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SURFACE MESH QUALITY EVALUATION

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SUMMARY

This paper proposes a method to evaluate the size quality as well as the shape quality of constrained surface meshes, the constraint being either a given metric or the geometric metric associated with the surface geometry. In the context of numerical simulations, the metric specifications are those related to the finite element method. The proposed measures allow to validate the surface meshes within a general mesh adaption scheme, the metric map being usually provided via an *a posteriori* error estimate. Several examples of surface meshes are proposed to illustrate the relevance of the approach. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: surface mesh; mesh quality; polyhedral approximation

1. INTRODUCTION

Surface triangulations are frequently employed in a wide variety of scientific applications (e.g. computer graphics, numerical simulations, etc., ...). In many applications however, surface triangulations must conform to specific properties, for instance those related to the geometry of the surface they represent or to the behavior of the physical phenomenon in numerical simulations. Therefore, a question arising frequently is to decide whether a given triangulation conforms or not to the desired properties. In other words, this question is '*how good is the surface triangulation*?'.

Surface triangulations can be classified into two classes according to the following definitions [1].

Definition 1.1. In a $\mathscr{G}$-mesh the distance between a mesh triangle and the underlying surface is bounded.

Commonly, a surface triangulation is simply a triangulation such that all the vertices belong to the surface and thus is not necessarily a faithful approximation of the true surface it represents. Sometimes, the requirement is to find an optimal piecewise planar approximation of the original surface, so that the maximal distance between the original surface and the element of the triangulation does not exceed a given tolerance threshold.

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Definition 1.2. A $\mathscr{H}$-mesh conforms to size specifications as well as to element shape quality requirements.

For finite element methods, the shape quality of the elements of the triangulation† is important, as it may affect the convergence of the computational scheme [3]. The element sizes must also conform to the desired accuracy of the numerical solutions (usually provided by an *a posteriori* error estimate).

Therefore, for a surface triangulation, two types of constraints have to be considered, the geometric conformity (for which the desired sizes corresponds to the principal radii of curvature) and the metric conformity (with respect to any arbitrary map, for instance one provided by an a posteriori error estimate). The purpose of this paper is to propose a method to decide whether a given triangulation $\mathscr{T}$ is a $\mathscr{G}$-mesh or a $\mathscr{H}$-mesh.

1.1. Outline

Preliminary definitions related to the definition of surface triangulations, metrics and element shape quality measures are reviewed in Section 2. Section 3, the definition of metric conformity related to surface meshes is provided and especially, the geometric conformity is defined as a particular case of the metric conformity. The concept of a finite element surface mesh is also introduced in Section 3. In Section 4, several geometric criteria are recalled to evaluate isotropic geometric triangulations and a global evaluation scheme (algorithm) is proposed to summarize this approach. Finally, extensions and future work are briefly mentioned in a conclusion section.

2. PRELIMINARY DEFINITIONS

Let us consider a surface triangulation $\mathscr{T}$ of the boundary Γ of a domain Ω of R^3, supplied as a list of entities (points co-ordinates and list of triangles).

Definition 2.1. A surface triangulation $\mathscr{T}$ is a set of simplices $\{K_i\}_{i=1}^{|\mathscr{T}|}$, whose vertices $\mathscr{V} = \{P_i\}_{i=1}^{|\mathscr{V}|}$ are members of the boundary Γ, where $|s|$ denotes the number of elements of the set s.

Definition 2.2. A triangulation $\mathscr{T}$ of Γ is a valid mesh if and only if:

(1) $\mathscr{T}$ is a conforming triangulation of Γ [4],
(2) $\mathscr{T}$ is geometrically conforming with respect to the tangent planes.

The first condition expresses the classical finite element requirement of conformity and requires the intersection of two triangles to be restricted to the empty set, a vertex or an edge. On the other hand, the second condition establishes at each vertex the conformity of the orthogonal projection of the elements sharing this vertex onto its tangent plane (for a geometrically non-conforming vertex, the trace of the boundary edges of its orthogonal projection is a non-simple polygonal segment, cf. Figure 1).

The notion of a metric play a central part in the evaluation of surface meshes. Therefore, we recall some definitions and results related to quadratic forms and metric operations.

† In this context, a surface mesh is intended to be the boundary description of a domain used in a three-dimensional finite element analysis

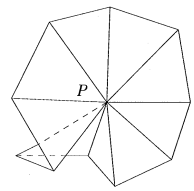

Figure 1. Vertex P is a non-conforming vertex

2.1. Matrix expression of a quadratic form

Let us consider a symmetric definite bilinear form $f(x,y)$ from $R^3 \times R^3$ with value in R and let e^k $(k=1,3)$ be the canonical basis of R^3; then $x \in R^3$ can be expressed as

$$x = \sum_k x_k e^k \tag{1}$$

thus leading to the following expression:

$$f(x,y) = f\left(\sum_k x_k e^k, \sum_k y_k e^k\right) = \sum_k \sum_l x_k y_l f(e^k, e^l) \tag{2}$$

which can be also written in the matrix form as $f(x,y) = {}^t x \mathcal{M} y$, where $m_{ij} = f(e^i, e^j)$ are the coefficients of the symmetric (invertible) matrix $\mathcal{M}$.

If the form f is the dot product function, the quadratic form $q(x) = f(x,x)$ associated with f can be written as the square of the norm:

$$q(x) = f(x,x) = \langle x,x \rangle_{\mathcal{M}} = \sum_k x^k x^k = \|x\|^2 \tag{3}$$

which yields

$$q(x) = {}^t x \mathcal{M} x \tag{4}$$

where $\mathcal{M}$ is a 3×3 symmetric positive-definite matrix.

2.2. Length of a segment

The length of a segment AB (i.e. the distance between point A and point B) can be written simply as

$$\|AB\|_{\mathcal{M}} = \sqrt{\langle AB, AB \rangle_{\mathcal{M}}} = \sqrt{{}^t AB \mathcal{M} AB} \tag{5}$$

Notice that the function $x \rightarrow \sqrt{q(x)}$ (and $\mathcal{M} = \mathrm{Id}$) defines the Euclidean norm used to compute the edge lengths. In the general case (i.e. non-Euclidean case), the aim is to compute the length of a segment for arbitrary metrics.

