C. R. Acad. Sci. Paris, Ser. I ••• (••••) •••–•••

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Numerical Analysis/Mathematical Problems in Mechanics

Topology and geometry optimization of elastic structures by exact deformation of simplicial mesh

Optimisation topologique et géométrique de structures élastiques par déformation exacte de maillage simplicial

Grégoire Allaire [a,1], Charles Dapogny [b,c], Pascal Frey [c]

[a] *Centre de mathématiques appliquées (UMR 7641), École Polytechnique, 91128 Palaiseau, France*
[b] *Renault DREAM-DELT'A, 78288 Guyancourt, France*
[c] *UPMC Univ Paris 06, UMR 7598, laboratoire J.-L. Lions, 75005 Paris, France*

ARTICLE INFO

Article history:
Received 16 June 2011
Accepted 18 August 2011
Available online xxxx

Presented by Olivier Pironneau

ABSTRACT

We propose a method for structural optimization that relies on two alternative descriptions of shapes: on the one hand, they are exactly meshed so that mechanical evaluations by finite elements are accurate; on the other hand, we resort to a level-set characterization to describe their deformation along the shape gradient. The key ingredient is a meshing algorithm for building a mesh, suitable for numerical computations, out of a piecewise linear level-set function on an unstructured mesh. Therefore, our approach is at the same time a geometric optimization method (since shapes are exactly meshed) and a topology optimization method (since the topology of successive shapes can change thanks to the power of the level-set method).

RÉSUMÉ

On présente dans cette note une méthode d'optimisation structurale qui s'appuie sur deux manières complémentaires de représenter des formes : d'une part, elles sont maillées exactement afin que l'évaluation des performances mécaniques par éléments finis soit précise ; d'autre part, on utilise leur représentation à l'aide d'une fonction de lignes de niveaux pour les déformer suivant le gradient de forme. L'ingrédient crucial est un algorithme de remaillage qui permet de construire un maillage, de qualité appropriée pour les calculs numériques, à partir d'une fonction ligne de niveaux continue et affine par morceaux sur un maillage non structuré. Par conséquent, notre approche peut être vue à la fois comme une méthode d'optimisation géométrique (puisque les structures sont maillées exactement) et comme une méthode d'optimisation topologique (puisque la topologie des formes successives peut changer grâce à l'utilisation de l'algorithme des lignes de niveaux).

E-mail addresses: gregoire.allaire@polytechnique.fr (G. Allaire), dapogny@ann.jussieu.fr (C. Dapogny), frey@ann.jussieu.fr (P. Frey).

[1] G. Allaire is a member of the DEFI project at INRIA Saclay Ile-de-France and is partially supported by the Chair "Mathematical modelling and numerical simulation, F-EADS – École polytechnique – INRIA".

1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.08.012

Please cite this article in press as: G. Allaire et al., Topology and geometry optimization of elastic structures by exact deformation of simplicial mesh, C. R. Acad. Sci. Paris, Ser. I (2011), doi:10.1016/j.crma.2011.08.012

Version française abrégée

Classiquement, l'optimisation structurale repose sur la méthode de Hadamard [1,13,17] qui prescrit la déformation de la frontière d'un domaine Ω (donnant lieu à une suite de formes Ω^k) pour que celui-ci minimise une certaine fonction-objectif. D'un point de vue technique, ceci se traduit par la nécessité de déformer le maillage de la forme courante (maillage qui permet d'effectuer des calculs par éléments finis), ce qui s'avère très difficile, voire impossible, notamment en trois dimensions. Pour remédier à cet inconvénient majeur, de récents développements [2,3,19] ont conduit à regarder le problème sous l'angle de la *méthode des lignes de niveaux* de Osher et Sethian [14]. Le bord du domaine est représenté comme ligne de niveau 0 d'une fonction implicite définie sur un domaine de calcul fixe $\mathcal{D}$ (maillé typiquement par une grille cartésienne) dont l'évolution est régie par une équation de type Hamilton–Jacobi. Cela nécessite de pouvoir donner un sens au problème mécanique considéré sur tout le domaine de calcul et pas seulement dans la forme Ω^k, ce qui dans le contexte de l'élasticité se fait en considérant que le milieu extérieur $\mathcal{D} \setminus \Omega^k$ n'est pas vide mais occupé par un matériau « ersatz » très mou.

