Chapter 2

Hilbert and Sobolev spaces

"The unified character of mathematics lies in its very nature; indeed, mathematics is the foundation of all exact natural sciences."
David Hilbert (1862-1943)

Hilbert spaces, named after the German mathematician D. Hilbert (1862-1943), are complete infinite-dimensional spaces in which distances and angles can be measured. These spaces have a major impact in analysis and topology and will provide a convenient and proper setting for the functional analysis of partial differential equations. The notion of smoothness of mathematical solutions is often related to the notion of continuity. We will see that a more interesting and stronger notion of smoothness is that of differentiability (derivatives must be continuous). Moreover, spaces like $C^k$ or $C^\infty$ have limitations when dealing with the solutions of partial differential equations of mathematical physics. The notion of weak derivative has emerged from the works of J. Leray (1906-1998) and S. Sobolev (1908-1989) to deal with non-continuous functions for which the derivatives exist almost everywhere. In a certain sense, functions that belong to Sobolev spaces represent a good compromise as they have some, but not too great, smoothness properties [Evans, 2002].

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2.1 Hilbert spaces

In this chapter, we will consider vector spaces on $K$, with the convention that the identity $\bar{u} = u$, if $K = \mathbb{R}$ and that it denotes the usual conjugation operator if $K = \mathbb{C}$. We recall some basic definitions related to the inner product in a vector space.

2.1.1 Definitions

**Definition 2.1** 1. A mapping $f : E \to F$ between two vector spaces on $K$ is said to be antilinear (or semilinear) if

$$f(\lambda x + y) = \bar{\lambda} f(x) + f(y), \quad \forall (x, y) \in E \times E, \forall \lambda \in K.$$
2. A sesquilinear form is a map \( f : E \times E \to K \) that is linear on the first argument and antilinear on the second:

\[
\forall \lambda \in K, \forall (x, y, z) \in E^3, \quad f(\lambda x + y, z) = \lambda f(x, y) + f(y, z) \\
f(z, \lambda x + y) = \bar{\lambda}f(z, x) + f(z).
\]

3. A mapping \( f : E \times E \to K \) is called an inner product on \( E \) if and only if it is sesquilinear and is a positive-definite hermitian form satisfying the following axioms:

\[
\begin{align*}
(i) & \quad \forall (x, y) \in E^2, \quad f(y, x) = \overline{f(x, y)} \quad \text{(hermitian)} \\
(ii) & \quad \forall x \in E, \quad f(x, x) \in \mathbb{R}^+, \quad \text{(nonnegativity)} \\
(iii) & \quad \forall x \in E, \quad f(x, x) = 0 \iff x = 0, \quad \text{(definite)}
\end{align*}
\] (2.1)

**Remark 2.1** In general and from now, the notation \( \langle \cdot, \cdot \rangle \) will be used to denote the inner product \(^1\).

A (complex) vector space with an inner product satisfying (i)-(iii) is sometimes called a pre-Hilbert space. Inner product spaces have a naturally defined norm.

**Theorem 2.1 (Cauchy-Schwarz inequality)** Consider a inner product \( \langle \cdot, \cdot \rangle \) on \( E \). The following inequality holds for all \( x, y \in E \):

\[
|\langle x, y \rangle|^2 \leq \langle x, x \rangle \cdot \langle y, y \rangle \tag{2.2}
\]

Furthermore, if the equality \( |\langle x, y \rangle|^2 = \langle x, x \rangle \cdot \langle y, y \rangle \) holds, the vectors \( x \) and \( y \) are colinear (linearly dependent).

**Proof.** For \( \lambda \in K, x, y \in E \), we have

\[
\langle x - \lambda y, x - \lambda y \rangle = \langle x, x - \lambda y \rangle - \lambda \langle y, x - \lambda y \rangle \\
= \langle x, x \rangle - \lambda \langle x, y \rangle - \bar{\lambda} \langle y, x \rangle + \lambda \bar{\lambda} \langle y, y \rangle \\
= \langle x, x \rangle - 2\Re(\bar{\lambda} \langle x, y \rangle) + |\lambda|^2 \langle y, y \rangle.
\]

There exists \( \rho \in \mathbb{R}^+ \) and \( \theta \in \mathbb{R} \) such that \( \langle x, y \rangle = \rho \exp(i\theta) \). For \( t \in \mathbb{R} \), we set \( \lambda = t \exp(i\theta) \). Then, we have:

\[
P(t) = \langle x, x \rangle - 2t\rho + t^2 \langle y, y \rangle \geq 0, \quad \forall t \in \mathbb{R}
\]

The second-order polynomial \( P \) has a negative or null discriminant. Hence, we deduce

\[
\Delta' = \rho^2 - \langle x, x \rangle \cdot \langle y, y \rangle \leq 0 \\
\text{or} \quad |\langle x, y \rangle|^2 \leq \langle x, x \rangle \cdot \langle y, y \rangle.
\]

In case of equality, the discriminant \( \Delta' = 0 \). Then, either \( y = 0 \) and thus \( x = 0 \cdot x \) or \( P \) has a double root \( t_0 = \langle x, y \rangle \exp(-i\theta)/\langle y, y \rangle \) which means that \( P(t_0) = \|x - t_0 \exp(i\theta)y\|^2 \) and thus \( x = t_0 \exp(i\theta)y \). \( \square \)

**Theorem 2.2** If \( \langle \cdot, \cdot \rangle \) is a inner product on \( E \), the mapping \( x \mapsto \|x\| = \sqrt{\langle x, x \rangle} \) is a norm on \( E \).

The norm \( \|x\| \) is thought as the length of the vector \( x \). It is well defined by the nonnegativity axiom of the definition.

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Proof. From the definition of the inner product, we have $\langle x, x \rangle = 0$ if and only if $x = 0$ and $\langle \lambda x, \lambda x \rangle = |\lambda|^2 \langle x, x \rangle$. In other words, $\|x\| = 0 \iff x = 0$ and $\|\lambda x\| = |\lambda| \|x\|$. Considering $x, y \in E$, we have
\[
\|x + y\|^2 = \langle x + y, x + y \rangle = \langle x, x \rangle + \langle y, y \rangle + \langle x, y \rangle + \langle y, x \rangle = \|x\|^2 + \|y\|^2 + 2 \text{Re}\langle x, y \rangle
\]
and using Cauchy-Schwarz inequality,
\[
\|x + y\|^2 \leq \|x\|^2 + \|y\|^2 + 2 |\langle x, y \rangle| \leq \|x\|^2 + \|y\|^2 + 2 \sqrt{\langle x, x \rangle \cdot \langle y, y \rangle}
\]
\[
\leq \|x\|^2 + \|y\|^2 + 2 \|x\| \cdot \|y\| = (\|x\| + \|y\|)^2.
\]
hence, $\|x + y\| \leq \|x\| + \|y\|$.

Definition 2.2 Two non-zero vectors $x, y$ of a pre-Hilbert space $E$ are orthogonal if their product is zero: $\langle x, y \rangle = 0$. We will then note $x \perp y$.

