Etude de l'ordre de convergence de la méthode des volumes finis

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Groupe de travail "Méthodes numériques" LJLL, 08/10/07

Supraconvergence

Definition :

Supraconvergence is observed when the global error of a Finite Differences scheme applied to an ODE or PDE has a better behavior than that indicated by the local error.

Examples :

- Order 2 scheme with order 1 local error
- Order 1 (therefore convergent) scheme with a non-zero convergent local error.

Consequences :

Lax Theorem is useless

Applications :

This loss of accuracy real for the local error and apparent for the global error is observed with non uniform grids

Lax Theorem

Finite differences scheme : ($h \longrightarrow 0$: parameter of discretization, L_h assumed linear)

$$L_h(u_h) = F$$

Stability : (c cst ind. on h)

$$\|L_h^{-1}\| \leq c$$

Truncation error : (u smooth exact sol.)

$$L_h(u) = \epsilon_h + F$$

Global error :

$$\|e_h\| := \|u_h - u\| = \|L_h^{-1}(L_h(e_h))\| \leq c\|L_h(e_h)\| = c\|\epsilon_h\|$$

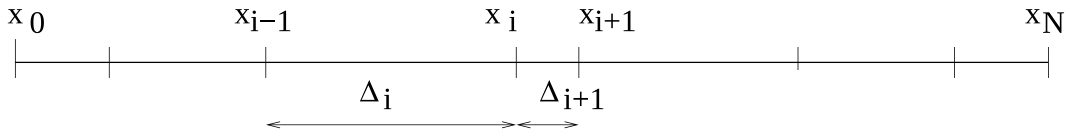
$$\Downarrow$$

Lax Theorem :

If the scheme is consistant (i.e. if $\|\epsilon_h\| \overset{h\rightarrow 0}{\longrightarrow} 0$) then $u_h \overset{h\rightarrow 0}{\longrightarrow} u$ with a rate at least equal to the local error rate.

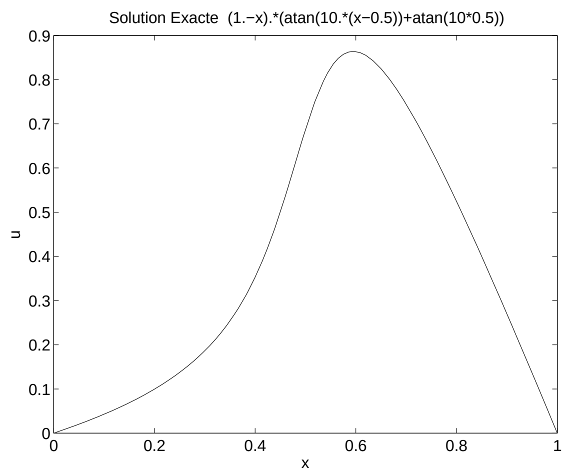
Diffusion in dimension 1

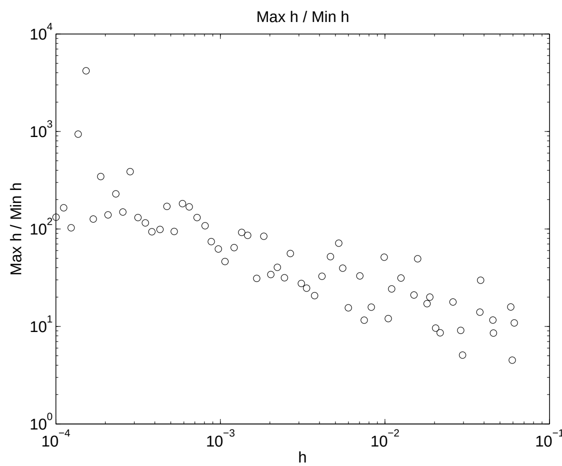
$-u'' = f$ with Dirichlet B.C.

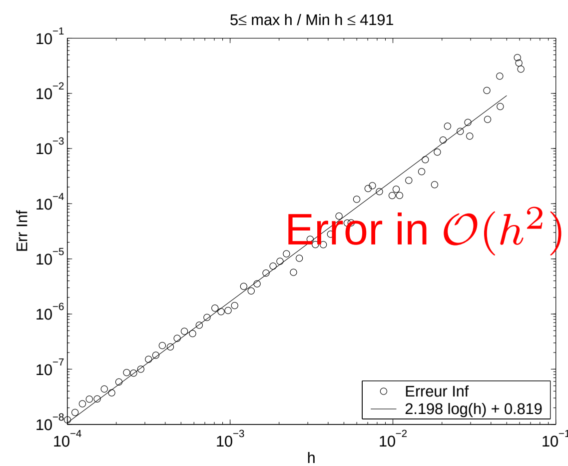


FD Scheme : $$-\frac{2u_{i-1}}{\Delta_i(\Delta_i+\Delta_{i+1})}+\frac{2u_i}{\Delta_i\Delta_{i+1}}-\frac{2u_{i+1}}{\Delta_{i+1}(\Delta_i+\Delta_{i+1})}=f(x_i)$$ et C.B.

Local Error : $$\epsilon_h^i=\frac{\Delta_i-\Delta_{i+1}}{3}u'''(x_i)+\mathcal{O}(h^2)=\mathcal{O}(h)$$ for non uniform grid







Keller, 78 : applied a FD scheme to a 1st order equiv. system $\Longrightarrow \mathcal{O}(h^2)$

Manteuffel et White, 86 : discretization of the 1st order system $\Longrightarrow$ 2nd order eq with $L_h = D_0 D_1$, then $\epsilon_h = D_1\gamma + \underline{\epsilon} \Longrightarrow \mathcal{O}(h^2)$

Mathematical Analysis

Supracongergence :

$\epsilon_h = \mathcal{O}(h^{p-1})$ and numerically $\|e_h\| = \mathcal{O}(h^p)$

Wendroff and White, Comp. Math. Appl., 89 :

$\Longrightarrow$ correction of the error for the mathematical analysis

if $\epsilon_h = \mathcal{O}(h^{p-1})$

but $\epsilon_h = L_h(\gamma) + \underline{\epsilon}$

with $\gamma = \mathcal{O}(h^p)$ and $\underline{\epsilon} = \mathcal{O}(h^p)$

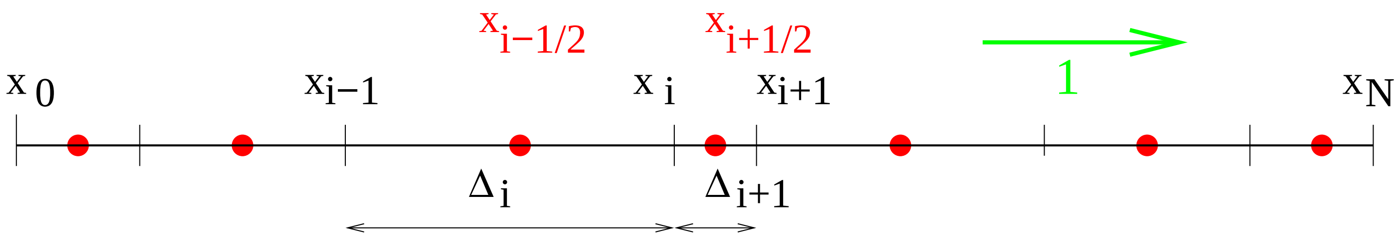
then $L_h(e_h + \gamma) = -\underline{\epsilon}$