Dans cette Note, on propose une nouvelle approche au confluent de ces deux cadres de travail : d'une part, comme dans l'approche classique, on continue à mailler exactement chaque forme Ω^k à l'itération k de l'algorithme d'optimisation ; d'autre part, l'évolution de la forme d'une itération à l'autre est toujours décrite par la méthode des lignes de niveaux, mais sur un maillage *non structuré* (simplicial) du domaine de calcul $\mathcal{D}$, que l'on s'autorise à modifier d'une itération à l'autre. Puisque les maillages sont non structurés la méthode des lignes de niveaux ne peut utiliser des schémas usuels de type différence finies : ici, on utilise une méthode des caractéristiques [15,18]. Il n'y a ensuite plus qu'à garder la partie intérieure à la forme de ce maillage pour procéder à l'évaluation de sa performance mécanique par un calcul d'éléments finis. Il est ainsi possible de décrire des changements importants (y compris des changements de topologie) de la forme alors que celle-ci reste maillée exactement à chaque étape. La méthode est présentée ici sur des exemples en 2d (voir les Figs. 1 et 2), mais a l'avantage de ne pas présenter d'obstacle conceptuel à une extension en 3d, contrairement à beaucoup d'heuristiques quant à l'évolution du maillage.

1. Introduction

Since [2,3] and [19], the level-set method of Osher and Sethian [14] has proved to be a very versatile tool in the context of structural optimization. Working on a fixed Cartesian grid of a large computational domain $\mathcal{D} \subset \mathbb{R}^d$, the authors used a consistant approximation of the mechanical problem at stake – namely the *ersatz material approach* – then applied classical shape sensitivity techniques (the so-called Hadamard method [1,13,16,17]) and described the evolution of the shape $\Omega \subset \mathcal{D}$ by a Hamilton–Jacobi equation for the associated level-set function. In this note, we propose a new approach where the shape Ω is exactly meshed and no ersatz material is necessary in the void region $\mathcal{D} \setminus \Omega$. We still rely on a larger computational domain $\mathcal{D}$ which is no longer meshed with a fixed Cartesian grid, but rather is endowed with an unstructured mesh that is notably changed at each iteration of the optimization process (using local mesh modification techniques [10]) so that the shape Ω is precisely captured, i.e. its boundary is a collection of internal edges (in 2d) or faces (in 3d) of the mesh. The level-set method is still a key ingredient for mesh deformation and, as such, allows for topology changes from one iteration to the next. However, we are inherently working on unstructured meshes, hence we cannot rely on finite difference schemes and we rather use a method of characteristics [15,18]. We emphasize that, even though all our numerical examples here are in the 2d setting, the whole method has been devised so that there is no additional conceptual difficulty for the 3d case, especially as regards the strategy for mesh evolution.

2. Description of the problem and notations

As a model problem, we are interested in the optimization of a shape Ω, that is, a bounded domain of $\mathbb{R}^d$, made of a linear isotropic material, with Hooke's law A. Such a shape is clamped on a part Γ_D of its boundary $\partial\Omega$, and submitted to surface loads $g \in H^2(\mathbb{R}^d)^d$ on the complementary part $\Gamma_N = \partial\Omega \setminus \Gamma_D$ (with Γ_D and Γ_N being of positive $(d-1)$-measure in $\partial\Omega$). For the sake of simplicity, we neglect body forces and restrict ourselves to linearized elasticity. In this context, the displacement field $u = u_\Omega$ of the shape is the unique solution in $H^1(\Omega)^d$ of the elasticity system

$$\begin{cases} -\operatorname{div}\big(Ae(u)\big) = 0 & \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma_D, \qquad Ae(u)\cdot n = g \quad \text{on } \Gamma_N, \end{cases} \tag{1}$$

where $e(u) = \frac{1}{2}((\nabla u)^t + \nabla u)$ is the *strain tensor* and n is the outer unit normal to $\partial\Omega$. We aim at finding a shape Ω that minimizes a given objective function $J(\Omega)$ in a set $\mathcal{U}_{ad}$ of *admissible shapes* which may involve geometric constraints such as $\Omega \subset \mathcal{D}$ and a fixed total volume $V(\Omega)$. In this note, we restrict ourselves to the *compliance* (which is a global measure of the rigidity of the structure Ω) and the volume constraint is taken into account through a Lagrangian with a fixed positive Lagrange multiplier ℓ, so that the optimization problem becomes

$$\inf_{\Omega\subset\mathcal{D}} \big(J(\Omega) + \ell V(\Omega)\big) \quad \text{with} \quad J(\Omega) = \int_{\Gamma_N} g\cdot u_\Omega \, ds. \tag{2}$$

Please cite this article in press as: G. Allaire et al., Topology and geometry optimization of elastic structures by exact deformation of simplicial mesh, C. R. Acad. Sci. Paris, Ser. I (2011), doi:10.1016/j.crma.2011.08.012

As explained in [3], there are no difficulties to extend our approach to more general objective functions, to additional constraints and to non-linear elasticity.