Proposition 2.1 If two vectors $x, y \in E$ are orthogonal, we have
\[
\|x + y\|^2 = \|x\|^2 + \|y\|^2 \quad \text{(Pythagorean theorem)}.
\]

Theorem 2.3 If $E$ is a pre-Hilbert space, the inner product is continuous from $E \times E$ in $K$.

Proof. For all $a, b, x, y \in E$, we have
\[
\langle x, y \rangle - \langle a, b \rangle = \langle x, y - b \rangle + \langle x - a, b \rangle = \langle a, y - b \rangle + \langle x - a, b \rangle + \langle x - a, y - b \rangle,
\]
and according to Cauchy-Schwarz inequality
\[
|\langle x, y \rangle - \langle a, b \rangle| \leq \|a\| \cdot \|y - b\| + \|b\| \cdot \|x - a\| + \|x - a\| \cdot \|y - b\|
\]
and for $\varepsilon > 0$ given, it is sufficient to take $\|x - a\| < \min \left(1, \frac{\varepsilon}{3\|b\|} \right)$ and $\|y - b\| < \min \left(\frac{\varepsilon}{3}, \frac{\varepsilon}{3\|a\|}\right)$ to ensure that all three terms are less than $\varepsilon/3$ and thus that $|\langle x, y \rangle - \langle a, b \rangle| \leq \varepsilon$. This proves the continuity.

Proposition 2.2 If $E$ is a pre-Hilbert space, the inner product of two vectors $x, y \in E$ is given by:
\[
\langle x, y \rangle = \frac{1}{4} \left(\|x + iy\|^2 + i\|x + iy\|^2 - \|x - y\|^2 - i\|x - iy\|^2\right).
\]

Proof. We have, for $k = 0, \ldots, 3$
\[
\|x + iy\|^2 = \langle x, x \rangle + \langle y, y \rangle + i^{-k} \langle x, y \rangle + i^k \langle y, x \rangle
\]
\[
i^k \|x + iy\|^2 = i^k \|x\|^2 + i^k \|y\|^2 + \langle x, y \rangle + (-1)^k \langle y, x \rangle
\]
\[
\sum_{k=0}^{3} i^k \|x + iy\|^2 = \|x\|^2 (1 + i + i^2 + i^3) + \|y\|^2 (1 + i + i^2 + i^3)
\]
\[
+ \langle x, y \rangle (1 + 1 + 1 + 1) + \langle y, x \rangle (1 - 1 + 1 - 1) = 4 \langle x, y \rangle.
\]
and this proves the desired equality.

Theorem 2.4 If $x, y, z$ are three vectors of a pre-Hilbert space $E$ and $u = \frac{x + y}{2}$ denotes the midpoint of the segment $[x, y]$, we have
\[
\frac{1}{2} \left(\|x\|^2 + \|y\|^2\right) = \left\|\frac{x + y}{2}\right\| + \left\|\frac{x - y}{2}\right\| \quad \text{(parallelogram law)}
\]
\[
\|z - x\|^2 + \|z - y\|^2 = 2\|z - u\|^2 + \frac{1}{2} \|x - y\|^2.
\]
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Definition 2.3 A pre-Hilbert space $E$ is a Hilbert space if and only if it is a complete normed space (i.e. a Banach space) under the norm associated with the inner product.

Remark 2.2 Actually, the hypothesis of completeness is weak; it is always possible to complete a space $K$ endowed with an inner product. The "completed" norm is then associated with the inner product.

Example 2.1 1. On $\mathbb{R}^d$, the inner product is usually denoted $x \cdot y$ and is defined by:

$$x \cdot y = \sum_{i=1}^{d} x_i y_i. \quad (2.3)$$

2. The vector space $\ell^2(\mathbb{C})$ of the square integrable sequences:

$$\ell^2(\mathbb{C}) = \left\{ (x_n)_{n \in \mathbb{N}} \in \mathbb{C}^N, \text{ s.t. } \sum_{n \in \mathbb{N}} |x_n|^2 < \infty \right\}$$

endowed with the inner product defined by:

$$\langle x, y \rangle = \sum_{n=0}^{\infty} x_n \overline{y_n} \quad (2.4)$$

is a Hilbert space. It plays an important role as it is isomorphic to all separable Hilbert spaces.

Proof. If $x$ is in $\ell^2$ so will be $\lambda x$. For $a, b \in \mathbb{C}$, we have $|a + b|^2 \leq 2(|a|^2 + |b|^2)$, then for $x, y \in \ell^2(\mathbb{C})$:

$$\sum_{n=0}^{\infty} |x_n + y_n|^2 \leq 2 \left( \sum_{n=0}^{\infty} |x_n|^2 + \sum_{n=0}^{\infty} |y_n|^2 \right) < +\infty,$$

hence, $x + y \in \ell^2(\mathbb{C})$.

The Cauchy-Schwarz inequality shows that, for every $m \in \mathbb{N}$:

$$\sum_{n=0}^{m} |x_n y_n| \leq \left( \sum_{n=0}^{m} |x_n|^2 \right)^{1/2} \left( \sum_{n=0}^{m} |y_n|^2 \right)^{1/2} \leq \left( \sum_{n=0}^{\infty} |x_n|^2 \right)^{1/2} \left( \sum_{n=0}^{\infty} |y_n|^2 \right)^{1/2} < +\infty,$$

thus yielding the convergence of the series of general term $(x_n y_n)$ and allowing to define $\langle x, y \rangle$. It is easy to show that the formula (2.4) defines an inner product on $\ell^2$ and that the associated norm is defined by:

$$\|x\| = \left( \sum_{n=0}^{\infty} |x_n|^2 \right)^{1/2}.$$

The completeness of the space $\ell^2(\mathbb{C})$ is established via the convergence of Cauchy sequences and is left as an exercise to the reader.

3. The integration theory gives a Hilbert space: the space $L^2(\Omega, d\mu)$ of the square-measurable integrable functions, for the inner product:

$$\langle f, g \rangle = \int_{\Omega} f(x)g(x)d\mu(x),$$

for any measure $d\mu(x)$, for example the Lebesgue measure $dx$. 
2.1.2 Projection on a closed convex

We recall that a set $C$ is said to be convex if, for all $x, y \in C$ and $t \in [0, 1]$, $tx + (1-t)y \in C$.

**Theorem 2.5** Suppose $E$ is a pre-Hilbert space and $A$ is a nonempty complete convex subset of $E$. Then, for all $x \in E$, there exists a unique point $y \in A$, called the orthogonal projection of $x$ onto $A$, such that

$$d(x, A) = \inf_{z \in A} \|x - z\| = \|x - y\|.$$

This point is often denoted as $P_A(x)$ and is characterized by the property:

$$\forall z \in A, \quad \Re(\langle x - y, z - y\rangle) \leq 0.$$

**Proof.** If $x \in A$, then $y = x$ is the only point that minimizes the distance to $x$. Moreover, we have $\langle x - x, z - x\rangle = 0$ for every $z \in A$. If for $y \in A$ we have $\Re(\langle x - y, z - y\rangle) \leq 0$ for all $z \in A$, then $\|x - y\|^2 = \Re(\langle x - y, x - y\rangle) \leq 0$, thus $x = y$.