Given a basis of three unit vectors u^k, $k=1,3$, and 3 positive real values λ_k and again a metric $\mathcal{M}$, we would like to check whether $x \in R^3$ conforms to all pairs of (λ_k, u^k), which is equivalent to check if $\|x\|_{\mathcal{M}} = 1$. For instance, if segments of length h_k in direction u^k are expected,

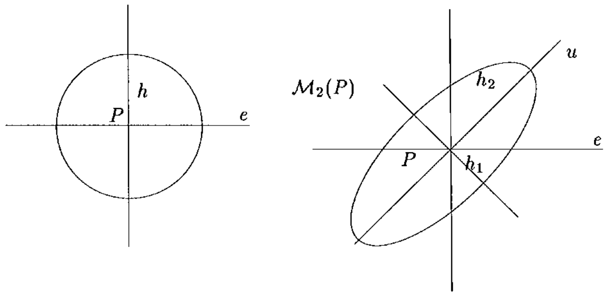

Figure 2. Geometric interpretation of a metric. Left: an isotropic metric corresponds to a circle, right: an anisotropic metric corresponds to an ellipse aligned with any arbitrary orthonormed vectors)

let us consider Λ the 3×3 diagonal matrix whose coefficients are $\lambda_k = h_k^{-1}$. A transformation $\mathcal{T}$ can be defined as

$$\mathcal{T}(u^k) = e^k \quad \text{and similarly} \quad \mathcal{T}^{-1}(e^k) = u^k, \quad 1 \leqslant k \leqslant 3 \tag{6}$$

where e^k are the vectors of the canonical basis and such that the matrix $\mathcal{M}$ is introduced as $\mathcal{M} = {}^t\mathcal{T}\Lambda\Lambda\mathcal{T}$ and

$$\|x\| = \sqrt{{}^t x \mathcal{M} x} \tag{7}$$

For the sake of simplicity, let us assume that segments of length h are desired in any direction (isotropic case). Then, the following relationship holds:

$$\|PX\|_{\mathcal{M}} = 1 \quad \Leftrightarrow \quad \|PX\| = h \tag{8}$$

2.3. Unit length

More precisely, the metric $\mathcal{M}$ is used to characterize the optimal (i.e. unit) length of any edge PX sharing P via the relationship

$$\|PX\|_{\mathcal{M}} = \sqrt{{}^t PX \mathcal{M} PX} = \sqrt{{}^t PX {}^t\mathcal{T}\Lambda^2\mathcal{T} PX} \tag{9}$$

Then, the geometric locus of all points X satisfying this equation is an ellipsoid, denoted as $\overline{\mathcal{M}(P)}$, centered at P, of radii h_k and aligned with the u^k (cf. Figure 2 in two dimensions).

2.4. Length of a curve

Let us consider a segment AB with the parameterization $\gamma(t) = AB = (A + tAB)_{0 \leqslant t \leqslant 1}$; then

$$\ell_{AB} = \int_0^1 \sqrt{{}^t AB\, \mathcal{M}(A + tAB) AB}\, \mathrm{d}t = \sqrt{\langle AB, AB \rangle_{\mathcal{M}}} \tag{10}$$

For instance, in the case of an isotropic metric, the aim is to define segments of length h in any direction (the locus of the points at distance h from P is a sphere of radius h, centered at P).

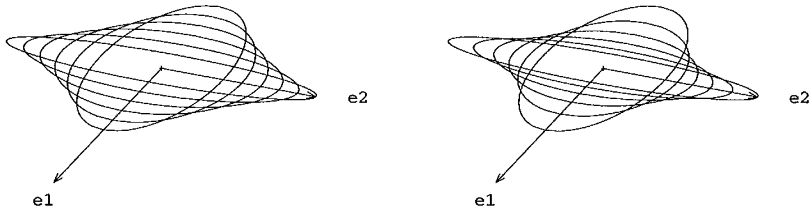

Figure 3. Metric interpolation. Left: linear interpolation, right: geometric interpolation

Hence, the metric $\mathcal{M}$ is defined as

$$\|AB\|_{\mathcal{M}} = \sqrt{{}^tAB\mathcal{M}AB} = \sqrt{{}^tAB\frac{\mathrm{Id}}{h^2}AB} = \frac{\|AB\|}{h} \tag{11}$$

2.5. Metric interpolation

Let $\mathcal{M}_P$ and $\mathcal{M}_Q$ be two metrics associated with two points P and Q. The aim is to define a continuous monotonous metric $\mathcal{M}(t)$ defined along the segment $PQ = (P + tPQ)_{0\leqslant t\leqslant 1}$ and such that $\mathcal{M}(0) = \mathcal{M}_P$ and $\mathcal{M}(1) = \mathcal{M}_Q$. Such a metric can be defined in two steps:

1. write $\mathcal{M}_P$ and $\mathcal{M}_Q$ in a diagonal form (using the simultaneous reduction form),
2. define $\mathcal{M}(t)$ using an interpolation function (linear, geometric, etc.)

Figure 3 shows an example of metric interpolation.

2.6. Metric intersection

Let us consider a point P, at which two metrics $\mathcal{M}_1$ and $\mathcal{M}_2$ are supplied. We consider the problem of finding a single metric $\mathcal{M}$ that preserves the nature of $\mathcal{M}_1$ and $\mathcal{M}_2$. A desired solution can be to consider the metric corresponding to the intersection of the two ellipsoids related to the metrics. The intersection may be forced to result in an ellipsoid (as it is usually not the case). Different solutions can be envisaged, for instance to consider the largest ellipsoid contained in the intersection region (cf. Figure 4, left) or an ellipsoid that preserves the directions of any of the initial metrics (cf. Figure 4, right).

2.7. Shape quality of a planar triangle

It is well-known that the accuracy of the finite element computations is mainly related to the shape quality of the mesh elements [3].

Definition 2.3. The shape quality $\mathcal{Q}_K$ of a two-dimensional triangle $K = P_1P_2P_3$ is given by

$$\mathcal{Q}_K = \min_{1\leqslant i\leqslant 3} Q_K^i \tag{12}$$

where Q_K^i is the shape quality of the triangle K computed in the metric $\mathcal{M}_i$ specified at vertex P_i of K.