3. Two complementary ways for representing shapes

We alternatively represent a shape $\Omega \subset \mathcal{D}$ as a mesh $\mathcal{T}_\Omega$ of the *whole* computational domain $\mathcal{D}$ in which Ω is *explicitly* discretized (so that a mesh of Ω is included in $\mathcal{T}_\Omega$ as a submesh – see Fig. 2below) and as a level-set function ψ_Ω, defined on $\mathcal{D}$ (in numerical practice it is a $\mathbb{P}^1$-Lagrange finite element function on an unstructured mesh), enjoying the properties

$$\Omega = \{x \in \mathcal{D} \setminus : \psi_\Omega(x) < 0\}; \qquad \partial\Omega = \{x \in \mathcal{D} \setminus \psi_\Omega(x) = 0\}. \tag{3}$$

Both representations are used at different steps of our method: thus, a crucial ingredient is an efficient algorithm for passing from one characterization to the other.

3.1. Generating a level-set function associated to a shape

Let $\Omega \subset \mathcal{D}$ be a subdomain, explicitly discretized in the mesh $\mathcal{T}$ of $\mathcal{D}$ (even though the method straightforwardly extends to the case of a non-discretized interface). It is classical to generate a corresponding level-set function by computing the signed distance function to Ω, at least near the interface $\partial\Omega$ [6]. To this end, we use a numerical scheme for unstructured (simplicial) meshes, based on some properties of the unique viscosity solution of the time-dependent Eikonal equation, which is described in detail in a previous work [7] (see e.g. [12] for an alternative).

3.2. Meshing the negative subdomain of a level set function, ensuring conformity with the positive subdomain

Given an initial triangular mesh $\mathcal{T}$ of $\mathcal{D}$, the 0 level-set of a $\mathbb{P}^1$ finite element function ψ is a piecewise affine curve (surface in 3d). To obtain a (new) mesh of the shape Ω, corresponding to ψ through (3), we proceed in two (or three) steps:

(i) Each simplex $K \in \mathcal{T}$, crossed by the 0 level-set function, is cut in such a way that $K \cap \partial\Omega$ belongs to the resulting mesh $\widetilde{\mathcal{T}}$, which has to remain conformal. To this end, a pattern which enumerates the various possible configurations is used [10]. Unfortunately, the intersections of $\partial\Omega$ with the mesh $\mathcal{T}$ are not controlled and the obtained mesh $\widetilde{\mathcal{T}}$ is bound to be of very poor quality as far as finite element computations are concerned (ill-shaped elements, e.g. very flat or small, are likely to appear).
(ii) A local mesh improvement is performed, so that a new improved quality mesh $\mathcal{T}'$ is created. This step relies on local mesh modification operators (collapse of close points, points relocations, ...) described in [10].
(iii) (Optional) The mesh $\mathcal{T}'$ is smoothed, especially near the boundary $\partial\Omega$, with a mesh regularization procedure [10] to remove small angles or bumps on $\partial\Omega$ that could impair the accuracy of the finite element computations to come.

4. Shape and topological sensitivity analysis

4.1. Shape sensitivity analysis

This is the so-called *Hadamard method* [1,13,17] which has already been implemented in the context of level-set methods [2,3]. Given a reference bounded domain Ω_0, for $\theta \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ small enough, $(I + \theta)$ is a Lipschitz diffeomorphism of $\mathbb{R}^d$, with Lipschitz inverse and we consider variations of the form $\Theta_{ad} \ni \theta \mapsto (I + \theta)\Omega_0 \in \mathbb{R}^d$, where Θ_{ad} is a subset of $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ corresponding to *admissible* variations of the shape. An objective function $J(\Omega)$ is called *shape-differentiable* at Ω_0 if the application $\theta \mapsto J((I + \theta)\Omega_0)$ is Fréchet-differentiable at 0 and the associated Fréchet differential $J'(\Omega_0)(\theta)$ is the *shape derivative* of J at Ω_0.