On the other hand, if $x \notin A$, we denote $\delta$ the distance from $x$ to $A$. For every $n \in \mathbb{N}$, we pose

$$C_n = \{z \in A, \|z - x\|^2 \leq \delta^2 + 2^{-2n}\}.$$

Hence, the sequence $(C_n)$ is a decreasing sequence of nonempty closed subsets of the complete space $A$. It is sufficient to show that the diameter of $C_n$ vanishes to conclude that the intersection of the $(C_n)$ is a singleton $\{y\}$. Let $z, w$ be two points in $C_n$. Since $A$ is convex, $u = \frac{z + w}{2}$ is in $A$, hence we deduce that $\|x - u\| \geq \delta$. According to theorem 2.4, we can write

$$2\delta^2 + \frac{1}{2}\|z - w\|^2 \leq 2\|x - u\|^2 + \frac{1}{2}\|z - w\|^2 = \|x - z\|^2 + \|x - w\|^2 \leq \delta^2 + 2^{-2n} + \delta^2 + 2^{-2n},$$

and we deduce that $\|z - w\|^2 \leq 4.2^{-2n}$, i.e. $\|z - w\| \leq 2^{1-n}$ and finally that $diam(C_n) \leq 2^{1-n}$. Now, we have established the existence of a point $y \in A$ such that $\bigcap_{n \in \mathbb{N}} C_n = \{y\}$. The point $y$ is the unique point in $A$ such that $\|x - y\| = d(x, A)$.

Let consider $z \in A$. Since $A$ is convex, we have $z_t := y + t(z - y) \in A$ for all $t \in [0, 1]$. Hence,

$$\delta^2 \leq \|x - z_t\|^2 = \langle x - y - t(z - y), x - y - t(z - y)\rangle$$

$$= \|x - y\|^2 + t^2\|z - y\|^2 - 2t\Re(\langle x - y, z - y\rangle)$$

$$= \delta^2 + t^2 \|z - y\|^2 - 2t\Re(\langle x - y, z - y\rangle)$$

We deduce that $2\Re(\langle x - y, z - y\rangle) \leq t\|z - y\|^2$, and since $t$ can be considered as arbitrarily small, we obtain that $\Re(\langle x - y, z - y\rangle) \leq 0$.

If a point $y' \in A$ satisfies this relation for every $z \in A$, we have for $z = y$,

$$0 \geq \Re(\langle x - y', y - y'\rangle) = \Re(\langle x - y, y - y'\rangle) + \Re(\langle y - y', y - y'\rangle)$$

$$= \|y - y'\|^2 - \Re(\langle x - y, y' - y\rangle) \geq \|y - y'\|^2$$

since $y$ is the projection of $x$ onto $A$. This leads to conclude that $\|y - y'\| = 0$ thus $y = y'$. The projection $P_A$ is continuous and 1-Lipschitz continuous as stated now.

**Proposition 2.3** Consider a nonempty complete convex subset $A$ of $E$. Then, For all $x, y \in E$, we have

$$\|P_A(x) - P_A(y)\| \leq \|x - y\|.$$

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Proof. Let denote $a = P_A(x) \text{ and } b = P_A(y)$. The characterization of the projection (Theorem 2.5) allows to write

$$0 \leq \langle a - x, b - a \rangle \quad 0 \leq \langle b - y, a - b \rangle.$$

By adding these relations, we find

$$0 \leq \langle a - x - b + y, b - a \rangle = -\|b - a\|^2 + \langle y - x, b - a \rangle,$$

hence

$$\|b - a\|^2 \leq \langle y - x, b - a \rangle \leq \|x - y\| \|b - a\|,$$

and the result follows. \hfill \Box

**Proposition 2.4** If $A$ is a subset of a pre-Hilbert space $E$, the orthogonal of $A$, denoted $A^\perp$ is the set of vectors $x \in E$ orthogonal to each element of $A$:

$$A^\perp \overset{\text{def}}{=} \{x \in E, \text{ s.t. } \forall y \in A, x \perp y \}.$$  

The set $A^\perp$ is a closed subspace of $E$.

Proof. For every $y \in A$, $y^\perp = \{x, \langle x, y \rangle = 0 \}$ is the kernel of the continuous linear form: $x \mapsto \langle x, y \rangle$, thus it is a closed subspace of $E$. This yields to conclude that:

$$A^\perp = \bigcap_{y \in A} y^\perp$$

is a closed subspace of $E$. \hfill \Box

**Theorem 2.6** If $F$ is a closed subspace of the Hilbert space $E$, there exists for every $x \in E$ a unique couple $(y, z)$ with $y \in F, z \in F^\perp$ such that $x = y + z$. We have then $\|y\| \leq \|x\|$ and $\|z\| \leq \|x\|$.

**Proposition 2.5** Let $F$ be a closed subspace of the Hilbert space $E$. Then, $F^\perp \perp = F$.

**Theorem 2.7** Let $X$ be a subspace of the Hilbert space $E$. Then, $X$ is dense if and only if $X^\perp = \{0\}$.
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Proof. If $X$ is dense and if $y \in X^\perp$, we have $X \subset y^\perp$. Since $y^\perp$ is closed, it contains the closure of $X$, i.e. $E$; in particular $y \in y^\perp$, thus $y = 0$. This shows that $X^\perp = \{0\}$. Conversely, if $X$ is not dense, its closure is a closed vector subspace $F$ distinct from $E$ and every vector $z \in F^\perp$ is in $X^\perp$. Having $F^\perp = \{0\}$ would imply that $F = F^\perp = \{0\} = E$, whic is in contradiction with the assumption. Hence, $X^\perp \neq \{0\}$. \hfill \Box

2.1.3 Hilbert bases, Bessel- Parseval identities

Recall that a subspace $E$ is separable if it contains a countable dense subset in $E$, i.e. a set with a countable number of elements whose closure is the space itself.

Definition 2.4 Consider $E$ a separable Hilbert space of infinite dimension. A Hilbert basis or orthonormal basis is a sequence $(e_n)_{n \in \mathbb{N}}$ of elements of $E$ that is complete (i.e. the linear span of $(e_n)_{n \in \mathbb{N}}$ is dense in $E$) and such that:

$$\langle e_i, e_j \rangle = \delta_{i,j}, \quad \text{with} \quad \delta_{i,j} \text{ the Kronecker symbol.} \quad (2.5)$$

Remark 2.3 A Hilbert basis is not an algebraic basis: the elements are linearly independent but do not form a spanning set of $E$.

Example 2.2 Consider the space $\ell^2(\mathbb{N})$. The sequence $(e_n)_{n \in \mathbb{N}}$ of elements of $\ell^2(\mathbb{N})$ defined by $e_n(k) = \delta_{n,k}$ is an Hilbert basis of $\ell^2(\mathbb{N})$.

Theorem 2.8 In any separable Hilbert space $E$, there exists Hilbert bases.