$\Longrightarrow \|e_h + \gamma\| = \mathcal{O}(h^p)$

$\Longrightarrow \|e_h\| = \mathcal{O}(h^p)$

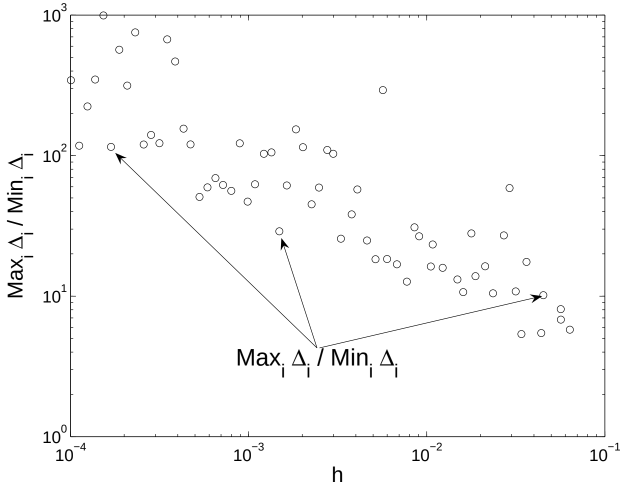
Equation of Transport in dimension 1 (1/2)

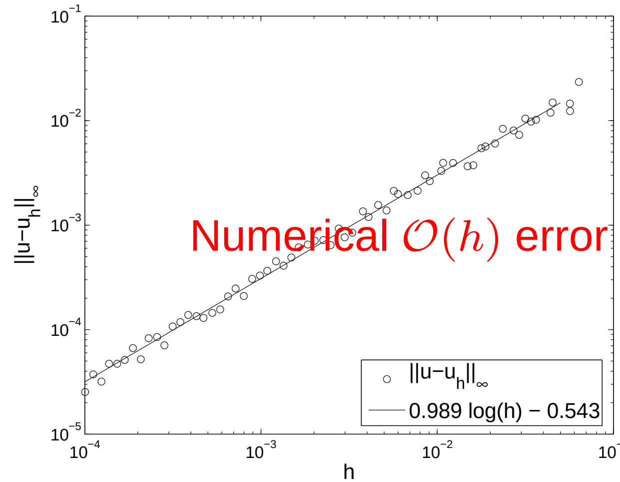
$-u' + f = 0$ and $u(0)$ given



FD Scheme : u_0 given and $-\dfrac{u_i - u_{i-1}}{\frac{1}{2}(\Delta_i + \Delta_{i+1})} + f(x_i) = 0$

Local error : $\epsilon_h^i = \dfrac{-\Delta_{i+1} + \Delta_i}{\Delta_{i+1} + \Delta_i} u'(x_i) + \mathcal{O}(h) = \mathcal{O}(1)$ for non uniform grid





Transport in dimension 1 (2/2)

Correction :

$$\epsilon_h^i = \frac{\gamma^i - \gamma^{i-1}}{\frac{1}{2}(\Delta_i + \Delta_{i+1})} + \underline{\epsilon}^i$$

with $$\gamma^i = -\frac{1}{2}\Delta_{i+1} u'(x_{i+1/2}) = \mathcal{O}(h)$$

and $$\underline{\epsilon}^i = \frac{\Delta_{i+1}(u'(x_{i+1/2}) - u'(x_i)) + \Delta_i(u'(x_i) - u'(x_{i-1/2}))}{\Delta_{i+1} + \Delta_i} = \mathcal{O}(h)$$

Linear Convection Problem

a constant vector

Stationnary problem : $0 < b_0 \leq b(x) \leq b_1$:

$$\begin{cases} bu + (a \cdot \nabla)u = f & \text{in } \Omega \\ u(x) = \psi(x) & \text{on } \partial\Omega^- \end{cases}$$

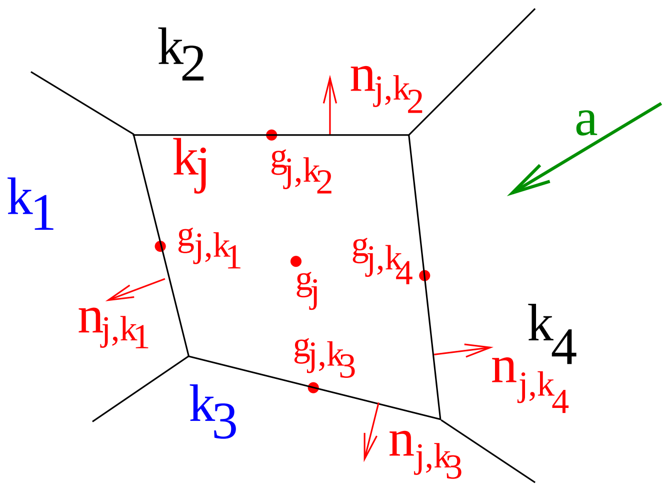
Unsteady problem :

$$\begin{cases} \dfrac{\partial u}{\partial t} + (a \cdot \nabla)u = 0 & \text{in } \Omega \\ u(x,0) = \phi(x) & \text{in } \Omega \\ u(x,t) = \psi(x,t) & \text{on } \partial\Omega^- \times [0,\infty[\end{cases}$$

Finite Volume Method

Explicit, Finite volume scheme : $u_j^n \approx \frac{1}{|K_j|}\int_{K_j} u(x,t^n)dx \equiv U_j^n$ such that $1 \le j \le N_v$

$$\frac{u_j^{n+1}-u_j^n}{\Delta t_n} + \frac{1}{|K_j|}\left(\sum_{k\in\mathcal{N}^+(j)} a\cdot N_{j,k} u_j^n + \sum_{k\in\mathcal{N}_0^-(j)} a\cdot N_{j,k} u_k^n + \sum_{k\in\mathcal{N}_b^-(j)} a\cdot N_{j,k}\phi(g_{j,k}) \right) = 0$$



$$N_{j,k} = |K_j \cap K_k| n_{j,k}$$

$k_1,\, k_3 \in \mathcal{N}^+(j)$ and $k_2,\, k_4 \in \mathcal{N}_0^-(j)$

Hyp : $\frac{h_j^{nd}}{|K_j|} \le \kappa_1$ and $\sharp\mathcal{N}(j) \le \kappa_2\,, \quad \forall K_j$

Local and Global error

FV scheme :

$$(1) \quad u_j^{n+1} - \mathcal{L}_j^n\left(\{u_k^n\}_k\right) + \mathcal{L}_{bord}^n = 0 \quad \text{where}$$

$$\mathcal{L}_j^n\left(\{\xi_k\}_k\right) = \xi_j - \frac{\Delta t_n}{|K_j|}\left(\sum_{k\in\mathcal{N}^+(j)} a\cdot N_{j,k}\xi_j + \sum_{k\in\mathcal{N}_0^-(j)} a\cdot N_{j,k}\xi_k\right)$$

Local error :

$$(2) \quad u(g_j, t^{n+1}) - \mathcal{L}_j^n\left(\{u(g_k, t^n)\}_k\right) + \mathcal{L}_{bord}^n = \Delta t_n\, \epsilon_j^n$$

Global error : $e_j^n = u_j^n - u(g_j, t^n)$

$$\Rightarrow (3) \quad e_j^{n+1} - \mathcal{L}_j^n\left(\{e_k^n\}_k\right) = -\Delta t_n\, \epsilon_j^n$$

Stability

Theorem 1 :

Under CFL $\frac{\Delta t_n}{\min_j \tau_j} \leq 1$ with $\tau_j = \frac{|K_j|}{\sum_{k \in \mathcal{N}^+(j)} a \cdot N_{j,k}}$

i) $\|\mathcal{L}^n(\xi)\|_p \leq \|\xi\|_p$ where $||\xi||_p = \left(\sum_{j=1}^{N_v} |K_j||\xi_j|^p\right)^{1/p}$ and $||\xi||_\infty = \max_{1 \leq j \leq N_v} |\xi_i|$

ii) $\|e^n\|_p \leq ||e^0||_p + \sum_{i=0}^{n-1} \Delta t_i ||\epsilon^i||_p$

Lax Theorem :