Let $\Omega \subset \mathbb{R}^d$ be a smooth domain, it is well-known [3] that the compliance $J(\Omega)$ is shape-differentiable at Ω. Denoting by κ the mean curvature of $\partial\Omega$, for any $\theta \in \Theta_{ad}$, its shape derivative reads

$$J'(\Omega)(\theta) = \int_{\Gamma_N} \left(2\left(\frac{\partial(g \cdot u_\Omega)}{\partial n} + \kappa g \cdot u_\Omega \right) - Ae(u_\Omega) : e(u_\Omega) \right) \theta \cdot n \, ds + \int_{\Gamma_D} Ae(u_\Omega) : e(u_\Omega) \theta \cdot n \, ds. \tag{4}$$

This yields a continuous velocity field θ (which is then to be numerically discretized in the finite elements framework), equal to minus the scalar integrand multiplied by the normal n, *a priori* defined on the boundary $\partial\Omega$, according to which this boundary has to be deformed so as to decrease the objective function under consideration. Note that because this deformation is accounted for by level set methods in our context, this velocity field has to be extended to the whole computational domain, following a regularization process described in [5,8].

Please cite this article in press as: G. Allaire et al., Topology and geometry optimization of elastic structures by exact deformation of simplicial mesh, C. R. Acad. Sci. Paris, Ser. I (2011), doi:10.1016/j.crma.2011.08.012

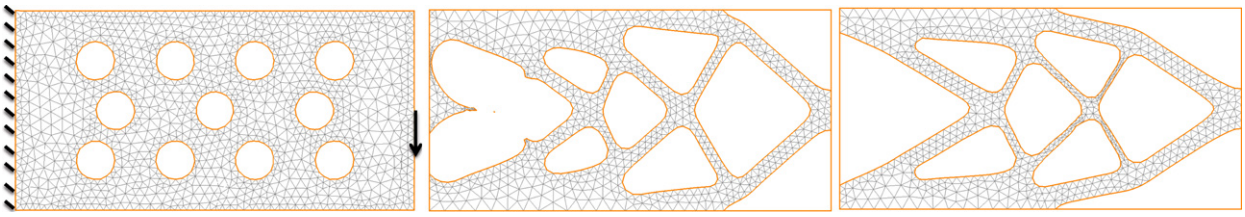

Fig. 1. Initial (left), intermediate (middle) and final (right) iterations of the optimization of a 2*d* cantilever.

4.2. Topological sensitivity analysis

The previous method produces a deformation of the boundary $\partial\Omega$ that allows us to decrease the value of $J(\Omega)$, but forbids the creation of new holes in the domain: the resulting shape is thus strongly dependent on the initialization of the algorithm. As proposed in [4] it should be coupled with the so-called topological gradient [9,11,17] which is a mechanism that evaluates the benefit of the formation of a small hole. This coupled strategy has the effect of making the optimization process less dependent on the initialization (especially in 2*d*). Its implementation is similar to that in [4]: every 5 or 10 iterations of the optimization process, we compute the topological gradient and select the 2 or 3% most negative locations where we change the sign of the level-set function, thus creating holes in the current shape. After discretizing on the mesh of $\mathcal{D}$ the resulting 0-level set of this modified function, we start again the geometric optimization process.

5. Numerical algorithm

Starting from an initial shape Ω^0 (e.g. the full computational domain $\mathcal{D}$), we get a decreasing sequence Ω^k of shapes with respect to function J by applying a shape-sensitivity analysis (Section 4) on the actual domain discretized under the form of a computational mesh, and evolve it with respect to the obtained shape derivative resorting to a level-set description. From times to times (say, every k_{top} step), we perform a topological sensitivity analysis instead of a shape sensitivity analysis so as to change the topology of the shape if need be. The proposed steepest-descent algorithm reads as follows (for clarity, we reported only the steps related to shape-sensitivity analysis, the other ones being easier):

For $k \geqslant 0$, until convergence, start with a shape $\Omega^k \subset \mathcal{D}$, the latter being equipped with a mesh $\mathcal{T}^k$ which encloses a mesh of Ω^k.