Proof. Consider a complete countable set $(a_n)_{n \in \mathbb{N}}$. We can assume that the family $(a_n)$ is linearly independent. We use the Gram-Schmidt orthogonalization procedure. To this end, we pose

$$e_1 = \frac{a_1}{\|a_1\|}.$$

Suppose the terms $(e_j)_{1 \leq j \leq n}$ satisfying the relation (2.5) have been defined and that

$$\text{Span}\{a_1,\ldots,a_n\} = \text{Span}\{e_1,\ldots,e_n\}.$$

We pose

$$e_{n+1} = \frac{f_{n+1}}{\|f_{n+1}\|} \quad \text{with} \quad f_{n+1} = a_{n+1} - \sum_{j=1}^{n} (a_{n+1}, e_j)e_j.$$

The relation (2.5) is immediately satisfied and the result follows. \hfill \Box

Definition 2.5 Let $(E_n)_{n \in \mathbb{N}}$ be a countable set of closed subspaces of $E$. The space $E$ is said to be the Hilbertian sum of $(e_n)$ and we denote $E = \bigoplus_{n \in \mathbb{N}} E_N$ if:

(i) the elements $e_i$ of the set $(e_n)$ are mutually orthogonal:

$$\langle x, y \rangle = 0, \quad \forall x \in e_n, v \in e_m, \quad n \neq m,$$

(ii) the vector space generated by the span of $(e_n)$ is dense in $E$.

Theorem 2.9 Let $E$ be a Hilbertian sum of $(e_n)_{n \in \mathbb{N}}$. For any $x \in E$, we denote $x_n = P_{e_n}(x)$ the orthogonal projection of $x$ onto $e_n$. Then, we have

(i) $x = \sum_{n \in \mathbb{N}} x_n$ (convergent series),

(ii) $x = \sum_{n \in \mathbb{N}} P_{e_n}(x)$ (convergent series),

(iii) $x = \sum_{n \in \mathbb{N}} e_n$ (convergent series),

(iv) $x = \sum_{n \in \mathbb{N}} x_n$ (convergent series).
(ii) moreover, \( \|x\|^2 = \sum_{n \in \mathbb{N}} \|x_n\|^2 \), (Parseval identity).

Conversely, given a sequence \( (x_n)_{n \in \mathbb{N}} \in E_n \) such that \( \sum_{n \in \mathbb{N}} \|x_n\|^2 < \infty \), then the series \( x = \sum_{n \in \mathbb{N}} x_n \) is convergent and \( x_n = P_{E_n}(x) \) for all \( n \geq 1 \).

**Remark 2.4** This result is fundamental, for at least two reasons:

(i) it indicates that every separable Hilbert space with a Hilbert basis \( (e_n)_{n \in \mathbb{N}} \) is isomorphic and isometric to the space \( l^2(\mathbb{N}) \): there exists a bijective linear map \( \phi : E \to l^2(\mathbb{N}) \), \( x \mapsto (\langle x, e_n \rangle)_{n \in \mathbb{N}} \);

(ii) it gives also a practical way of getting the best approximation in a finite dimensional subspace by using an orthonormal basis. This will reveal extremely useful in (least-squares) polynomial approximations, orthogonal polynomials, Fourier series, etc.

### 2.1.4 Duality and Riesz theorem

Consider a Hilbert space \( E \) on \( K \). Recall that a continuous linear form on \( E \) is a linear mapping \( f : E \to K \), continuous on \( E \).

**Definition 2.6** The vector space of all continuous linear forms on \( E \) is called the topological dual of \( E \) and is denoted \( E' \).

**Theorem 2.10 (Riesz-Fréchet)** For every continuous linear form \( f \) on the Hilbert space \( E \), there exists a unique \( y \in E \) such that for all \( x \in E \) the following identity holds \( f(x) = \langle x, y \rangle \). Moreover, we have \( \|y\| = \|f\| \).

The mapping from \( E \) to \( E' \) that associates to \( y \) the linear form \( x \mapsto \langle x, y \rangle \) is an antilinear isomorphism from \( E \) to \( E' \) (i.e. \( E \) and \( E' \) are isometrically anti-isomorphic).

**Proof.** Let denote \( f_y \) the linear form \( x \mapsto \langle x, y \rangle \), for every \( y \in E \). According to the definition of the inner product, the mapping \( y \mapsto f_y \) is antilinear from \( E \) to \( E' \). Cauchy-Schwarz inequality allows to write

\[
|f_y(x)| = |\langle x, y \rangle| \leq \|x\| \cdot \|y\|
\]

thus \( \|f_y\| \leq \|y\| \). Moreover, since \( f_y(y) = \langle y, y \rangle = \|y\|^2 \), we have \( \|f_y\| \geq \|y\| \). Hence, \( \|f_y\| = \|y\| \).

We have to show that every continuous linear form \( f \) on \( E \) is of the form \( f_y \) for a certain \( y \). If \( f = 0 \), the trivial setting \( y = 0 \) is appropriate. If \( f \neq 0 \), the kernel \( H \) of \( f \) is a closed hyperplane of \( E \). Let \( a \notin H \). We pose \( \alpha = f(a) \) and if we denote \( b \) the orthogonal projection of \( a \) onto \( H \), the vector \( y = \frac{a}{\|a-b\|^2}(a-b) \) is orthogonal to \( H \) since \( a-b \) is. In particular, \( \langle b, a-b \rangle = 0 \), this \( \langle a, a-b \rangle = \langle a-b, a-b \rangle = \|a-b\|^2 \). Moreover,

\[
\langle a, y \rangle = \frac{\alpha}{\|a-b\|^2} \langle a, a-b \rangle = \alpha.
\]

This yields \( f_y(a) = f(a) \). Hence, \( f \) and \( f_y \) coincide on \( a \) and on \( K \) since they both vanishes. Since every vector of \( E \) can be written as the sum of a vector of \( H \) and of a vector colinear to \( a \), \( f \) and \( f_y \) coincide on \( E \): \( f = f_y \). \( \square \)

**Remark 2.5** The identification of \( E \) and \( E' \) may be erroneous and misleading, when considering the inner product of \( E = L^2 \). For example, let consider a dense subspace \( X = L^2(\mathbb{R}; (1 + |t|) \, dt) \subset L^2(\mathbb{R}) \), endowed with the inner product:

\[
\langle x, y \rangle = \int_{\mathbb{R}} (1 + |t|) x(t) y(t) \, dt.
\]

A linear form \( f \in L^2(\mathbb{R})' \) is also an element of \( X' \). The identification of \( f \) to an element \( f' \in L^2(\mathbb{R}) \) does not define a linear form on \( X \) and the identity \( \tilde{f}(y) = \langle f', y \rangle \) makes no sense on \( E \). Here, we must write

\[
X \subset E = E' \subset X'.
\]
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**Definition 2.7** Let \((x_n)_{n \in \mathbb{N}}\) be a sequence of elements in a Hilbert space \(E\) and consider a point \(x \in E\). The sequence \((x_n)_{n \in \mathbb{N}}\) is said to converge weakly to \(x\) and we note \((x_n)_{n \in \mathbb{N}} \rightharpoonup x\) if and only if
\[
\forall y \in E, \quad \lim_{n \to \infty} \langle y, x_n \rangle = \langle y, x \rangle.
\]

Any weakly convergent sequence is bounded. In addition, if \((x_n)_{n \in \mathbb{N}} \rightharpoonup x\) then
\[
\|x\| \leq \liminf_{n \to \infty} \|x_n\|.
\]

### 2.1.5 Lax-Milgram and Stampacchia theorems

We consider a Hilbert space \(E\) and we will give two important results for bilinear forms on Hilbert spaces.

