\begin{aligned}
W_h &= \{\, v \in W \mid v|_K \in P_{k+1}(K)\ \forall K \in \mathcal{C}_h \,\}, \\
\boldsymbol{V}_h &= \{\, \boldsymbol{\eta} \in \boldsymbol{V} \mid \boldsymbol{\eta}|_K \in [P_k(K) + B_{k+3}(K)]^2\ \forall K \in \mathcal{C}_h \,\}, \\
\boldsymbol{Q}_h &= \{\, \boldsymbol{r} \in [L^2(\Omega)]^2\ |\boldsymbol{r}_{|K} \in [P_k(K)]^2\ \forall K \in \mathcal{C}_h \,\}.
\end{aligned}
$$

Here we have (a quite big) bubble space

$$B_{k+3}(K) = P_{k+3}(K) \cap H_0^1(K).$$

Stability follows from Brezzis conditions, Z_h-ellipticity and inf-sup.

Z_h-ellipticity

$$Z_h = \{(v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h \mid (\nabla v - \boldsymbol{\eta}, \boldsymbol{r}) = 0 \ \forall \boldsymbol{r} \in \boldsymbol{Q}_h \ \}.$$

Since it holds

$$\nabla W_h \subset \boldsymbol{Q}_h$$

we have

$$Z_h = \{(v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h \mid \nabla v = \boldsymbol{\Pi}_h \boldsymbol{\eta} \ \}.$$

From this the Z_h-ellipticity easily follows:

$$a(\boldsymbol{\eta}, \boldsymbol{\eta}) \geq C \|\boldsymbol{\eta}\|_1^2 \geq C' |||(v, \boldsymbol{\eta})|||_h^2.$$

Inf-Sup

Since we have included so many bubbles we can, for $\boldsymbol{r} \in \boldsymbol{Q}_h$ given, choose $\boldsymbol{\eta} \in \boldsymbol{V}_h$ such that

$$\boldsymbol{\eta}_{|K} = b_K \boldsymbol{r}_{|K} \qquad \forall K \in \mathcal{C}_h,$$

where b_K is the cubic bubble on K.

By scaling we get

$$\frac{(\boldsymbol{\eta}, \boldsymbol{r})}{\|\boldsymbol{\eta}\|_1} \geq \beta \Big(\sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{r}\|_0^2 \Big)^{1/2}.$$

A-priori estimates

We now have, with C independent of t,

$$|||(w-w_h, \boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_h + \|\boldsymbol{q}-\boldsymbol{q}_h\|_h \le C\big\{|||(w-v, \boldsymbol{\beta}-\boldsymbol{\eta})|||_h + \|\boldsymbol{q}-\boldsymbol{r}\|_h\big\}$$

for all $(v, \boldsymbol{\eta}, \boldsymbol{r}) \in W_h \times \boldsymbol{V}_h \times \boldsymbol{Q}_h$.

Recall

$$|||(v, \boldsymbol{\eta})|||_h^2 = \|\boldsymbol{\eta}\|_1^2 + \sum_{K \in \mathcal{C}_h} (t^2 + h_K^2)^{-1} \|\nabla v - \boldsymbol{\eta}\|_{0,K}^2.$$

This is sharp (in contrast to FT).

When estimating the right hand side we write $w = w_0 + w_r$, and use the shift theorem

$$\|w_0\|_3 + t^{-1}\|w_r\|_2 + \|\boldsymbol{\beta}\|_2 \le C\big(\|g\|_{-1} + t\|g\|_0\big).$$

We get

$$\begin{aligned}
&|||(w-\tilde{w}, \boldsymbol{\beta} - \tilde{\boldsymbol{\beta}})|||_h \\
&\quad \le \quad |||(w_0 - \tilde{w}_0, \mathbf{0})|||_h + |||(w_r - \tilde{w}_r, \mathbf{0})|||_h + |||(0, \boldsymbol{\beta} - \tilde{\boldsymbol{\beta}})|||_h \\
&\quad \le \quad h^{-1}\|w_0 - \tilde{w}_0\|_1 + t^{-1}\|w_r - \tilde{w}_r\|_1 + h^{-1}\|\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}\|_0 \\
&\quad \le \quad C\big(h\|w_0\|_3 + t^{-1}h\|w_r\|_2 + h\|\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}\|_2\big) \\
&\quad \le \quad Ch\big(\|g\|_{-1} + t\|g\|_0\big).
\end{aligned}$$

The final a-priori estimate for a convex region:

$$|||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h + \|\boldsymbol{q} - \boldsymbol{q}_h\|_h \le Ch\big(\|g\|_{-1} + t\|g\|_0\big).$$

A-posteriori estimate

Claes Johnson: "The two legs of error analysis. A priori estimates are based on the stability of the discrete problem. A posteriori estimates are based on the stability of the continuous problem."

The problem is now the stability of the continuous problem, i.e. the norms!

We are forced to rely on one discrete leg.

We use a saturation assumption. Let $(\tilde{w}_h, \tilde{\boldsymbol{\beta}}_h)$ be the solution of the FE equations with elements of one degree higher.

There exist a positive constant $\alpha < 1$ such that

$$|||(w - \tilde{w}_h, \boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}_h)|||_h \leq \alpha |||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h.$$

The a posteriori estimate

It holds

$$|||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h \approx \Big(\sum_{K \in \mathcal{C}_h} \eta_K^2 \Big)^{1/2},$$

with

$$\begin{aligned} \eta_K^2 \quad = \quad & h_K^2 (h_K^2 + t^2) \| \operatorname{div} \boldsymbol{q}_h + g \|_{0,K}^2 \\ & + h_K^2 \| \mathbf{div}\, \boldsymbol{\varepsilon}(\boldsymbol{\beta}_h) + \boldsymbol{q}_h \|_{0,K}^2 \\ & + (h_K^2 + t^2)^{-1} \| t^2 \boldsymbol{q}_h - (\nabla w_h - \boldsymbol{\beta}_h) \|_{0,K}^2 \\ & + \sum_{E \subset K} h_E \| [\![\boldsymbol{\varepsilon}(\boldsymbol{\beta}_h) \boldsymbol{n}]\!] \|_{0,E}^2 \\ & + \sum_{E \subset K} h_E (h_E^2 + t^2) \| [\![\boldsymbol{q}_h \cdot \boldsymbol{n}]\!] \|_{0,E}^2 . \end{aligned}$$