Local error estimate transfers to global error estimate

Consistency

Local error : $\epsilon_j^n = G_j^n + I_j^n$ + B. C. discretization error

- Error on B.C. discretization assumed small

- Centered part :

$$G_j^n = \frac{u(g_j, t_{n+1}) - u(g_j, t_n)}{\Delta t_n} + \frac{1}{|K_j|} \sum_{k \in \mathcal{N}(j)} a \cdot N_{j,k} u(g_{j,k}, t_n) = \mathcal{O}(h)$$

- Upwind part :

$$I_j^n = \frac{1}{|K_j|} \left(\sum_{k \in \mathcal{N}^+(j)} a \cdot N_{j,k} \Big(u(g_j, t_n) - u(g_{j,k}, t_n) \Big) + \sum_{k \in \mathcal{N}_0^-(j)} a \cdot N_{j,k} \Big(u(g_k, t_n) - u(g_{j,k}, t_n) \Big) \right) = \mathcal{O}(1)$$

FVM is not consistent in the FD sense

Correction

Can we write $\epsilon_j^n = \frac{1}{\Delta t_n}\Big(\gamma^{n+1} - \mathcal{L}_j^n(\gamma^n)\Big) + \underline{\epsilon}_j^n$ with $\gamma^n \equiv \{\gamma_j^n\}_j = \mathcal{O}(h)$ and $\underline{\epsilon}_j^n = \mathcal{O}(h)$?

Geometric corrector : $\gamma_j^n = -\Gamma_j \cdot \nabla u(g_j, t^n)$ where

$\Gamma \equiv \{\Gamma_j\}_j$ depends only on the mesh and vector a and is solution of

$$\sum_{k\in\mathcal{N}^+(j)} a\cdot N_{j,k}\Gamma_j + \sum_{k\in\mathcal{N}_0^-(j)} a\cdot N_{j,k}\Gamma_k = \sum_{k\in\mathcal{N}^+(j)} a\cdot N_{j,k}(g_{j,k}-g_j) + \sum_{k\in\mathcal{N}_0^-(j)} a\cdot N_{j,k}(g_{j,k}-g_k)$$

$\Longleftrightarrow (I-B)\Gamma = \Delta$ with $\quad (B\psi)_j = \dfrac{\displaystyle\sum_{k\in\mathcal{N}_0^-(j)} a\cdot N_{j,k}\psi_k}{\displaystyle\sum_{k\in\mathcal{N}^+(j)} a\cdot N_{j,k}}$

Existence of the geometric corrector

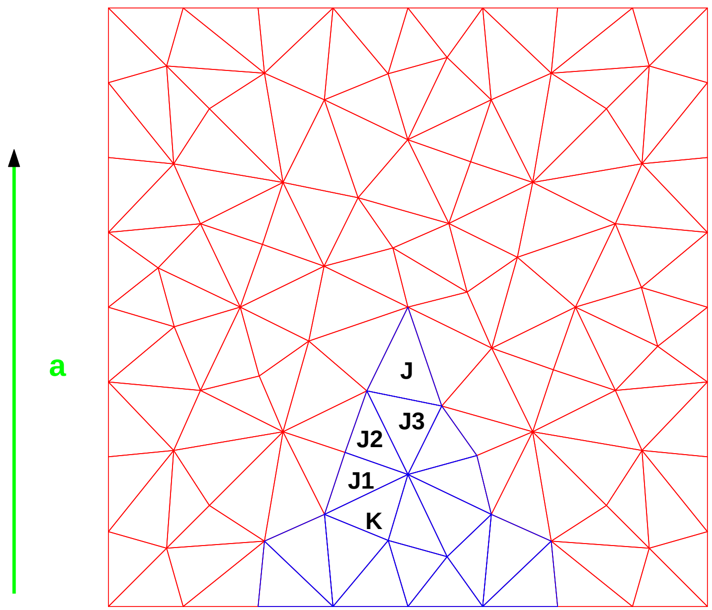
Theorem :

$$\sigma(B) \subset \{z \in \mathbf{C}\,,\ |z| < 1\}$$

$$et \qquad (I - B)^{-1} = \sum_{i=0}^{\infty} B^i$$

Dependence cone : let J a volume of control

$$\mathcal{C}(J) = \{K\,/\,\exists\, J_1, \cdots, J_n\,,\ a \cdot N_{J_1,K} < 0,\ a \cdot N_{J_2,J_1} < 0,\ \cdots\,,\ a \cdot N_{J,J_n} < 0\}$$



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Proof :

- a) $\|Bx\|_\infty \leq \|x\|_\infty$ since $\displaystyle\sum_{k \in \mathcal{N}(j)} a \cdot N_{j,k} = 0$
- b) $\forall J \in \mathcal{T}$, there is at least $K \in \mathcal{C}(J)$ sharing a face with $\partial\Omega^-$

  Input Boundary plays an important role
- c) By contradiction

  Let λ / $|\lambda| = 1$ et x / $Bx = \lambda x$.

  For K_j / $|x_j| = \max_k |x_k|$, we get $\mathcal{N}_b^-(K_j) = \emptyset$ and $\forall k \in \mathcal{N}_0^-(j)$, $|x_k| = |x_j|$.

  By induction, $\forall K \in \mathcal{C}(K_j)$, $\mathcal{N}_b^-(K) = \emptyset$ contradiction with b)

Main result

Theorem : let $\gamma_j^n = -\Gamma_j \cdot \nabla u(g_j, t^n)$ and let assume u smooth enough

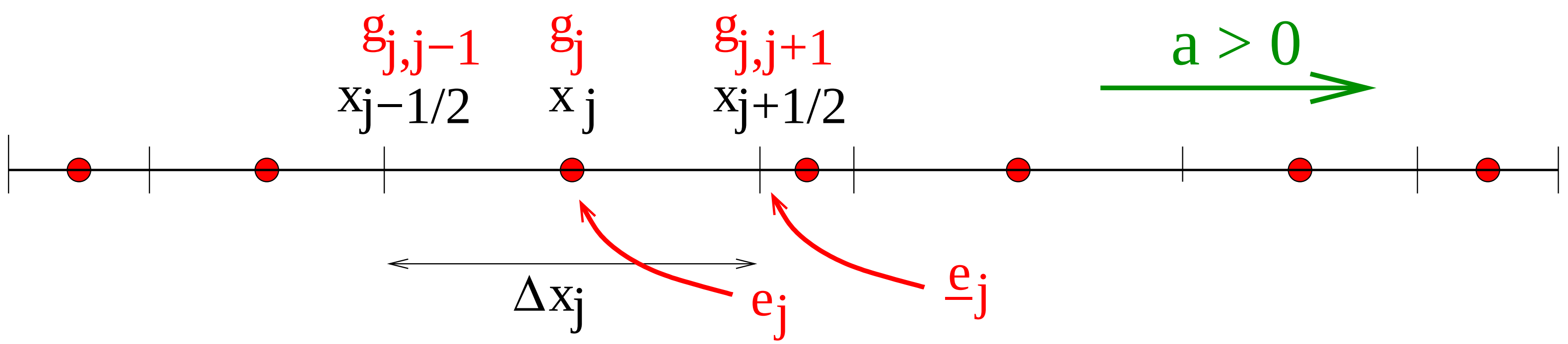
$i)$	local quasi-uniformity of the mesh
	$\frac{1}{\kappa_3}|K_k| \leq |K_j| \leq \kappa_3 |K_k|, \quad \forall h < h_0, \forall K_j \in \mathcal{T}^h, \forall k \in \mathcal{N}(j)$
$ii)$	I.C.. : $\|(u_j^0 - \varphi(g_j))_j\|_p \leq \kappa_4 h$,
$iii)$	B.C. : $|u_k^n - \psi(g_{j,k}, t_n)| \leq \kappa_5 h^2$
then	under CFL , $\forall p \in [1, +\infty], \quad (\alpha \in]0,1])$
	$\exists c_p > 0 \,/\, \|\Gamma\|_p \leq c_p h^\alpha \quad \Longrightarrow \quad \|u(\cdot, t^n) - (u_j^n)_j\|_p \leq c_p h^\alpha \quad \forall t^n \leq T$

Extension :

- Implicit scheme
- Mesh where interfaces are not hyperplane
- The center of gravity g_j can be replaced by any point at a distance less than h from g_j.