**Definition 2.8** Let \(a(\cdot, \cdot): E \times E \to \mathbb{R}\) be a bilinear form. It is said to be

(i) continuous if there exists a constant \(C > 0\) such that
\[
|a(x, y)| \leq C \|x\| \|y\|, \quad \forall x, y \in E,
\]

(ii) \(\alpha\)-elliptic (or coercive) if there exists a constant \(\alpha > 0\) such that
\[
a(x, x) \geq \alpha \|x\|^2, \quad \forall x \in E.
\]

**Theorem 2.11 (Lax-Milgram)** Suppose \(a(\cdot, \cdot): E \times E \to \mathbb{R}\) is a continuous and coercive bilinear form. For every linear form \(l \in E'\), there exists a unique \(x \in E\) such that:
\[
a(x, y) = l(y), \quad \forall y \in E.
\]

Moreover, if \(a(\cdot, \cdot)\) is symmetric, then \(x \in E\) is characterized by:
\[
\frac{1}{2} a(x, x) - l(x) = \min_{y \in E} \left( \frac{1}{2} a(y, y) - l(y) \right).
\]

**Proof.**

(i) (Existence and uniqueness of \(x\)) Riesz-Fréchet Theorem 2.10 allows to define a linear mapping \(A : E \to E\) by:
\[
a(x, y) = (A(x), y), \quad \text{and} \quad \|A(x)\| \leq C\|x\|.
\]

Similarly, we associate with \(l \in E'\) an element \(f \in E\) as \(l(y) = \langle f, y \rangle\). Hence, we have to solve \(A(x) = f\) or
\[
x - rA(x) + rf = x, \quad \text{in} \ E.
\]

It is sufficient to prove that the mapping \(T : E \to E\)
\[
T(x) = x - rA(x) + rf
\]
is a strict contraction for \(r > 0\) sufficiently small. This can be achieved using the Banach fixed-point theorem and is left as an exercise. From the existence and uniqueness of a fixed point of \(T\) we deduce the solution \(x\) to the problem.
(ii) (variational principle) If the bilinear form $a(\cdot, \cdot)$ is symmetric, we write:

$$
\frac{1}{2}a(x, x) - \frac{1}{2}a(y, y) - l(x - y) = \frac{1}{2}a(x, x) - \frac{1}{2}a(y, y) - a(x, x - y) \\
= -\frac{1}{2}a(x, x) - \frac{1}{2}a(y, y) + a(x, y) \\
= -\frac{1}{2}a(x - y, x - y) \leq 0.
$$

\[ \square \]

**Corollary 2.1** Let $A \in \mathcal{L}(E, E)$ (i.e. a continuous linear mapping) such that $\langle A(x), x \rangle \geq \nu\|x\|^2$, for all $x \in E$ with $\nu > 0$. Then, $A \in \text{Isom}(E, E)$.

An extension of Lax-Milgram Theorem is given by the following theorem.

**Theorem 2.12 (Stampacchia)** Under the assumptions of Theorem 2.11, let $C$ be a closed nonempty convex set. Then, for all $l \in E'$, there exists a unique $x \in C$ such that

$$
a(x, y - x) \geq l(y - x), \quad \forall y \in C.
$$

Moreover, if the bilinear form $a(\cdot, \cdot)$ is symmetric, then $x$ is characterized by

$$
x \in C, \quad \frac{1}{2}a(x, x) - l(x) = \min_{y \in C} \left( \frac{1}{2}a(x, y) - l(y) \right).
$$

As we will see in Chapter 3, Lax-Milgram Theorem has an interesting application in the resolution of partial differential equations of the type:

$$
\begin{array}{l}
-\Delta u(x) = -\sum_{i=1}^{d} \frac{\partial^2 u(x)}{\partial x_i^2} = f(x) \in L^2(\Omega), \quad \forall x \in \Omega, \\
u(x) = 0 \quad \forall x \in \partial \Omega,
\end{array}
$$

where $\Omega$ denotes a open subset of $\mathbb{R}^d$ and $\partial \Omega$ its boundary. Indeed, this problem can be reduced to solving:

$$
a(u, v) = \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx = \int_{\Omega} f(x)v(x) \, dx, \quad \forall v \in E,
$$

and the difficulty is then to select the appropriate Hilbert space $E$. And this is where and why Sobolev spaces appeared.

### 2.2 Sobolev spaces

We are considering functions for which all the derivatives, in the distributional sense, belongs to $L^2$ space and such that most of the classical derivation rules can be applied. Before turning to Sobolev spaces, we introduce Hölder spaces.
2.2.1 Hölder spaces
Suppose \( \Omega \subset \mathbb{R}^d \) is an open set and \( 0 < \gamma \leq 1 \). Recall that \( k \)-Lipschitz continuous functions \( f : \Omega \to \mathbb{R} \) satisfy by definition the following estimate:
\[
|f(x) - f(y)| \leq k\|x - y\|, \quad \forall x, y \in \Omega, k \in \mathbb{R}_+.
\]
This relation provides a uniform modulus of continuity. It is often useful to consider functions \( f \) satisfying
\[
|f(x) - f(y)| \leq k\|x - y\|^\gamma, \quad \forall x, y \in \Omega.
\]
Such functions are said to be Hölder continuous with exponent \( \gamma \in \mathbb{R}_+ \).

**Definition 2.9** Consider \( f : \Omega \to \mathbb{R} \).

(i) if \( f \) is bounded and continuous, we write
\[
\|f\|_{C^0(\Omega)} \overset{\text{def}}{=} \sup_{x \in \Omega} |f(x)|.
\]

(ii) the \( \gamma \)-Hölder seminorm of \( f \) is:
\[
[f]_{C^{0,\gamma}(\Omega)} \overset{\text{def}}{=} \sup_{x, y \in \Omega, x \neq y} \left( \frac{|f(x) - f(y)|}{\|x - y\|^\gamma} \right).
\]

and the \( \gamma \)-Hölder norm of \( f \) is:
\[
\|f\|_{C^{0,\gamma}(\Omega)} \overset{\text{def}}{=} \|f\|_{C^0(\Omega)} + [f]_{C^{0,\gamma}(\Omega)}.
\]

**Definition 2.10** The Hölder space \( C^{k,\gamma}(\Omega) \) consists of all functions \( f \in C^k(\Omega) \) for which the norm
\[
\|f\|_{C^{k,\gamma}(\Omega)} \overset{\text{def}}{=} \sum_{|\alpha| \leq k} \| \partial^\alpha f \|_{C^0(\Omega)} + \sum_{|\alpha| = k} [\partial^\alpha f]_{C^{0,\gamma}(\Omega)},
\]
is finite.