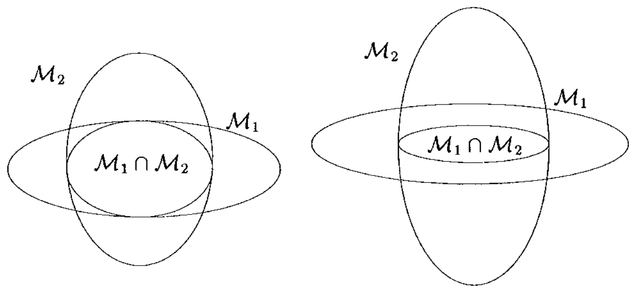

Figure 4. Metric intersection. Left: using the simultaneous reduction of $\mathcal{M}_1$ and $\mathcal{M}_2$, right: preserving the direction of $\mathcal{M}_1$

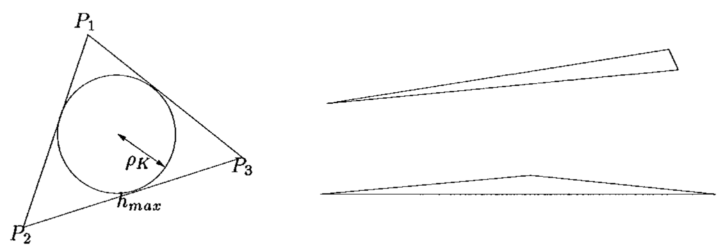

Figure 5. Triangle aspect ratio (isotropic case): well-shaped (left) and degenerated triangles (right)

Definition 2.4. The shape quality Q_K^i of K computed in the metric $\mathcal{M}_i$ can be expressed as

$$Q_K^i = \alpha \frac{|\sqrt{\mathrm{Det}(\mathcal{M}_i)}\mathrm{Det}(\overrightarrow{P_1P_2}, \overrightarrow{P_1P_3})|}{\max_{1\leqslant j<k\leqslant 3}\sqrt{{}^t\overrightarrow{P_jP_k}\mathcal{M}_i\overrightarrow{P_jP_k}}\sqrt{\sum_{1\leqslant j<k\leqslant 3}{}^t\overrightarrow{P_jP_k}\mathcal{M}_i\overrightarrow{P_jP_k}}} \tag{13}$$

where $\alpha = 2\sqrt{3}$ is a normalization coefficient so that $Q_K^i = 1$ for an equilateral triangle (in the metric $\mathcal{M}_i$, cf. [5]).

The shape quality varies in the interval $[0, 1]$, where 0 (resp. 1) is associated with a flat triangle (resp. equilateral triangle) and is capable of discriminating well-shaped from degenerated elements (cf. Figure 5). Moreover, the quality $\mathcal{Q}_{\mathcal{T}}$ and average quality $\overline{\mathcal{Q}_{\mathcal{T}}}$ of the surface triangulation $\mathcal{T}$ are related to the quality of the mesh elements and are defined as

$$\mathcal{Q}_{\mathcal{T}} = \min_{K\in\mathcal{T}} \mathcal{Q}_K, \quad \overline{\mathcal{Q}_{\mathcal{T}}} = \frac{1}{|\mathcal{T}|}\sum_{K\in\mathcal{T}} Q_K \tag{14}$$

Remark 2.1. In the isotropic case, the shape quality $\mathcal{Q}_K$ is reduced to the classical shape quality measure [5]:

$$\mathcal{Q}_K = \alpha\frac{\rho_K}{h_{\max}} = \alpha\frac{S_K}{h_{\max}\, p_K} \tag{15}$$

where $h_{\max}$ represents the length of the longest edge, ρ_K is the inradius of K, S_K is the surface area of K and p_K is the half-perimeter of K.

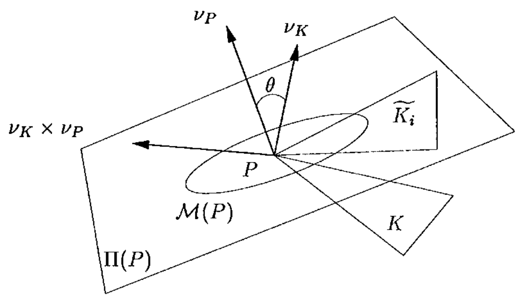

Figure 6. Shape quality of a surface triangle measured in the tangent plane $\Pi(P)$ associated with vertex P ($\mathcal{M}(P)$ denotes the metric at P)

Indeed, by considering that all the matrices $\mathcal{M}_i$ are equal to Id, the identity matrix in R^2, it yields

$$\sqrt{\mathrm{Det}(\mathcal{M}_i)}\mathrm{Det}(\overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}) = \mathrm{Det}(\overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}) = 2S_K$$

$$\sum_{1 \leqslant j < k \leqslant 3} \sqrt{{}^t\overrightarrow{P_jP_k}\mathcal{M}_i\overrightarrow{P_jP_k}} = \sum_{1 \leqslant j < k \leqslant 3} \|\overrightarrow{P_jP_k}\| = 2p_K \tag{16}$$

$$\max_{1 \leqslant i \leqslant 3} \sqrt{{}^t\overrightarrow{P_jP_k}\mathcal{M}_i\overrightarrow{P_jP_k}} = \max_{1 \leqslant i \leqslant 3} \|\overrightarrow{P_jP_k}\| = h_{\max}$$

2.8. Shape quality of a surface triangle

The shape quality measure $\mathcal{Q}_K$ can be extended to surface triangles as well. The metric $\mathcal{M}_i$ at a vertex P_i is then defined in the associated tangent plane $\Pi(P)$ and the expression of Q_K^i is modified accordingly.

Let us consider the vertex P_i of triangle K; let v_K be a normal vector to the plane supporting K and let v_{P_i} be a normal vector to the tangent plane $\Pi(P_i)$. The angle between the two vectors v_K and v_{P_i} is denoted by θ. Let $\widetilde{K}_i$ be the image of K by the rotation around the supporting axis of the vector $v_K \times v_{P_i}$, θ being the angle of rotation. The triangle $\widetilde{K}_i$ belongs to the tangent plane $\Pi(P_i)$ and its shape quality can then be measured with respect to the metric associated with P_i, defined in this plane (cf. Figure 6).

Definition 2.5. The shape quality $\mathcal{Q}_K$ of a surface triangle K is defined as

$$\mathcal{Q}_K = \min_{1 \leqslant i \leqslant 3} Q_{\widetilde{K}_i}^i \tag{17}$$

3. METRIC CONFORMITY

Let us consider a surface triangulation $\mathcal{T}$ and let $\mathcal{M}_3(\mathcal{V})$ be a three-dimensional metric map associated with the set of vertices $\mathcal{V}$. As mentioned in the previous section, the metric $\mathcal{M}_3(P)$ at vertex P of $\mathcal{T}$ is represented by a symmetric positive-definite matrix. By interpolating the discrete

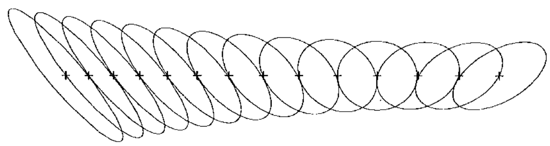

Figure 7. Geometric interpolation along a segment AB

metric map $\mathcal{M}_3(\mathcal{V})$ on $\mathcal{T}$, a continuous map $\mathcal{M}_3^*(\mathcal{V})$ is obtained in R^3. Thus, a Riemannian structure on R^3 is defined using the continuous field of metric $\mathcal{M}_3^*(\mathcal{V})$.