Study of the Geometric corrector (1)

Dimension 1 :



$$\frac{u_j^{n+1} - u_j^n}{\Delta t_n} + a\frac{u_j^n - u_{j-1}^n}{\Delta x_j} = 0$$

$$\begin{cases} a\,\Gamma_1 = a\,(x_{\frac{3}{2}} - x_1)\,, & j = 1 \\ a\,(\Gamma_j - \Gamma_{j-1}) = a\,(x_{j+\frac{1}{2}} - x_j) - a\,(x_{j-\frac{1}{2}} - x_{j-1})\,, & j \geq 2 \end{cases} \Longrightarrow \quad \Gamma_j = \frac{\Delta x_j}{2}$$

Remark : $\gamma_j^n = -\frac{\Delta_j}{2}\frac{\partial u(x_j, t_n)}{\partial x} \Longrightarrow \underline{e}_j^n = u_j^n - u(x_{j+1/2}, t_n) + \mathcal{O}(h^2)$

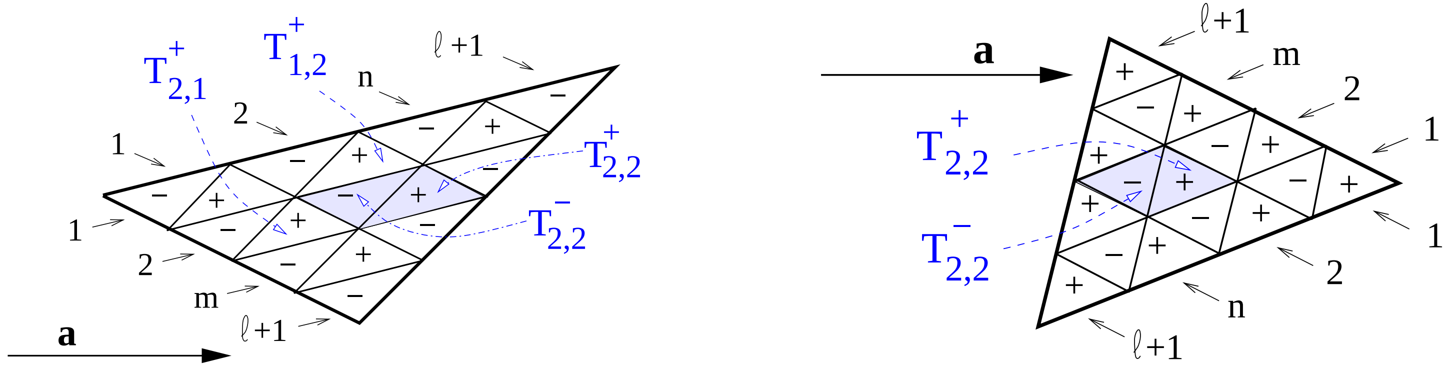
Study of the Geometric corrector (2)

Theorem :

> $-$Dimension 2
>
> $-\mathcal{T}_0$ unstructured coarse mesh of triangles or quadrangles
>
> $-\mathcal{T}_h$ obtained by uniformly refinement
>
> then $\|\Gamma\|_p \leq ch$

Proof :

a) Start with one triangle : $\sharp \mathcal{T}_0 = 1$



$$\Longrightarrow \quad |\Gamma^{\epsilon}_{m,n}| \leq \frac{C(a,T)}{\ell+1}$$

Study of the Geometric corrector (2)

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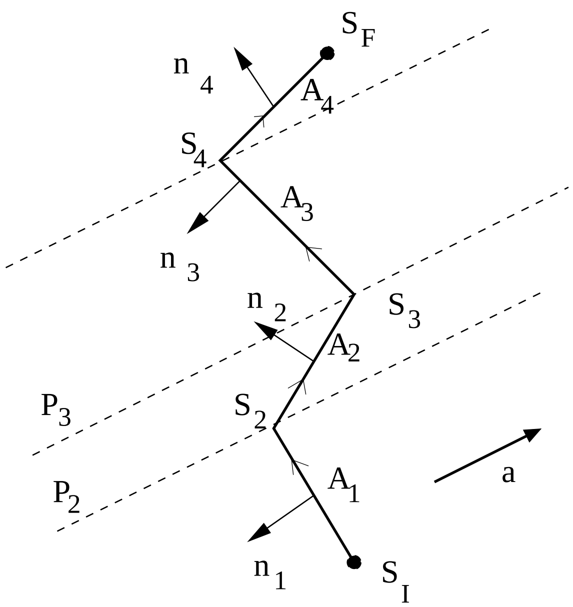
Proof :