(i) Consider only the part related to Ω^k in the mesh $\mathcal{T}^k$, and compute the solution u_{Ω^k} to the elasticity system (1) on this submesh.
(ii) Generate the signed distance function ψ_{Ω^k} associated to Ω^k, on mesh $\mathcal{T}^k$.
(iii) Infer from (4) the vector-valued velocity field θ^k for the advection of the shape to come.
(iv) Solve the following level set advection equation on the time interval $[0, \tau^k]$ ($\tau^k > 0$ being a descent step for the gradient algorithm)

$$\begin{cases} \dfrac{\partial \psi}{\partial t}(t,x) + \theta^k(x).\nabla\psi(t,x) = 0 \quad \text{in } \left(0,\tau^k\right) \times \mathcal{D}, \\ \psi(0,x) = \psi_{\Omega^k}(x) \quad \text{in } \mathcal{D} \end{cases} \tag{5}$$

with the *method of characteristics* [15] (which can be interpreted as a linearly implicit scheme for the true non-linear Hamilton–Jacobi equation [18]) to get the level set function $\psi(\tau^k, \cdot)$ which corresponds to the new shape Ω^{k+1}.
(v) Discretize the 0-level set of $\widetilde{\psi_{\Omega^{k+1}}} = \psi(\tau^k, \cdot)$ in the mesh $\mathcal{T}^k$ as in Section 3, to get the new mesh $\mathcal{T}^{k+1}$ of $\mathcal{D}$, the interior part of which yields a mesh of Ω^{k+1}.

Note that while this algorithm is quite similar to a mesh adaptation technique, it does not require any interpolation whatsoever between two successive iterations.

Fig. 1 depicts a classical numerical example for the compliance objective function (details of the test-case are reported on the first picture) – the so-called *cantilever problem*. Here we take a normalized Young modulus $E = 1$ and a Poisson ratio $\nu = 0.3$. The Lagrange multiplier is set to $l = 3$ and we perform 200 iterations of the above algorithm, without using the notion of topological gradient. Each mesh $\mathcal{T}^k$ has about 1500 vertices (≈ 3000 triangles) and the whole process takes around 3 minutes on a laptop computer. Fig. 2 focuses on a single iteration of the process. Fig. 3 presents another benchmark test-case, where we use the topological gradient every $k_{top} = 10$ iterations.

References

[1] G. Allaire, Conception optimale de structures, Mathématiques & Applications, vol. 58, Springer Verlag, Heidelberg, 2006.

Please cite this article in press as: G. Allaire et al., Topology and geometry optimization of elastic structures by exact deformation of simplicial mesh, C. R. Acad. Sci. Paris, Ser. I (2011), doi:10.1016/j.crma.2011.08.012

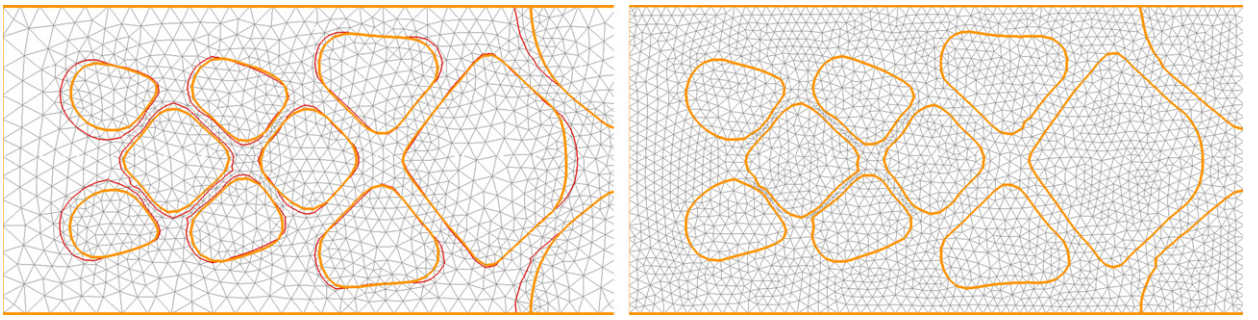

Fig. 2. The 0-level set of $\widetilde{\psi_{\Omega^{k+1}}}$ (in thin line), after advection on the mesh $\mathcal{T}^k$ (thick line; left), and the mesh $\mathcal{T}^{k+1}$, with its associated shape Ω^{k+1} (right).