The problem we face is to decide whether or not the given surface triangulation $\mathcal{T}$ conforms to the map $\mathcal{M}_3(\mathcal{V})$.

Definition 3.1. The triangulation $\mathcal{T}$ conforms to the map $\mathcal{M}_3(\mathcal{V})$, without any geometric requirement (no edge is considered traced onto the surface), if and only if

$$\forall AB \in \mathcal{T}, \quad \frac{1}{\sqrt{2}} \leqslant \ell_{AB} \leqslant \sqrt{2} \tag{18}$$

where AB is an edge and ℓ_{AB} is the length of AB (according to Definition 10) with respect to the given metric $\mathcal{M}_3^*(\mathcal{V})$.

3.1. Isotropic metric

If the size specifications are constant in any direction (isotropic case), equation (9) leads to

$$\|\overrightarrow{AB}\|_{\mathcal{M}_3} = 1 \;\Leftrightarrow\; \|\overrightarrow{AB}\| = h_A \tag{19}$$

where h_A is a real value representing the desired element size at vertex A. Furthermore, if Id denotes the identity matrix in R^3, the metric at a vertex P can be expressed as

$$\mathcal{M}_3(P) = \frac{\mathrm{Id}}{h_P^2} \tag{20}$$

Considering a monotonic size interpolation function $H_{AB}(t)$ such that $H_{AB}(0) = h_A$ and $H_{AB}(1) = h_B$, the metric $\mathcal{M}_3^*(A + t\overrightarrow{AB})_{0 \leqslant t \leqslant 1}$ at the point $A + t\overrightarrow{AB}$ along the edge AB can be defined as

$$\mathcal{M}_3^*(A + t\overrightarrow{AB}) = \frac{\mathrm{Id}}{H_{AB}(t)} \tag{21}$$

Hence, using equation (10), the edge length formula yields the following simplified expression:

$$\ell_{AB} = \|\overrightarrow{AB}\| \int_0^1 \frac{1}{H_{AB}(t)}\,\mathrm{d}t \tag{22}$$

Remark 3.1. Equation (22) can be computed exactly, if the function $H_{AB}(t)$ is explicitly known.

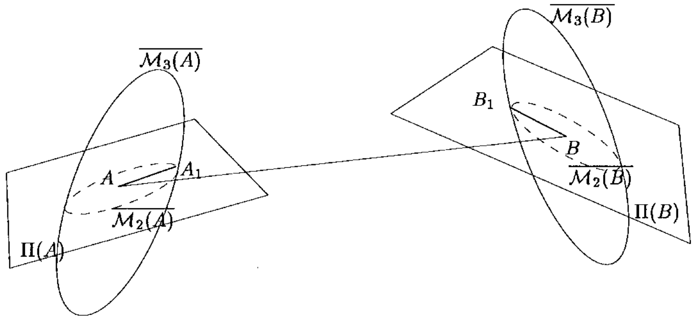

Figure 8. Definition of the metric $\mathcal{M}_2$ in the tangent planes $\Pi(A)$ and $\Pi(B)$ associated with points A and B

3.2. *Geometric size interpolation*

For the sake of simplicity, let us consider a geometric size progression along the edge AB, thus leading to the following expression:

$$H_{AB}(t) = H_{AB}(0)\left(\frac{H_{AB}(1)}{H_{AB}(0)}\right)^t = h_A\left(\frac{h_B}{h_A}\right)^t \tag{23}$$

with $H_{AB}(0) = \|\overrightarrow{AA_1}\|$ (resp. $H_{AB}(1) = \|\overrightarrow{BB_1}\|$), where $A_1 = AB \cap \overline{\mathcal{M}_3(A)}$ (resp. $B_1 = BA \cap \overline{\mathcal{M}_3(B)}$) represent the intersections of the segment AB with the ellipsoid $\overline{\mathcal{M}_3(A)}$ (resp. $\overline{\mathcal{M}_3(B)}$).

Consequently, the length of segment AB can be computed as

$$\ell_{AB} = \|\overrightarrow{AB}\| \frac{h_B - h_A}{h_B\, h_A\, (\log h_B - \log h_A)} \tag{24}$$

3.3. *Edges traced on the surface*

If we consider the edges being traced onto the surface, then we are interested in checking whether the triangulation T conforms to the metric map $\mathcal{M}_2(\mathcal{V})$ representing the trace of the metric map $\mathcal{M}_3(\mathcal{V})$ in the tangent planes to the surface, defined at any point $P \in \mathcal{V}$ by the relationship

$$\overline{\mathcal{M}_2(P)} = \overline{\mathcal{M}_3(P)} \cap \Pi(P) \tag{25}$$

where $\Pi(P)$ denotes the tangent plane to the surface at P.

Remark 3.2. The map $\mathcal{M}_2(\mathcal{V})$ is defined only in the tangent planes.

The length of edge AB can be approached using formula (22) by considering $A_1 = A\widetilde{B} \cap \overline{\mathcal{M}_2(A)}$ (resp. $B_1 = B\widetilde{A} \cap \overline{\mathcal{M}_2(B)}$), $\widetilde{B}$ (resp. $\widetilde{A}$) being the projection of B (resp. A) in the tangent plane $\Pi(A)$ (resp. $\Pi(B)$).

Based on these results, we now have a framework to decide whether a surface triangulation is conforming a given metric map.

Definition 3.2. A triangulation $\mathcal{T}$ conforming the metric map $\mathcal{M}_2(\mathcal{V})$ (according to Definition 2.1) is called a $\mathcal{M}$-mesh.

Remark 3.3. If $\mathcal{M}_2(\mathcal{V})$ is an isotropic metric, the following assertions are trivially equivalent:

$$\mathcal{T} \text{ conforms to the map} \mathcal{M}_2(\mathcal{V}) \Leftrightarrow \mathcal{T} \text{ conforms to the map} \mathcal{M}_3(\mathcal{V})$$

3.4. Geometric conformity

In many applications, a common requirement for surface triangulations is to represent an optimal piecewise planar approximation of the original surface, so that the maximal distance between the original surface and any element of the triangulation is bounded by a given (user-specified) tolerance value. This type of triangulation is called a *geometric* mesh and denoted as $\mathcal{G}$-mesh [1, 2].