b) By induction on the number of volumes

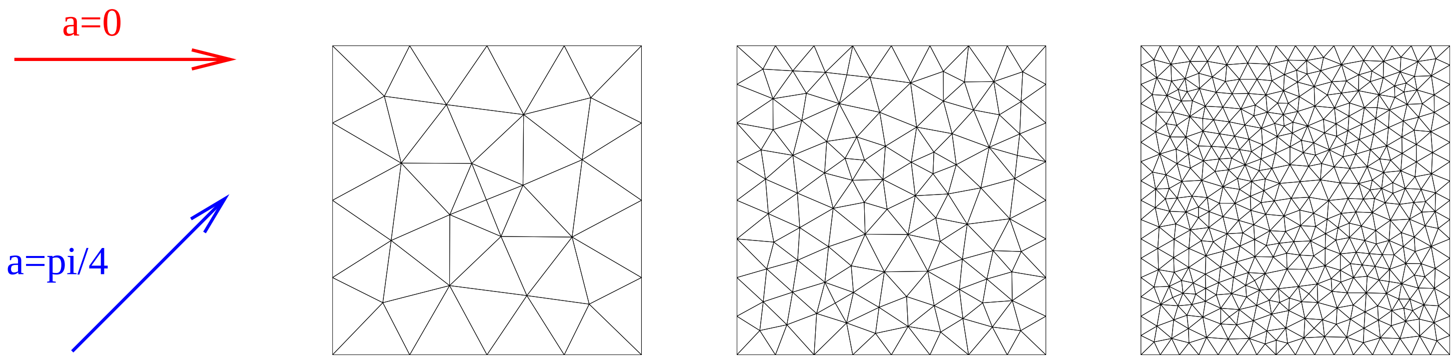


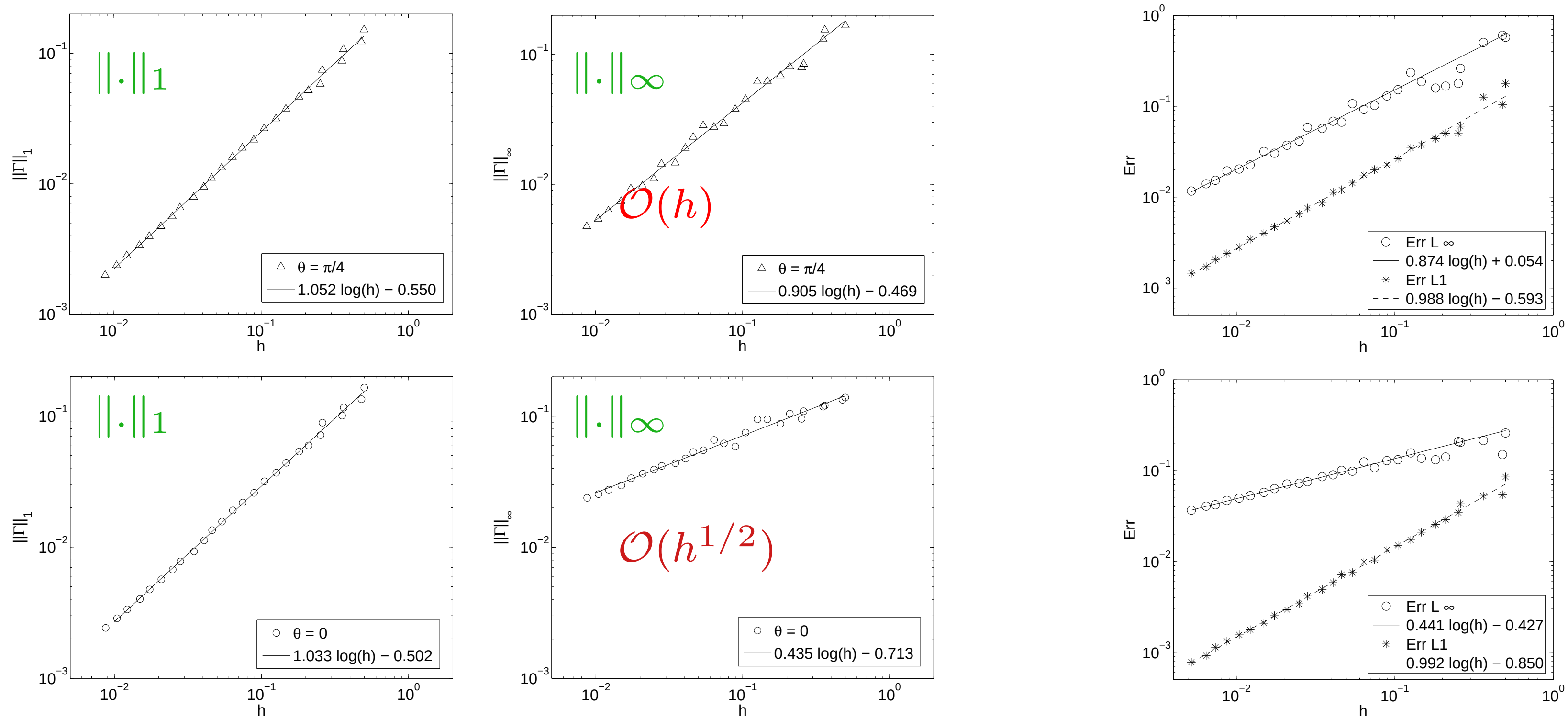
Lemma : convex volumes in 2d

$\exists$ a broken line of interfaces that are "en-lighted" on the same side which devide the domain in 2

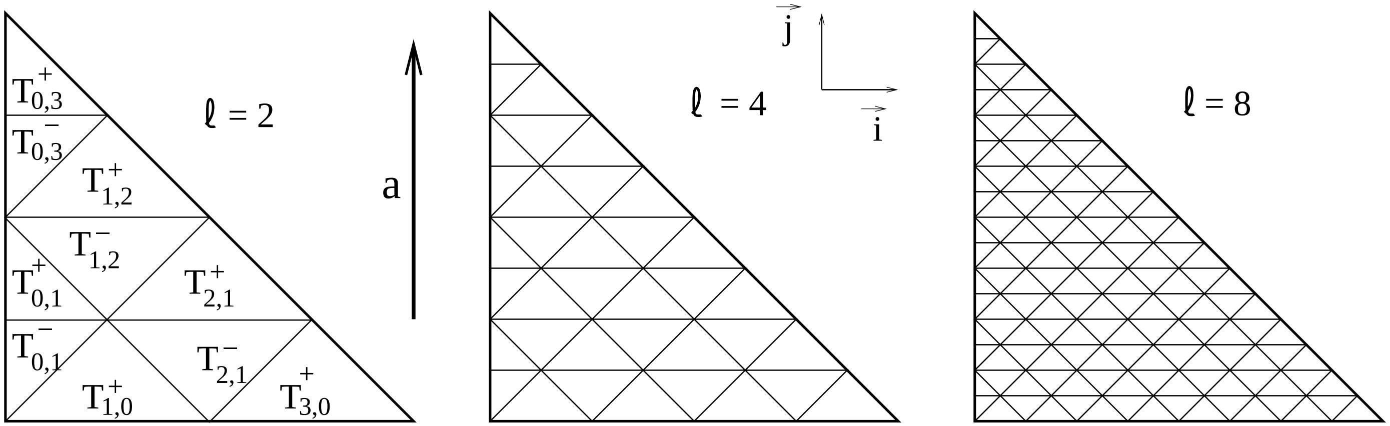
Study of the Geometric corrector (3)

Numerical test with independent meshes :

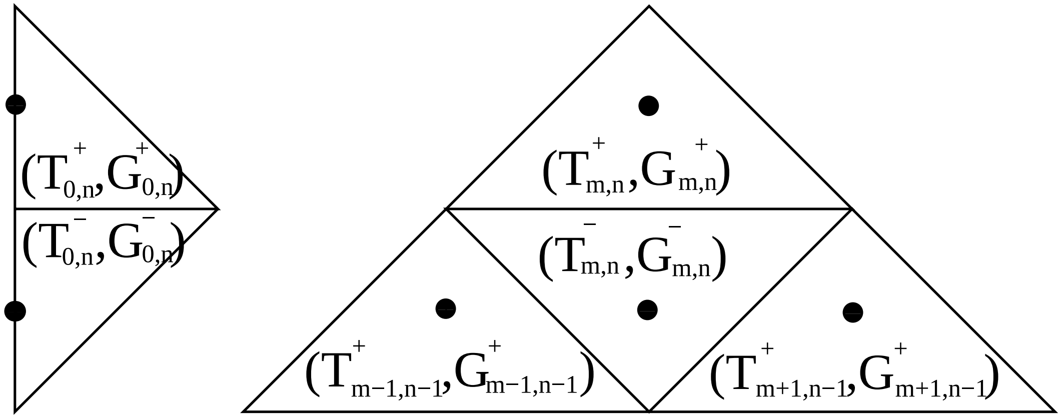




Counter Example Peterson (*Sinum 91*)



Numbering of triangles and definition of g_j :