**Theorem 2.13** The space of functions \( C^{k,\gamma}(\Omega) \) is a Banach space.

2.2.2 Sobolev spaces of integer order
Let \( \Omega \subset \mathbb{R}^d \) and let \( v \) be a function of \( L^2(\Omega) \), it can be identified to a distribution on \( \Omega \) as a function of \( L^1_{\text{loc}}(\Omega) \), also denoted as \( v \) and we can define its derivatives \( \partial_{x_i} v, 1 \leq i \leq n \) as distributions on \( \Omega \). In general, \( \partial_{x_i} v \) is not an element of \( L^2(\Omega) \). Hence, we introduce

**Definition 2.11** We call Sobolev space of order \( 1 \) on \( \Omega \) the space
\[
H^1(\Omega) = \{ v \in L^2(\Omega), \quad \partial_{x_i} v \in L^2(\Omega), \quad 1 \leq i \leq d \}.
\]

The space \( H^1 \) is endowed with the norm associated to the inner product:
\[
\langle u, v \rangle_{1,\Omega} = \int_{\Omega} (uv + \sum_{i=1}^d \partial_{x_i} u \partial_{x_i} v) \, dx,
\]
and we note the corresponding norm:
\[
\|v\|_{1,\Omega} = \sqrt{\langle v, v \rangle_{1,\Omega}} = \left( \int_{\Omega} |u|^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx \right)^{1/2}.
\]
And we have the generalization of such spaces.

**Definition 2.12** Let \( m \in \mathbb{N} \). A function \( v \in L^2(\Omega) \) belongs to the Sobolev space of order \( m \), denoted \( H^m(\Omega) \), if all the derivatives of \( v \) up to order \( m \), in the distributional sense, belong to \( L^2(\Omega) \). By convention, we note \( H^0(\Omega) = L^2(\Omega) \).

**Theorem 2.14** The spaces \( H^m(\Omega) \), \( m \geq 0 \) endowed with the following inner product are Hilbert spaces:

\[
\langle u, v \rangle_{m, \Omega} = \sum_{|\alpha| \leq m} \int_{\Omega} \partial^\alpha u(x) \overline{\partial^\alpha v(x)} \, dx ,
\]}

(2.6)

with the associated norm

\[
\| u \|_{m, \Omega} = \sqrt{\sum_{|\alpha| \leq m} \| \partial^\alpha u \|_{L^2(\Omega)}^2}.
\]

**Proof.** The expression (2.6) defines an inner product. It is sufficient to show that the space is complete for the associated norm. Let consider a Cauchy sequence \( (u_n)_{n \in \mathbb{N}} \) for this norm. We have

\[
\lim_{j,k \to \infty} \sum_{|\alpha| \leq m} \| \partial^\alpha u_j - \partial^\alpha u_k \|_{L^2(\Omega)}^2 = 0,
\]

which means that, for each \( \alpha \) such that \( |\alpha| \leq m \), the sequence \( (\partial^\alpha u_j) \) is a Cauchy sequence in \( L^2(\Omega) \). Since \( L^2(\Omega) \) is complete, there exists \( v_\alpha \) such that \( \partial^\alpha u_j \to v_\alpha \) in \( L^2(\Omega) \). Since the convergence in \( L^2 \) implies the convergence in the distributional sense, we have \( u_j \to v_\alpha \). Hence, we have \( \partial^\alpha v_0 = v_\alpha \in L^2(\Omega) \) and consequently \( v_0 \in H^m(\Omega) \). Moreover, we have

\[
\| v_0 - u_j \|_{m, \Omega}^2 = \sum_{|\alpha| \leq m} \| v_\alpha - \partial^\alpha u_j \|_{L^2(\Omega)}^2 \to 0,
\]

and thus \( u_j \to v_0 \) for the \( H^m(\Omega) \) norm. \( \Box \)

**Definition 2.13** More generally, we can define for every \( 1 \leq p \leq \infty \) and for every \( m \in \mathbb{N} \), \( m \geq 1 \), the Sobolev spaces as:

\[
W^{m,p} = \left\{ u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega), \forall \alpha \in \mathbb{N}^d, |\alpha| \leq m \right\},
\]

endowed with the norm:

\[
\| u \|_{W^{m,p}} = \left( \sum_{|\alpha| \leq m} \int_{\Omega} |\partial^\alpha u|^p \, dx \right)^{1/p}.
\]

All these spaces are Banach spaces. However, we will consider here the spaces \( W^{m,2}(\Omega) = H^m(\Omega) \).

**Remark 2.6** The inclusion \( H^m(\Omega) \subset L^2(\Omega) \) is continuous. Moreover, since \( D(\Omega) \) is dense in \( L^2(\Omega) \) (cf. Chapter 1) and \( D(\Omega) \subset H^m(\Omega) \), we have that \( H^m(\Omega) \) is dense in \( L^2(\Omega) \).

**Example 2.3** To apprehend the concept of regularity for function in \( H^m(\Omega) \), we consider the following example\(^2\). Consider, for every \( \alpha \in \mathbb{R} \) the function defined by:

\[
u_\alpha(x) = \begin{cases} 
 x^\alpha \exp(-x) & x > 0 \\
 -(-x)^\alpha \exp(x) & x \leq 0
\end{cases}
\]

According to the definition of Sobolev spaces, we can establish that

\(^2\)This example is taken from the lecture notes of A. Munnier, *Espaces de Sobolev et introduction aux EDP*, Institut Elie Cartan, Nancy (2006).
2.2. Sobolev spaces

\begin{figure}[h]
\centering
\begin{subfigure}{0.4\textwidth}
\includegraphics[width=\textwidth]{graph1}
\caption{\(\alpha = -1/2\)}
\end{subfigure}
\begin{subfigure}{0.4\textwidth}
\includegraphics[width=\textwidth]{graph2}
\caption{\(\alpha = 0\)}
\end{subfigure}
\begin{subfigure}{0.4\textwidth}
\includegraphics[width=\textwidth]{graph3}
\caption{\(\alpha = 1/2\)}
\end{subfigure}
\begin{subfigure}{0.4\textwidth}
\includegraphics[width=\textwidth]{graph4}
\caption{\(\alpha = 1\)}
\end{subfigure}
\caption{Graphs of the function \(u_\alpha\) for different values of the parameter \(\alpha\).}
\end{figure}

- if \(\alpha > m\), \(m \in \mathbb{N}\) then \(u_\alpha \in C^m(\mathbb{R})\)
- if \(\alpha > m - 1/2\), \(m \in \mathbb{N}\), then \(u_\alpha \in H^m(\mathbb{R})\);
  - if \(1/2 < \alpha < 1\), \(u_\alpha \in H^1(\mathbb{R})\) and \(u_\alpha \in C^0(\mathbb{R})\) but \(u_\alpha \notin C^1(\mathbb{R})\).

The figure 2.2 shows the graphs of this function for different values of \(\alpha\).

**Theorem 2.15** Suppose \(\Omega\) is an open set of class \(C^1\) and its boundary \(\partial \Omega\) is bounded (or \(\Omega = \mathbb{R}^d\)). Then, for every \(m \in \mathbb{N}\), there exists a linear extension operator \(P : H^m(\Omega) \rightarrow H^m(\mathbb{R}^d)\) such that for every \(u \in H^m(\Omega)\) we have:

1. \(P|_\Omega = u\),
2. \(\|Pu\|_{H^m(\mathbb{R}^d)} \leq C \|u\|_{H^m(\Omega)}\), where \(C\) depends upon \(\Omega\) only.