From the practical point of view, it is possible to construct a geometric metric map $\mathcal{G}_3$ associated with the points of a surface, such that in any triangulation conforming to this map the gap between the triangles and the surface is controlled [6]. More precisely, the metric $\mathcal{G}_3$ prescribes the sizes at any point according to its main basis formed by the two vectors director along the principal directions (in the tangent plane) and the normal vector. The prescribed sizes along the principal directions are proportional to the corresponding principal radii of curvature and the prescribed size along the normal direction is arbitrary. A procedure to estimate the principal basis and the principal radii of curvature has been described in [7], based on the sole surface triangulation and can be efficiently used to determine the metric $\mathcal{G}_3$.

Remark 3.4. As the prescribed size along the normal direction is arbitrary, we rather consider the trace $\mathcal{G}_2$ of $\mathcal{G}_3$ in the tangent planes to the surface, thus verifying for any point P,

$$\overline{\mathcal{G}_2(P)} = \overline{\mathcal{G}_3(P)} \cap \Pi(P) \tag{26}$$

Under this assumption, the following definition provides a simple way of checking the geometric nature of a given surface triangulation.

Definition 3.3. A surface triangulation $\mathcal{T}$ conforming to the map $\mathcal{G}_2(\mathcal{V})$ (i.e., defined in the tangent planes) associated with the set of vertices $\mathcal{V}$ is a $\mathcal{G}$-mesh.

Notice that this geometric conformity is related to the maximal gap between the triangulation and the surface. Hence, for a given gap value, a surface triangulation is either a $\mathcal{G}$-mesh or not. Practically, for a given gap value, it is sufficient to look at the maximal edge length at a vertex to determine if the triangulation is locally geometric.

3.4.1. Isotropic case. In this case, the metric map $\mathcal{G}_3(\mathcal{V})$ corresponds to the metric map associated with r_P, the smallest of the principal radii of curvature at the vertices of $\mathcal{V}$. More precisely, at any point P:

$$\mathcal{G}_3(P) = \frac{\mathrm{Id}}{r_P^2} \tag{27}$$

Notice that r_P can be computed independently of the calculation of the principal radii of curvature and approached by the formula [1, 2]

$$r_P = \min_{P_i} \frac{1}{2} \frac{\langle \overrightarrow{PP_i}, \overrightarrow{PP_i} \rangle}{\langle v_P, \overrightarrow{PP_i} \rangle} \tag{28}$$

where P_i covers the set of endpoints of all edges sharing P. Thus, it yields trivially to the relationship

$$\mathscr{G}_2(P) = \frac{\mathrm{Id}}{r_P^2} \tag{29}$$

Remark 3.5. Any reduced map $\widetilde{\mathscr{G}_2}$ of $\mathscr{G}_2$, verifying $\forall P \in \mathscr{T}$, $\widetilde{r_P} \leqslant r_P$, is a geometric map.

Definition 3.4. The intrinsic size $\widetilde{h_P}$ at a point P represents the average value of the Euclidean lengths of all edges sharing P.

This average value corresponds to the solution of a minimization problem in which the distance between the desired size and the edge lengths is minimized. More precisely, the problem is to find $\widetilde{h_P}$ for any vertex P to minimize the following function:

$$\sum_Q (\widetilde{h_P} - \|\overrightarrow{PQ}\|)^2 \tag{30}$$

where Q covers the set of endpoints of all edges sharing P. The rectification procedure replaces r_P by $\min(r_P, \widetilde{h_P})$.

Finally, if the intrinsic size $\widetilde{h_P}$ is smaller than r_P then $\widetilde{h_P}$ represents locally a geometric metric. The map $\mathscr{G}_2$ is then rectified according to the intrinsic size map $\widetilde{\mathscr{G}_2}$.

3.5. *Combined metric conformity*

Definition 3.5. A triangulation $\mathscr{T}$ conforming to the metric $\mathscr{M}_2(\mathscr{V})$ as well as to $\mathscr{G}_2(\mathscr{V})$ is called a $\mathscr{H}$-mesh.

Checking if the triangulation $\mathscr{T}$ is a $\mathscr{H}$-mesh is equivalent to checking whether $\mathscr{T}$ conforms to the intersection metric map between the maps $\mathscr{M}_2$ and $\mathscr{G}_2$, both maps being defined in the tangent planes [6].

3.5.1. Isotropic case. The metrics $\mathscr{M}_2(\mathscr{V})$ and $\mathscr{G}_2(\mathscr{V})$ are defined, for any point P, as

$$\mathscr{M}_2(P) = \frac{\mathrm{Id}}{h_P^2}, \quad \mathscr{G}_2(P) = \frac{\mathrm{Id}}{r_P^2} \tag{31}$$

Hence, the combined metric, denoted $\mathscr{H}_2(\mathscr{V})$ is defined as

$$\forall P \in \mathscr{T}, \quad \mathscr{H}_2(P) = \frac{\mathrm{Id}}{\min(r_P, h_P)^2} \tag{32}$$

Remark 3.6. By definition, the map $\mathscr{H}_2(\mathscr{V})$ is geometric.

On the other hand, if h_P is large enough as compared to r_P, the map $\mathscr{M}_2$ is locally discarded to the benefit of the geometry. To overcome this problem, larger gap values between the mesh and the surface are tolerated. Under this assumption, the gap value is locally controlled and no longer globally controlled, and is therefore related to the map $\mathscr{M}_2(\mathscr{V})$ (usually related to the physics of the problem analysed).

4. GEOMETRIC CRITERIA

Several geometric criteria have been proposed to characterize a surface triangulation [1, 2]. These measures (defined in case of an isotropic metric only) allow to locally assess the geometric nature of the triangulation at a vertex and evaluate if the triangulation is coherent with the intrinsic size map and, if yes, if the triangulation is a finite element mesh. For sake of consistency, the measures have been normalized to range between 0 (locally 'bad') and 1 (locally 'good').

Let $\{P_i\}$ be the set of vertices adjacent to vertex P and let ν_P (resp. ν_{P_i}) be the unit normal at P (resp. P_i). The following criteria are evaluated at vertex P:

(i) *Planarity*, $\mathscr{P}_P$. The planarity at vertex P is related to the maximum angle between the normal ν_P and the normals $\{\nu_{P_i}\}$, and is given by

$$\mathscr{P}_P = \frac{1}{2}\left(1 + \min_{P_i} \langle \nu_P, \nu_{P_i}\rangle\right) \tag{33}$$

Similarly the edge planarity $\mathscr{P}_{PQ}$ is defined as

$$\mathscr{P}_{PQ} = \tfrac{1}{2}(1 + \langle \nu_{K_1}, \nu_{K_2}\rangle) \tag{34}$$

where K_1 and K_2 are two faces sharing edge PQ.