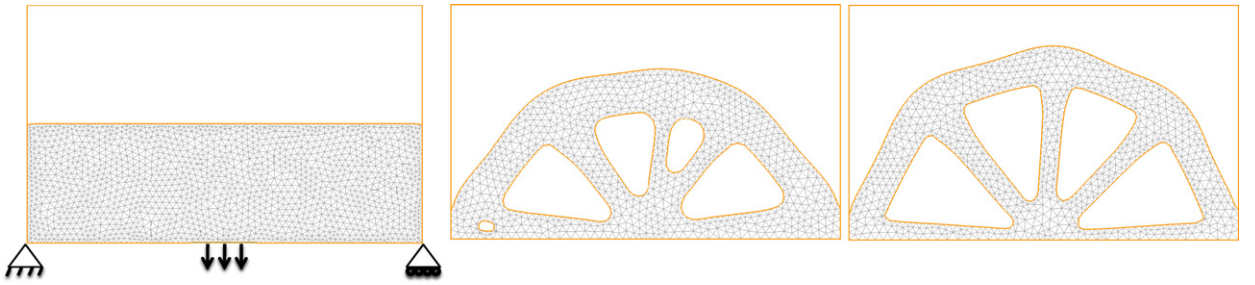

Fig. 3. Initial (left), intermediate (middle), and final shape (right) of the bridge optimization problem.

[2] G. Allaire, F. Jouve, A.M. Toader, A level-set method for shape optimization, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1125–1130.
[3] G. Allaire, F. Jouve, A.M. Toader, Structural optimization using shape sensitivity analysis and a level-set method, J. Comput. Phys. 194 (2004) 363–393.
[4] G. Allaire, F. de Gournay, F. Jouve, A.M. Toader, Structural optimization using topological and shape sensitivity analysis via a level-set method, Control Cybernet. 34 (2005) 59–80.
[5] M. Burger, A framework for the construction of level-set methods for shape optimization and reconstruction, Interfaces Free Bound. 5 (2003) 301–329.
[6] D. Chopp, Computing minimal surfaces via level-set curvature flow, J. Comput. Phys. 106 (1993) 77–91.
[7] C. Dapogny, P. Frey, Computation of the signed distance function to a discrete contour on adapted triangulation, Calcolo, 2010, submitted for publication.
[8] F. de Gournay, Velocity extension for the level-set method and multiple eigenvalues in shape optimization, SIAM J. Control Optim. 45 (1) (2006) 343–367.
[9] H. Eschenauer, V. Kobelev, A. Schumacher, Bubble method for topology and shape optimization of structures, Struct. Optimization 8 (1994) 42–51.
[10] P.J. Frey, P.L. George, Mesh Generation: Application to Finite Elements, second ed., Wiley, London, 2008.
[11] S. Garreau, Ph. Guillaume, M. Masmoudi, The topological asymptotic for PDE systems: the elasticity case, SIAM J. Control. Optim. 39 (2001) 1756–1778.
[12] R. Kimmel, J.A. Sethian, Computing geodesic paths on manifolds, Proc. Nat. Acad. Sci. 95 (1998) 8431–8435.
[13] F. Murat, J. Simon, Sur le contrôle par un domaine géométrique, Technical report RR-76015, Laboratoire d'Analyse Numérique, 1976.
[14] S.J. Osher, J.A. Sethian, Fronts propagating with curvature-dependent speed: algorithms based on Hamilton–Jacobi formulations, J. Comput. Phys. 79 (1988) 12–49.
[15] O. Pironneau, The Finite Element Methods for Fluids, Wiley, New York, 1989.
[16] J. Sokołowski, A. Żochowski, Topological derivatives of shape functionals for elasticity systems, Mech. Structures Mach. 29 (3) (2001) 331–349.
[17] J. Sokolowski, J.-P. Zolesio, Introduction to Shape Optimization: Shape Sensitivity Analysis, Springer Ser. Comput. Math., vol. 10, Springer, Berlin, 1992.
[18] J. Strain, Semi-Lagrangian methods for level set equations, J. Comput. Phys. 151 (1999) 498–533.
[19] M.Y. Wang, X. Wang, D. Guo, A level set method for structural topology optimization, Comput. Methods Appl. Mech. Engrg. 192 (2003) 227–246.

Please cite this article in press as: G. Allaire et al., Topology and geometry optimization of elastic structures by exact deformation of simplicial mesh, C. R. Acad. Sci. Paris, Ser. I (2011), doi:10.1016/j.crma.2011.08.012