**Proof.** The proof is established in the particular case where \(N = 1, m = 1\) and \(\Omega\) is a subset of \(\mathbb{R}^+\). Let consider \(\Omega = ]0, +\infty[\) and \(\in H^1(\Omega)\). We define

\[(Pu)(x) = u^*(x) = \begin{cases} u(x) & x \geq 0 \\ u(-x) & x < 0 \end{cases}\]

and we pose

\[v(x) = \begin{cases} u'(x) & x > 0 \\ -u'(-x) & x < 0 \end{cases}\]

It is easy to see that \(v \in L^2(\mathbb{R})\). To conclude, we have simply to show that \((Pu)' = v\) in \(\mathcal{D}'(\mathbb{R})\). Indeed, under this assumption, we deduce that \(Pu \in H^1(\mathbb{R})\) and that \(\|Pu\|_{H^1(\mathbb{R})} \leq 2\|u\|_{H^1(\Omega)}\). To prove that the hypothesis \((Pu)' = v\) in \(\mathcal{D}'(\mathbb{R})\) holds, we introduce the sequence \((\eta_k)\) of functions of \(C^\infty(\mathbb{R})\) defined for every \(k \in \mathbb{N}^*\) by

\[\eta_k(t) = \eta(kt), \quad t \in \mathbb{R}, \quad k \in \mathbb{N}^*,\]
where \( \eta \in C^\infty(\mathbb{R}) \) is a given function such that (Figure 2.3):

\[
\eta(t) = \begin{cases} 
0 & t < \frac{1}{2} \\
1 & t > 1 
\end{cases}
\]

Let consider \( v \in \mathcal{D}(\mathbb{R}) \). We have

\[
\int_0^{+\infty} u^*(x)v'(x) \, dx = \int_0^{+\infty} u(x)\chi'(x) \, dx,
\]

where \( \chi(x) = v(x) - v(-x) \). Since \( \eta_k(x) \in \mathcal{D}(0, \infty) \) we have

\[
\int_0^{+\infty} u(\eta_k\chi)' \, dx = \int_0^{+\infty} u'(\eta_k\chi) \, dx.
\]  (2.7)

However, we have the inequalities \( (\eta_k\chi)'(x) = \eta_k(x)\chi'(x) + k\eta'(kx)\chi(x) \) and

\[
\left| \int_0^{+\infty} ku(x)\eta'(kx) \chi(x) \, dx \right| \leq kMC \int_0^{1/k} x|u(x)| \, dx \leq MC \int_0^{1/k} |u(x)| \, dx,
\]

with the constants \( C = \sup_{t \in [0,1]} |\eta'(t)| \) and \( \chi(x) \leq M|x| \), hence

\[
\lim_{k \to \infty} \left| \int_0^{+\infty} ku(x)\eta'(kx) \, dx \right| = 0.
\]

Thus, we can deduce from the previous relations:

\[
\int_0^{+\infty} u\chi' \, dx = -\int_0^{+\infty} u'\chi \, dx = -\int_\mathbb{R} u'v \, dx.
\]

From the Relation (2.7) and the previous relation, we deduce that \( (Pu)' = v \) in \( \mathcal{D}'(\mathbb{R}) \). Now, we consider a bounded interval, say \( I = ]0,1[ \). We define \( \eta \in C^\infty(\mathbb{R}) \) such that \( \eta(x) \in [0,1] \), for all \( x \in \mathbb{R} \) and

\[
\eta(x) = \begin{cases} 
1 & x < \frac{1}{3} \\
0 & x > \frac{2}{3} 
\end{cases}
\]

If \( u \in H^1(I) \), we define \( \tilde{u} :]0, +\infty[ \to \mathbb{R} \) as:

\[
\tilde{u}(x) = \begin{cases} 
u(x) & 0 < x < 1 \\
0 & x > 1 \end{cases}
\]
Thus, \( \eta \hat{u} \in H^1([0, +\infty[) \) and \((\eta \hat{u})' = \eta' \hat{u} + \eta \hat{u}'\). The function \( u \in H^1(I) \) can be written as
\[
u = \eta u + (1 - \eta)u.
\]
At first, the function \( \eta u \) is extended to \([0, +\infty[\) to the function \( \eta \hat{u} \in H^1([0, +\infty[) \) and then extended to \( \mathbb{R} \) by reflection. In this manner, we obtain a function \( v_1 \in H^1(\mathbb{R}) \) that extends \( \eta u \) and such that
\[
\|v_1\|_{L^2(\mathbb{R})} \leq 2 \|u\|_{L^2(I)}, \quad \|v_1\|_{H^1(\mathbb{R})} \leq C \|u\|_{H^1(I)},
\]
where the constant \( C \) depends on \( \|\eta'\|_{L^\infty} \). Similarly, we extend the function \((1-\eta)u\) to obtain a function \( v_2 \in H^1(\mathbb{R}) \) such that
\[
\|v_2\|_{L^2(\mathbb{R})} \leq 2 \|u\|_{L^2(I)}, \quad \|v_2\|_{H^1(\mathbb{R})} \leq C \|u\|_{H^1(I)},
\]
Finally, \( Pu = v_1 + v_2 \) gives the result. We can show that this result still holds if the boundary \( \partial \Omega \) is only piecewise \( C^1 \) continuous. Nonetheless, the following counter-example shows that for some open subsets \( \Omega \), smooth functions on \( \Omega \) cannot be extended in a regular manner.

**Example 2.4** Consider the domain \( \Omega \) defined as \([-1, 2[ \times ] -1, 1[ \) without the segment \([0, 2]\times0). On \( \Omega \), we consider the function \( u \) defined as:
\[
u(x, y) = \begin{cases}
\exp(-1/x^2) & x > 0, y > 0 \\
-\exp(-1/x^2) & x > 0, y < 0 \\
0 & \text{in the other cases}
\end{cases}
\]
We can show that \( u \in C^\infty(\Omega) \) and \( \partial^\alpha u \in L^\infty(\Omega) \) for all \( \alpha \in \mathbb{N}^2 \). Hence, this yields \( u \in H^m(\Omega) \) for every \( m \in \mathbb{N} \). However, the function \( u \) cannot be extended to a regular function on all \( \mathbb{R}^2 \).

The previous result has been extended to Sobolev spaces \( W^{m,p} \).

**Corollary 2.2 (Stein)** Consider a Lipschitz domain \( \Omega \). There exists a linear extension operator \( P \) bounded of \( W^{m,p}(\Omega) \) in \( W^{m,p}(\mathbb{R}^d) \) such that for all \( u \in W^{m,p}(\Omega) \):

1. \( Pu|_{\Omega} = u \),
2. \( \|Pu\|_{W^{m,p}(\mathbb{R}^d)} \leq C\|u\|_{W^{m,p}(\Omega)} \), where \( C \) depends on \( \Omega \).

Using the extension operator introduced above, we deduce the following result.