(ii) *Smoothness*, $\mathscr{S}_P$. The smoothness at P represents the minimum value of the edge planarity of all edges sharing PQ and is expressed as

$$\mathscr{S}_{PQ} = \min_{P_i} \mathscr{P}_{PQ} \tag{35}$$

(iii) *Deviation*, $\mathscr{D}_P$. The deviation at vertex P is defined as the maximum angle between the edges PP_i and the tangent plane $\Pi(P)$:

$$\mathscr{D}_P = 1 - \min_{P_i} \left| \left\langle \nu_P, \frac{\overrightarrow{PP_i}}{\|\overrightarrow{PP_i}\|} \right\rangle \right| \tag{36}$$

(iv) *Size quality*, $\mathscr{L}_P$. The size quality checks whether the triangulation conforms to the intrinsic size map and is then defined as

$$\mathscr{L}_{PQ} = \begin{cases} \ell_{PQ} & \text{if } \ell_{PQ} \leqslant 1 \\ \frac{1}{\ell_{PQ}} & \text{if } \ell_{PQ} > 1 \end{cases} \tag{37}$$

where ℓ_{PQ} is the length of PQ (according to formula 10). By extension, the size quality at vertex P is given by a mean value of the lengths $\mathscr{L}_{PP_i}$ of all edges PP_i sharing P.

(v) *Shape quality*, $\mathscr{Q}_P$. The shape quality at vertex P is defined as

$$\mathscr{Q}_P = \min_{i} \mathscr{Q}_{K_i} \tag{38}$$

where $\{K_i\}$ is the set of triangles sharing P and $\mathscr{Q}_K$ is the shape quality of triangle K according to Remark 2.1.

Remark 4.1. The above criteria can be combined into a single weighted criterion $\mathscr{C}(P)$ defined as

$$\mathscr{C}_P = \mathscr{P}_P^{\alpha_1}\,\mathscr{D}_P^{\alpha_2}\,\mathscr{S}_P^{\alpha_3}\,\mathscr{L}_P^{\alpha_4}\,\mathscr{Q}_P^{\alpha_5} \tag{39}$$

where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 1$ (the context of the geometric evaluation will naturally precises the relevant weighted coefficients). A global measure is then defined as

$$\mathscr{C}_T = \min_{P \in \mathscr{V}} \mathscr{C}_P \quad \text{and} \quad \overline{\mathscr{C}_T} = \frac{1}{|\mathscr{V}|} \sum_{P \in \mathscr{V}} \mathscr{C}_P \tag{40}$$

4.1. *General scheme for quality assessment*

The method proposed to evaluate a given surface triangulation $\mathscr{T}$ with respect to a given metric map $\mathscr{M}(\mathscr{V})$ specified at the vertices $\mathscr{V}$ of $\mathscr{T}$ can be summarized in the following scheme (in the isotropic case):

1. Initializations,
   - (a) compute the metric $\mathscr{M}_2(\mathscr{V})$ in $\Pi(\mathscr{V})$,
   - (b) compute the metric $\mathscr{G}_2(\mathscr{V})$ in $\Pi(\mathscr{V})$,

     $$\forall P \in \mathscr{V}, \quad \mathscr{G}_2(P) = \frac{\text{Id}}{(\alpha\, r_P)^2}$$

     where r_P is the minimal of the principal radii of curvature and $\alpha < 1$ is a real coefficient used to control the gap between the surface and the edges [6],
   - (c) compute the intrinsic size map $\widetilde{\mathscr{G}}_2(\mathscr{V})$, and rectify $\mathscr{G}_2(\mathscr{V})$ according to $\widetilde{\mathscr{G}}_2(\mathscr{V})$.
   - (d) rectify the map $\mathscr{M}_2(\mathscr{V})$ according to $\mathscr{G}_2$.
2. Loop over all vertices P of $\mathscr{V}$ to compute
   - (a) $(\ell_P)_{\mathscr{G}_2} = \min_{PX} \ell_{PX}$ with respect to $\mathscr{G}_2$,
   - (b) $(\ell_P)_{\mathscr{M}_2} = \min_{PX} \ell_{PX}$ with respect to $\mathscr{M}_2$,
   - (c) $\mathscr{P}_P$, $\mathscr{D}_P$, $\mathscr{S}_P$ (geometric criteria),
   - (d) $\mathscr{L}_P$ and $\mathscr{Q}_P$ with respect to the intrinsic size map.
3. Analysis of the results.
   - (a) $\mathscr{T}$ is a $\mathscr{G}$-mesh $\Leftrightarrow \forall P \in \mathscr{V},\ (1/\sqrt{2}) \leqslant (\ell_P)_{\mathscr{G}_2} \leqslant \sqrt{2}$
   - (b) $\mathscr{T}$ is a $\mathscr{H}$-mesh $\Leftrightarrow \forall P \in \mathscr{V},\ (1/\sqrt{2}) \leqslant (\ell_P)_{\mathscr{M}_2} \leqslant \sqrt{2}$
   - (c) $\mathscr{T}$ is a geometric triangulation $\Leftrightarrow \forall P \in \mathscr{V},\ (\mathscr{P}_P\,\mathscr{D}_P\,\mathscr{S}_P)^{1/3} > \tau$, τ being a given threshold value
   - (d) $\mathscr{T}$ is a finite element mesh $\Leftrightarrow \forall P \in \mathscr{V},\ \mathscr{L}_P \approx 1$ and $\mathscr{Q}_P \approx 1$.

5. APPLICATION EXAMPLES

In this section, several examples of surface triangulations from different fields of applications (e.g. numerical simulation, computer graphic) are analysed to demonstrate the relevance of the proposed approach.