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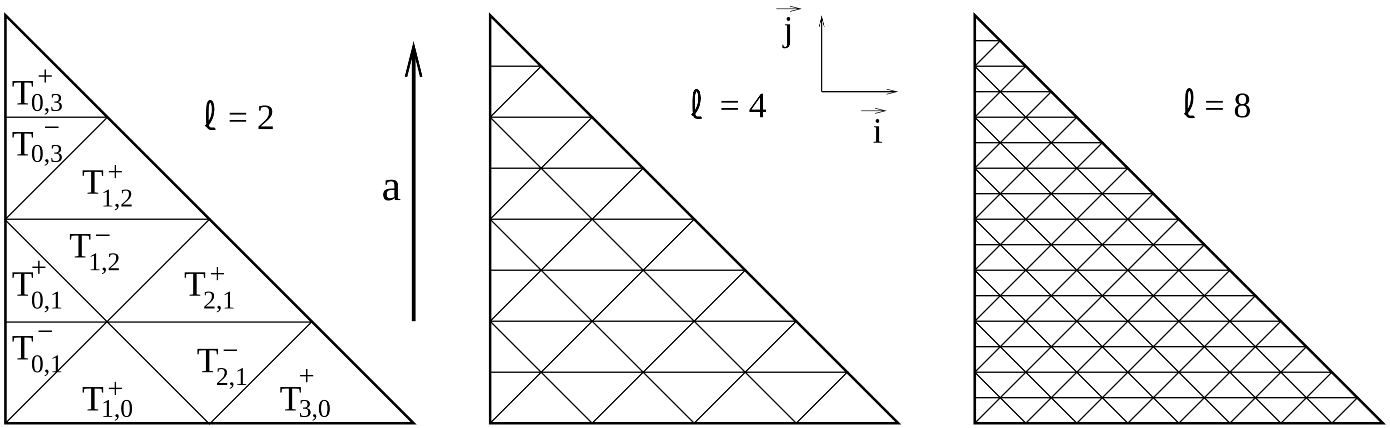
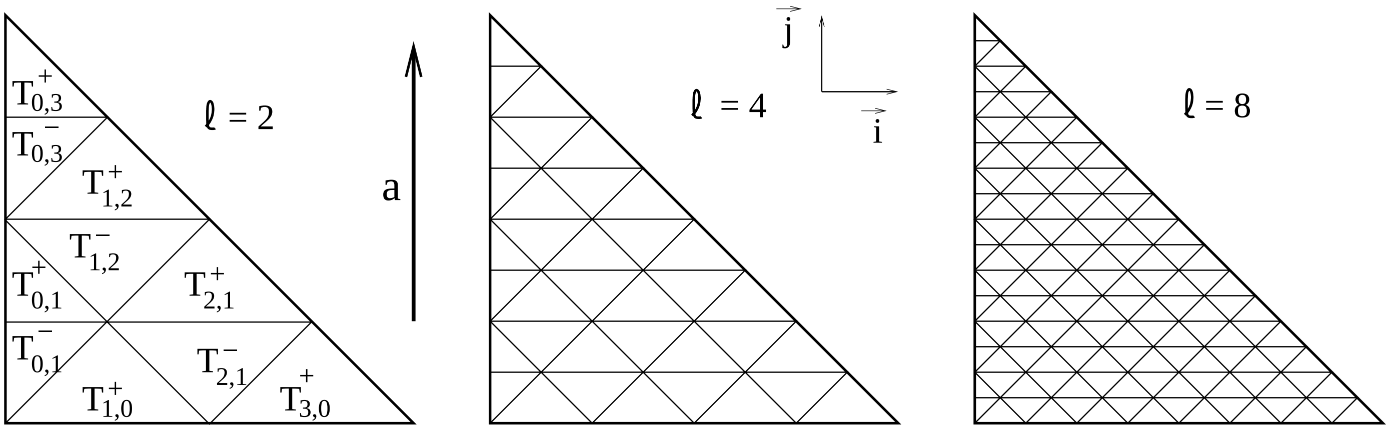


Geometric Corrector :

$$
\begin{aligned}
\mathbf{\Gamma}^+_{m,n} &= \tfrac{1}{2}(\mathbf{\Gamma}^+_{m-1,n-1} + \mathbf{\Gamma}^+_{m+1,n-1}), \quad & m \geq 1\,,\ n \geq 1 \\
\mathbf{\Gamma}^+_{0,n} &= \mathbf{\Gamma}^+_{1,n-1} + \tfrac{h}{2}\vec{i}, & n = 1, 3, \ldots, 2\ell - 1 \\
\mathbf{\Gamma}^+_{m,0} &= 0\,, & m = 1, 3, \ldots, 2\ell - 1\,.
\end{aligned}
$$

Counter Example Peterson (*Sinum 91*)

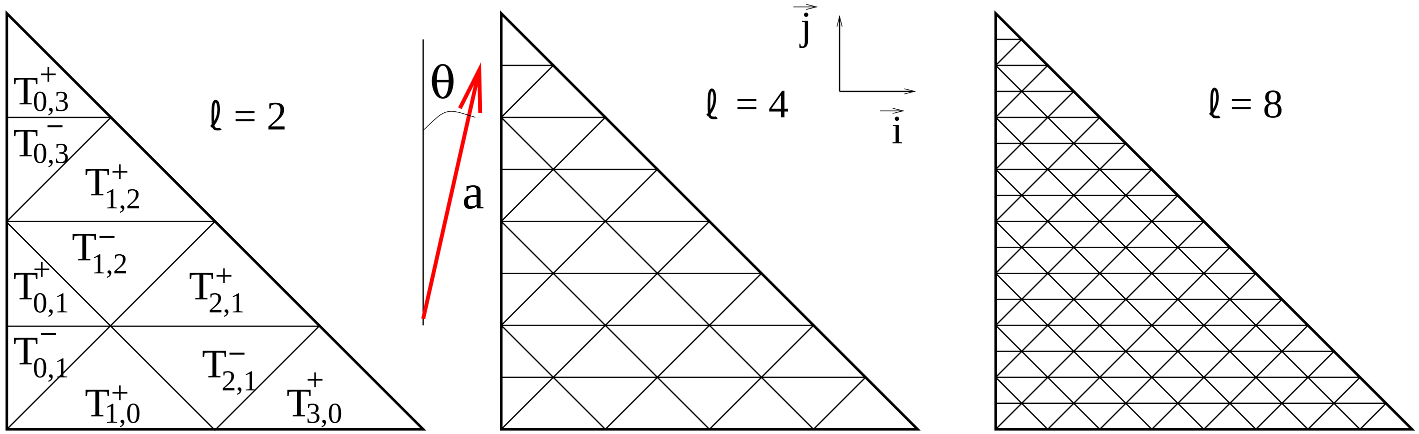


Explicit estimation equal to the analytical error estimation :

$$\mathbf{\Gamma}^+_{2p,2q+1} = \begin{cases} 0\,, & 0 \le q \le \frac{\ell}{2} - 1\,,\ q+1 \le p \le \ell - q - 1 \\ \displaystyle\sum_{k=p}^{q} \frac{1}{2^{2k+1}} \binom{2k}{k-p} h\vec{i}\,, & 0 \le q \le \ell - 1\,,\ 0 \le p \le \min(q, \ell - q - 1) \end{cases}$$

$$||\mathbf{\Gamma}^{(\ell)}||_\infty = \frac{1}{\sqrt{\pi}} h^{1/2} + \mathcal{O}(h) \quad \text{et} \quad ||\mathbf{\Gamma}^{(\ell)}||_1 = \mathcal{O}(h)$$

Counter Example Peterson: *oblique incidenc*

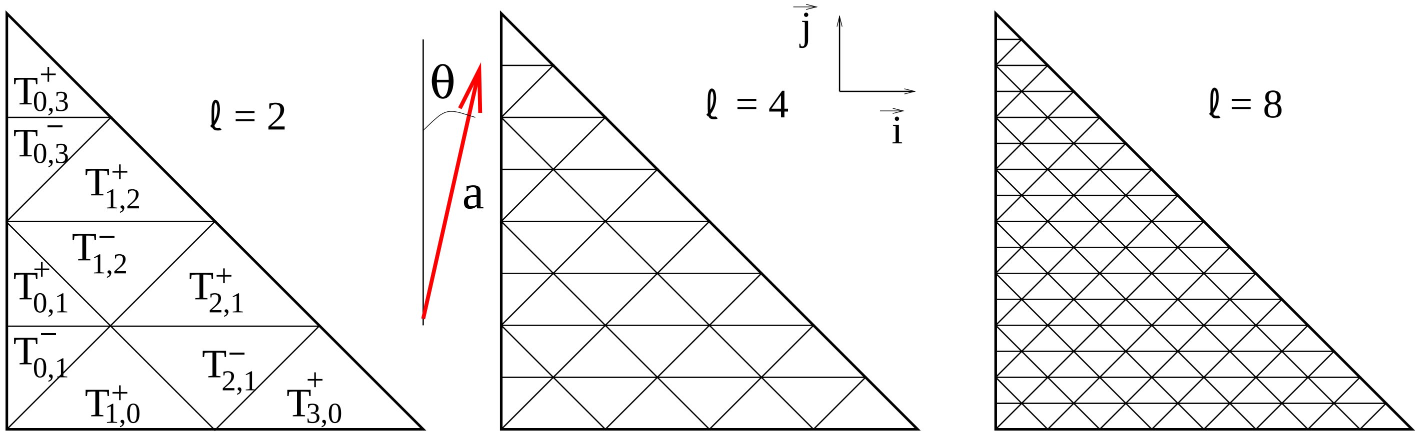


Geometric corrector :

$$\begin{aligned}
\mathbf{\Gamma}^+_{m,n} &= p\mathbf{\Gamma}^+_{m-1,n-1} + q\mathbf{\Gamma}^+_{m+1,n-1}\,, && m \neq 0\,, n \geq 1 \\
\mathbf{\Gamma}^+_{0,n} &= \tfrac{q}{p}\mathbf{\Gamma}^+_{1,n-1} + h\vec{\beta}\,, && n = 1, 3, \ldots, 2\ell - 1 \\
\mathbf{\Gamma}^+_{m,0} &= 0\,, && m = 1, 3, \ldots, 2\ell - 1\,,
\end{aligned}$$

$$p = \frac{\cos\alpha}{\cos\alpha + \sin\alpha}\,, \quad p + q = 1\,, \quad q < 1/2 < p\,, \quad \alpha = \frac{\pi}{4} - \theta\,, \quad \theta \in]0, \pi/4]\,, \quad \vec{\beta} \text{ constant}$$

Counter Example Peterson: *oblique incidenc*



Results :

$$||\mathbf{\Gamma}^{(\ell)}||_\infty = \mathcal{O}(h) \quad \text{et} \quad ||\mathbf{\Gamma}^{(\ell)}||_1 = \mathcal{O}(h)$$

A matrix formulation (square of a matrix of transition of a Markov process)

A probabilist approach : a sum of random walk

Second order scheme (1d)

1st order : $\dfrac{du_j}{dt} + a\dfrac{u_j - u_{j-1}}{\Delta x_j} = 0$

$$\Gamma_j - \Gamma_{j-1} = \frac{\Delta x_j}{2} - \frac{\Delta x_{j-1}}{2}$$

2nd order : $\dfrac{du_j}{dt} + a\dfrac{f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}}}{\Delta x_j} = 0$ with $f_{j+\frac{1}{2}} = a\left(u_j + \dfrac{\Delta x_j}{2}\dfrac{u_{j+1} - u_{j-1}}{\Delta x_{j+\frac{1}{2}} + \Delta x_{j-\frac{1}{2}}}\right)$

$$\gamma_j = -\Gamma_j \frac{\partial^2 u(x_j, t)}{\partial x^2}$$

$$\begin{cases} \Gamma'_j - \Gamma'_{j-1} = \rho_j - \rho_{j-1} \quad \text{with} \quad \rho_j = \dfrac{\Delta x_j(\Delta x_{j-1} + \Delta x_j - \Delta x_{j+1})}{8} \\ \\ \Gamma'_j = \Gamma_j + \alpha_j(\Gamma_{j+1} - \Gamma_{j-1}) \quad \text{with} \quad \alpha_j = \dfrac{\Delta x_j}{\Delta x_{j-1} + 2\Delta x_j + \Delta x_{j+1}} \end{cases}$$

$\Gamma'_j = \mathcal{O}(h^2)$ and $\|\Gamma\|_p \leq c\|\Gamma'\|_p$

Non constant vector a (1d)

A Riemann finite volume approach :

$$\frac{u_j^{n+1} - u_j^n}{\Delta t_n} + \frac{1}{\Delta x_j}\left(\Phi_{j+\frac{1}{2}}(u^n) - \Phi_{j-\frac{1}{2}}(u^n)\right) = 0$$

with $\Phi_{j+\frac{1}{2}}(u^n) = \frac{a_{j+\frac{1}{2}}(u_j^n + u_{j+1}^n)}{2} - \sigma_{j+\frac{1}{2}} \frac{a_{j+\frac{1}{2}}(u_{j+1}^n - u_j^n)}{2}$

$$= \frac{\sigma_{j+\frac{1}{2}} + 1}{2} a_{j+\frac{1}{2}} u_j^n - \frac{\sigma_{j+\frac{1}{2}} - 1}{2} a_{j+\frac{1}{2}} u_{j+1}^n$$

$$= a^+_{j+\frac{1}{2}} u_j^n - a^-_{j+\frac{1}{2}} u_{j+1}^n$$

$$u_j^{n+1} = u_j^n - \frac{\Delta t_n}{\Delta x_j}\left(a^+_{j+1/2} u_j^n - a^-_{j+1/2} u_{j+1}^n - a^+_{j-1/2} u_{j-1}^n + a^+_{j-1/2} u_j^n\right)$$

with $a_{j+1/2} \equiv a(x_{j+1/2}) = a^+_{j+1/2} - a^-_{j+1/2}$ and $\sigma_{j+\frac{1}{2}} = \text{sign}(a_{j+\frac{1}{2}})$

Analysis

Stability : Under $\frac{\Delta t_n}{\Delta x_j}(a^+_{j+\frac{1}{2}} + a^-_{j-\frac{1}{2}}) \le 1$ then $\|\mathcal{L}^n(\xi)\|_p \le (1 + \Delta t_n \|a_x\|_\infty)\ \|\xi\|_p$

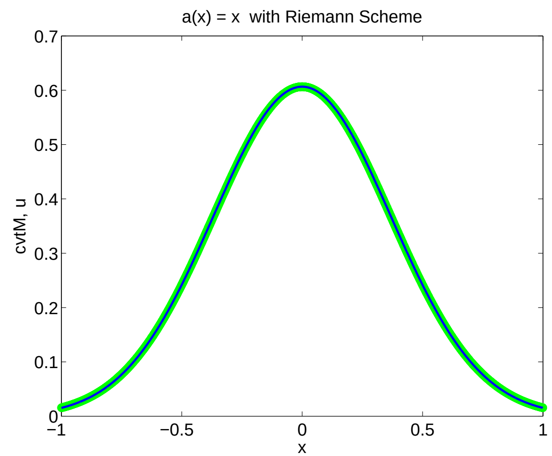
Consistency : $\epsilon^n_j = G^n_j + I^n_j$

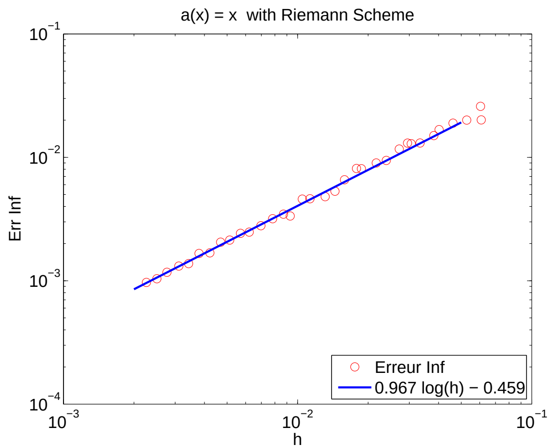
$$G^n_j = \frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{\Delta t_n} + \frac{f(x_{j+\frac{1}{2}}, t_n) - f(x_{j-\frac{1}{2}}, t_n)}{\Delta x_j} = \mathcal{O}(\Delta t) + \mathcal{O}(h)$$

$$I^n_j = \frac{\Phi_{j+\frac{1}{2}}(U^n) - \Phi_{j-\frac{1}{2}}(U^n) + f(x_{j-\frac{1}{2}}, t_n) - f(x_{j+\frac{1}{2}}, t_n)}{\Delta x_j}.$$