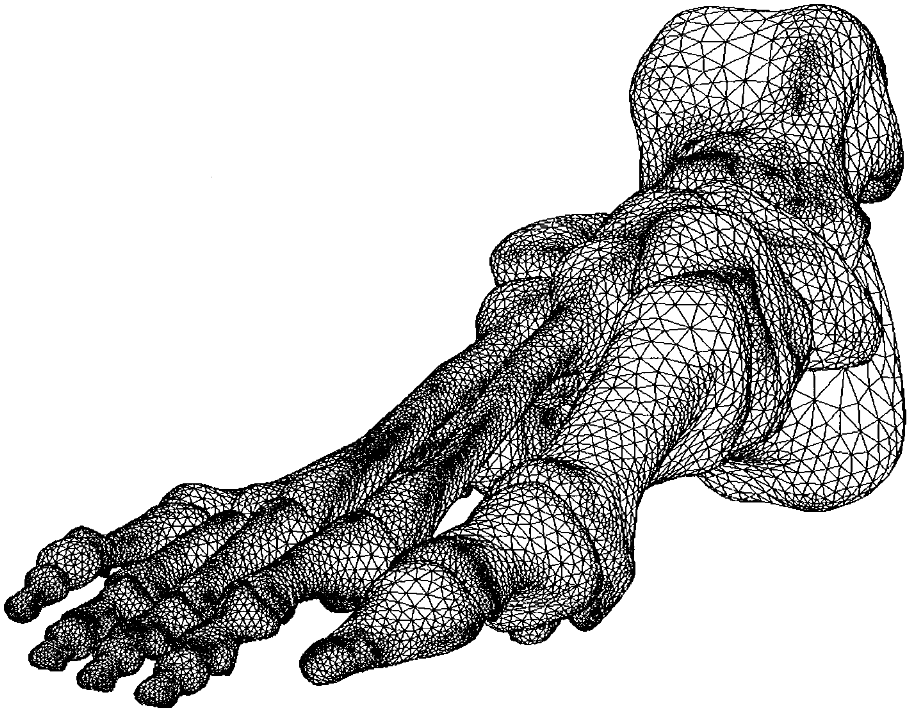

Figure 9. Optimized surface triangulation of a foot skeleton

5.1. Description

foot: Triangulation of a biomedical surface (data courtesy Naval Air Warfare Center Weapons Div.). The original triangulation (2154 vertices and 4204 triangles) has been optimized using a geometric surface mesh optimization algorithm developed by the authors [2].
bunny: Surface reconstruction from range data (data courtesy Computer Graphics Lab., Stanford Univ.). The original surface triangulation, 35 947 vertices and 69 451 faces, has been simplified using a decimation algorithm corresponding to a gap value of 14° (leading to a vertex saving of 76 per cent) [1, 2].
shuttle: Triangulation of the Columbia shuttle (data courtesy of ONERA) as provided by a CAD system. It corresponds to the image, via rational functions of a rectangular regular grid in the parametric space and is globally smooth (i.e. G^1 continuous).
head: Biomedical iso-surface reconstruction of a human head obtained from volumetric data (data courtesy of Low Temp. Lab., Helsinki Univ. of Technology, Finland). The original surface contained 67 106 vertices and 134 208 triangles. A simplified triangulation is analysed corresponding to a gap value of 3° between the surface and the triangulation.
device: Triangulation of a mechanical device provided by a CAD modelling system (data courtesy of MacNeal-Schwendler Corp.). The original triangulation contained 3895 vertices and 7794 triangles. An optimized geometric mesh has been generated, corresponding to a gap value of 5°.

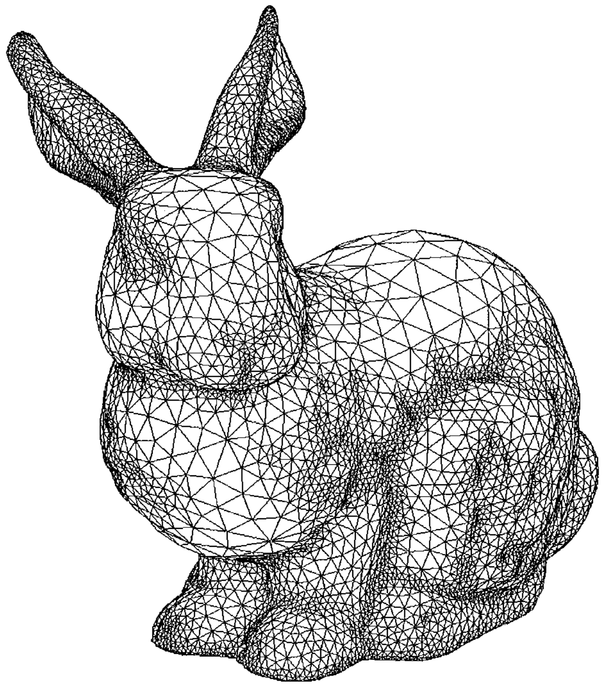

Figure 10. Simplified surface triangulation of Stanford bunny (gap 14°)

Table I. Statistics relative to the numerical evaluation of the surface triangulations

Mesh	N_p	N_e	$\mathcal{Q}_{\mathcal{T}}$	$\overline{\mathcal{Q}_{\mathcal{T}}}$	$\ell_{\min}$	$\ell_{\max}$	$\overline{\ell}$	Figure
foot	71,370	142,636	0·51	0·94	0·56	1·97	1·02	9
bunny	8,381	16,651	0·3	0·92	0·44	7·19	1·11	10
shuttle	6,348	12,692	0·05	0·42	0·16	230·4	4·90	12
head	11,814	23,624	0·53	0·92	0·6	1·81	1·02	11
device	13,877	27,758	0·47	0·94	0·46	3·54	1·02	13
adaption	22,553	45,122	0·23	0·91	0·14	1·94	1·01	14

adaption: The last example concerns the mesh adaption of a mechanical device (data courtesy of MacNeal-Schwendler Corp.) with respect to a prescribed analytical size field. The triangulation has been generated in an adaption loop, where the metric used to govern the mesh generation step is composed of the discrete size map associated with the mesh vertices at the previous step enriched by interpolation on the current mesh. The last triangulation (iteration 3) is analysed with respect to the specified size specifications.

Table I reports statistics about the proposed surfaces triangulations. In this table, N_p, N_e denote the number of vertices and triangles, $\mathcal{Q}_{\mathcal{T}}$, $\overline{\mathcal{Q}_{\mathcal{T}}}$ correspond to the worst and average element shape quality and $\ell_{\min}$, $\ell_{\max}$ and $\overline{\ell}$ represent the minimal, maximal and average normalized edge lengths.

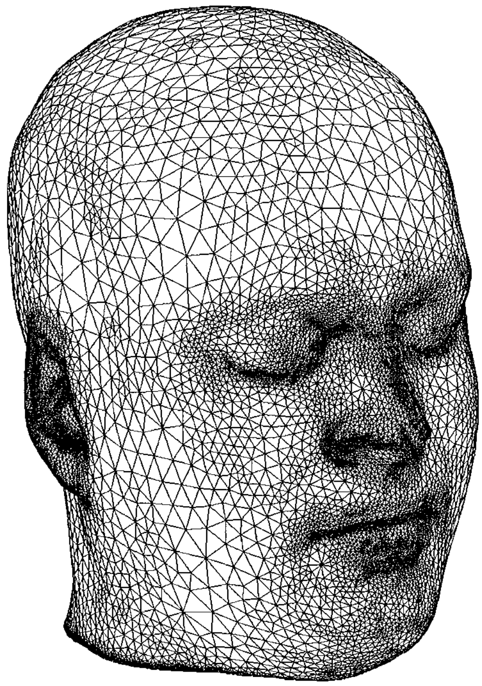

Figure 11. Optimized surface triangulation of a human head (gap 3°)