Corrector : $\gamma^n_j = u(x_j, t_n) - u(x_j + \sigma_j \frac{\Delta x_j}{2}, t_n) = -\sigma_j \frac{\Delta x_j}{2} \frac{\partial u(x_j, t^n)}{\partial x} + \mathcal{O}(h^2)$

$\sigma_j = \text{sign}(a(x_j))$





For $\gamma^n_j = -\Gamma_j \frac{\partial u(x_j, t^n)}{\partial x}$, the equation satisfied by Γ_j has the same form as constant a but is no longer 0 but $\mathcal{O}(h^2)$ near a change of sign of a

Non constant vector $a(x)$ (1d)

A flux scheme :

$$\frac{u_j^{n+1}-u_j^n}{\Delta t_n}+\frac{1}{\Delta x_j}\left(\Psi_{j+\frac12}(u^n)-\Psi_{j-\frac12}(u^n)\right)=0$$

with
$$\begin{aligned}\Psi_{j+\frac12}(u^n) &= \frac{a_j u_j^n + a_{j+1}u_{j+1}^n}{2} - \sigma_{j+\frac12}\frac{a_{j+1}u_{j+1}^n - a_j u_j^n}{2}\\ &= \frac{\sigma_{j+\frac12}+1}{2}a_j u_j^n - \frac{\sigma_{j+\frac12}-1}{2}a_{j+1}u_{j+1}^n\end{aligned}$$

$$\left\{u_j^{n+1}=u_j^n-\frac{\Delta t_n}{\Delta x_j}\left(\frac{1-\sigma_{j+1/2}}{2}(a_{j+1}u_{j+1}^n-a_j u_j^n)-\frac{1+\sigma_{j-1/2}}{2}(a_{j-1}u_{j-1}^n-a_j u_j^n)\right)\right\}\times a_j$$

with $\sigma_{j+1/2}=\text{sign}(a(x_{j+1/2}))$

In term of fluxes : $\varphi_j^n = a_j u_j^n$ satisfy

$$\frac{\varphi_j^{n+1}-\varphi_j^n}{\Delta t_n}+\frac{a_j}{\Delta x_j}\left(\frac{1-\sigma_{j+\frac12}}{2}(\varphi_{j+1}^n-\varphi_j^n)+\frac{1+\sigma_{j-\frac12}}{2}(\varphi_j^n-\varphi_{j-1}^n)\right)=0$$

Analysis

Stability on the flux : under $\dfrac{\Delta t_n}{\Delta x_j} a_j \dfrac{\sigma_{j+\frac{1}{2}} + \sigma_{j-\frac{1}{2}}}{2} \leq 1$ then

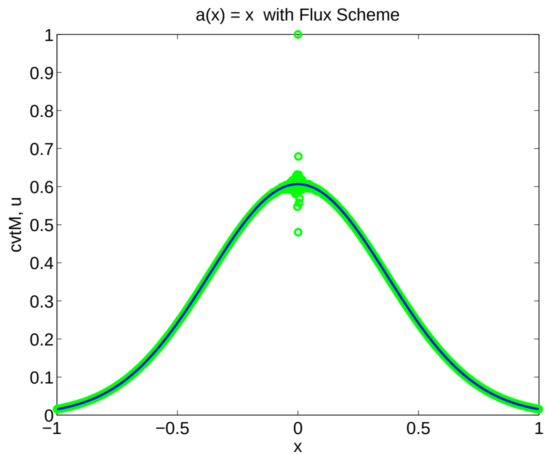
$$\|\mathcal{L}_F^n(\xi)\|_p \leq (1 + c\Delta t_n \|a_x\|_\infty)\ \|\xi\|_p$$

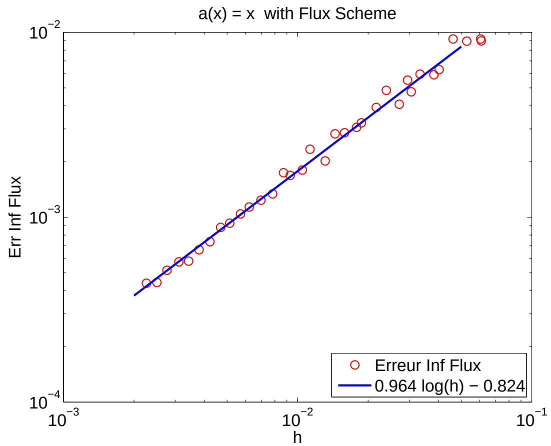
Consistency : $\epsilon_j^n = G_j^n + a_j I_j^n$

$$G_j^n = \frac{f(x_j, t_{n+1}) - f(x_j, t_n)}{\Delta t_n} + a_j \frac{f(x_{j+\frac{1}{2}}, t_n) - f(x_{j-\frac{1}{2}}, t_n)}{\Delta x_j} = \mathcal{O}(\Delta t) + \mathcal{O}(h)$$

$$I_j^n = \frac{\Psi_{j+\frac{1}{2}}(F^n) - \Psi_{j-\frac{1}{2}}(F^n) + f(x_{j-\frac{1}{2}}, t_n) - f(x_{j+\frac{1}{2}}, t_n)}{\Delta x_j}.$$

Correction on flux : $\gamma_j^n = f(x_j, t_n) - f(x_j + \sigma_j \dfrac{\Delta x_j}{2}, t_n) = -\sigma_j \dfrac{\Delta x_j}{2} \dfrac{\partial f(x_j, t^n)}{\partial x} + \mathcal{O}(h^2)$





Error estimate on the flux

The nonlinear case

Nonlinear problem : $$\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u(x,t))}{\partial x} = 0$$

Scheme : $$\frac{u_j^{n+1} - u_j^n}{\Delta t_n} + \frac{1}{\Delta x_j}(\Phi^n_{j+\frac{1}{2}}(u^n) - \Phi^n_{j-\frac{1}{2}}(u^n)) = 0$$

$$\Phi^n_{j+\frac{1}{2}}(u^n) = \frac{f(u^n_{j+1}) + f(u^n_j)}{2} - \sigma^n_{j+\frac{1}{2}} \frac{f(u^n_{j+1}) - f(u^n_j)}{2}$$

with $\sigma^n_{j+\frac{1}{2}} = \text{sign}(s^n_{j+\frac{1}{2}})$ where $s^n_{j+\frac{1}{2}} = \begin{cases} \dfrac{f(u^n_{j+1}) - f(u^n_j)}{u^n_{j+1} - u^n_j} & \text{if } u^n_{j+1} \neq u^n_j \\ f'(u^n_j) & \text{if } u^n_{j+1} = u^n_j \end{cases}$

Error analysis : $e_j^n = u_j^n - u(x_j + \delta_j \dfrac{\Delta x_j}{2}, t_n)$ with $\delta_j = \text{sign}(f'(u(x_j,\cdot))$

Under $\frac{\Delta t_n}{\Delta x_j}\left(\frac{\sigma^n_{j+\frac{1}{2}} - 1}{2} s^n_{j+\frac{1}{2}} + \frac{\sigma^n_{j-\frac{1}{2}} + 1}{2} s^n_{j-\frac{1}{2}}\right) \leq 1$

$$||(e_j^n)_{j=1}^N||_\infty \leq C'_\infty h$$

$||(e_j^n)_{j=1}^N||_1 \leq C'_1 h$ (for global quasi-uniform meshes)

Conclusion

- Geometric corrector helps in finding optimal error estimates
- Bounded domains and B.C. are taken into account
- Lost of order of convergence for non smooth solution is not due to the mesh

Open question and Perspective :

- Probabilist approach to study the corrector
- Extension in 2d for non constant vector and 2nd order scheme
- Study of the norm of the corrector is open in 2d