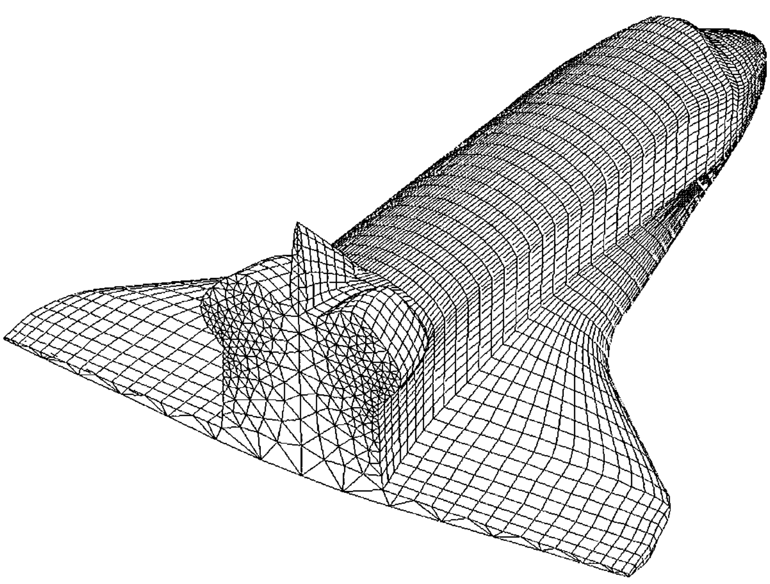

Figure 12. Surface triangulation of the Columbia shuttle

Table II. Numerical evaluation of the surface meshes

Mesh	$\mathcal{P}_{\mathcal{T}}$	$\overline{\mathcal{P}_{\mathcal{T}}}$	$\mathcal{D}_{\mathcal{T}}$	$\overline{\mathcal{D}_{\mathcal{T}}}$	$\mathcal{S}_{\mathcal{T}}$	$\overline{\mathcal{S}_{\mathcal{T}}}$	$\mathcal{L}_{\mathcal{T}}$	$\overline{\mathcal{L}_{\mathcal{T}}}$
foot	0·93	0·98	0·66	0·87	0·91	0·98	0·65	0·95
bunny	0·62	0·96	0·12	0·81	0·38	0·96	0·25	0·90
shuttle	0·72	0·99	0·32	0·95	0·76	0·99	0·01	0·49
head	0·96	0·98	0·77	0·88	0·96	0·99	0·72	0·96
device	0·97	0·99	0·72	0·94	0·94	0·99	0·48	0·96
adaption	0·98	0·99	0·85	0·99	0·99	1·00	0·77	0·94

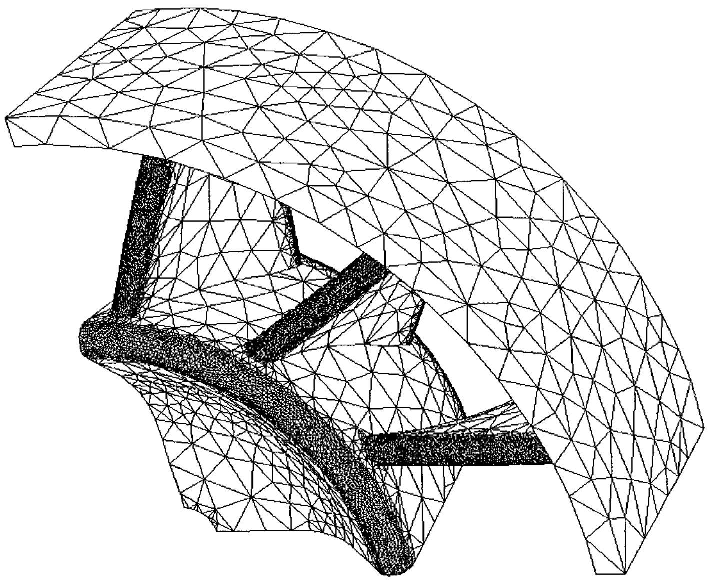

Figure 13. Optimized mesh of a mechanical device

5.2. Numerical evaluation

Table II shows the result of the evaluation of the surface triangulations according to the geometric criteria recalled in Section 4. For each example, the Table indicates the minimum and average values of each criterion.

Evaluation:

1. *foot*: the triangulation is a $\mathcal{G}$-mesh.
2. *bunny*: the triangulation is a $\mathcal{G}$-mesh, although the geometric approximation can be locally improved as emphasized by the deviation value.
3. *shuttle*: the analysis of the normalized edge lengths indicates that the triangulation needs local refinement. The triangulation is a $\mathcal{G}$-mesh.
4. *head*: the triangulation is a good $\mathcal{G}$-mesh.
5. *device*: the triangulation is a $\mathcal{G}$-mesh.

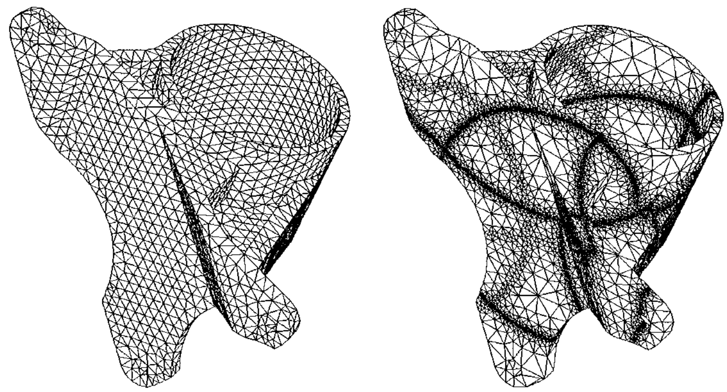

Figure 14. Adapted surface mesh, iterations 0 and 3

6. *adaption*: the combined geometric criterion with equally weighted coefficient reports the mean value of 0·96. The triangulation is a $\mathscr{H}$-mesh with respect to the given analytical size map.

6. CONCLUSIONS

In this paper, we have proposed a method suitable to analyse whether a given surface triangulation satisfies the requirements of a prescribed size map. In particular, this procedure allows to check if the geometric constraints are satisfied by the triangulation. The proposed measures allow in particular to validate the geometric nature of surface meshes. Several examples have been provided to illustrate the proposed approach.

This paper was devoted exclusively to isotropic geometric metric and to isotropic analytical metric specifications. This approach has still to be validated, in the isotropic case, on realistic numerical examples, for which the metric map is provided by an a posteriori error estimate. The second part of this study will focus on anisotropic metric specifications.

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