**Lemma 2.1** If \( \Omega \) is of class \( C^1 \) with \( \partial \Omega \) bounded (or if \( \Omega = \mathbb{R}^d \)), then for every \( u \in H^1(\Omega) \) there exists a sequence of functions \( (u_n)_{n \in \mathbb{N}} \subset \mathcal{D}(\mathbb{R}^d) \) such that
\[
\|u_n|_{\Omega} - u\|_{H^1(\Omega)} \to 0, \quad n \to \infty.
\]
This lemma states that only the restrictions of the functions of \( \mathcal{D}(\mathbb{R}^d) \) are dense in \( H^1(\Omega) \) (and not the functions of \( \mathcal{D}(\Omega) \)). Actually, in general \( \mathcal{D}(\Omega) \) is not dense in \( H^1(\Omega) \).

**Theorem 2.16 (Meyer-Serrin)** Let \( \Omega \) be an open subset of \( \mathbb{R}^d \). Then, the space \( C^\infty(\Omega) \) is dense in \( H^m(\Omega) \) for every \( m \in \mathbb{N} \).

**Definition 2.14** We denote by \( H^1_0(\Omega) \) the closure of \( \mathcal{D}(\Omega) \) in \( H^1(\Omega) \). By extension, we note \( H^m_0(\Omega) \) the closure of \( \mathcal{D}(\Omega) \) in \( H^m(\Omega) \) (for the norm \( \|\cdot\|_{H^m(\mathbb{R}^d)} \)).

**Theorem 2.17** The space \( \mathcal{D}(\mathbb{R}^d) \) is dense in \( H^1(\mathbb{R}^d) \), i.e.
\[
H^1_0(\mathbb{R}^d) = H^1(\mathbb{R}^d).
\]
The proof is obtained by truncation and regularization. It can be found in [Raviart-Thomas, 1998].

**Theorem 2.18** If \( v \in H^1_0(\Omega) \), the function \( \tilde{v} \), extension of \( v \) by 0 in \( \mathbb{R}^d \setminus \Omega \) is a function of \( H^1(\mathbb{R}^d) \).

**Proof.** If \( v \in \mathcal{D}(\Omega) \), the function \( \tilde{v} \), extension of \( v \) by 0 in \( \mathbb{R}^d \setminus \Omega \), belongs to \( \mathcal{D}(\mathbb{R}^d) \). Moreover, we have

\[
\| \tilde{v} \|_{1,\mathbb{R}^d} = \| v \|_{1,\Omega}.
\]

Hence, the mapping \( v \mapsto \tilde{v} \) is continuous from \( \mathcal{D}(\Omega) \), endowed with the norm induced by \( \| \cdot \|_{1,\Omega} \), in \( H^1(\mathbb{R}^d) \). It can be extended by continuity into a linear mapping \( v \mapsto \tilde{v} \) from \( H^1_0(\Omega) \) to \( H^1(\mathbb{R}^d) \) and we have

\[
\tilde{v} = \begin{cases} v & \text{a.e. in } \Omega \\ 0 & \text{a.e. in } \mathbb{R}^d \setminus \Omega \end{cases}
\]

The result follows.

**Definition 2.15** For \( k \in \mathbb{N} \), the Sobolev spaces \( W^{-k,p}(\Omega) \) are defined as dual spaces \( (W_0^{-k,q}(\Omega))^\prime \), where \( q \) is conjugate to \( p \): \( \frac{1}{p} + \frac{1}{q} = 1 \). Their elements are distributions:

\[
W^{-k,p}(\Omega) = \left\{ u \in \mathcal{D}'(\Omega) : u = \sum_{|\alpha| \leq k} \partial^\alpha u_\alpha, \text{ for some } u_\alpha \in L^p(\Omega) \right\}.
\]

Naturally, \( W^{-k,p}(\Omega) \) is a Banach space with the norm:

\[
\| u \|_{W^{-k,p}(\Omega)} = \sup_{v \in W_0^{-k,q}(\Omega), \| v \|_{W_0^{-k,q}(\Omega)} \neq 0} \frac{|\langle u, v \rangle|}{\| v \|_{W^{-k,q}(\Omega)}}.
\]

For any integer \( k \), \( \partial^\alpha \) is a bounded operator from \( W^{k,p} \) to \( W^{k-|\alpha|,p} \).

### 2.2.3 Poincaré’s inequality

Poincaré’s inequality is a fundamental tool for solving partial differential equations.

**Theorem 2.19 (Poincaré’s inequality)** If \( \Omega \) is bounded, there exists a constant \( C = C(\Omega) > 0 \) such that the following inequality holds:

\[
\forall v \in H^1_0(\Omega), \quad \| v \|_{0,\Omega} \leq C(\Omega) \left( \sum_{j=1}^{d} \| \partial x_j v \|_{0,\Omega}^2 \right)^{1/2}.
\]

**Proof.** Let consider a function \( v \in \mathcal{D}(\Omega) \) and let \( \tilde{v} \) be the extension of \( v \) by 0 outside \( \Omega \). We verify easily that \( \tilde{v} \in \mathcal{D}(\mathbb{R}^d) \). Moreover, the support of \( \tilde{v} \) is included in \( \Omega \) that is bounded in at least one direction, say \( x_d \). Thus, we have

\[
\text{Supp}(\tilde{v}) \subset \Omega \subset \{ x = (x', x_d), \quad x' \in \mathbb{R}^{d-1}, \quad a < x_d < b \}.
\]

We write, since \( \tilde{v}(\cdot, a) = 0 \):

\[
\tilde{v}(x', x_d) = \int_a^{x_d} \partial x_d \tilde{v}(x', t) dt, \quad a \leq x_d \leq b.
\]

Cauchy-Schwarz inequality leads us to write

\[
|\tilde{v}(x', x_d)|^2 \leq (x_d - a) \int_a^{x_d} |\partial x_d \tilde{v}(x', t)|^2 dt \leq (b - a) \int_a^{b} |\partial x_d \tilde{v}(x', t)|^2 dt.
\]
Integrating with respect to the variable \( x' \) in \( \mathbb{R}^{d-1} \) yields
\[
\int_{-\infty}^{+\infty} |\tilde{v}(x', x_d)|^2 \, dx' \leq (b-a)\|\partial_{x_d} \tilde{v}\|_{L^2(\mathbb{R}^d)}^2.
\]
Integrating with respect to the variable \( x_d \) between \( a \) and \( b \) leads to
\[
\|v\|_{L^2(\Omega)}^2 \leq (b-a)^2\|\partial_{x_d} v\|_{L^2(\Omega)}^2 \leq (b-a)^2|v|_{1,\Omega}^2,
\]
and the result follows with \( C = (b-a) \).

Poincaré’s inequality is also known under the following form:

**Proposition 2.6** Consider an open subset \( \Omega \) bounded in one direction. Then, there exists a constant \( C = C(\Omega) > 0 \) (depending only upon \( \Omega \)) such that:
\[
\forall v \in H^1_0(\Omega), \quad \|v\|_{L^2(\Omega)} \leq C(\Omega) \|\nabla v\|_{L^2(\Omega)}.
\]

**Corollary 2.3** Consider an open subset of \( \Omega \subset \mathbb{R}^d \), bounded in one direction. Then,
\[
\|u\|_{H^1_0} = \|\nabla u\|_{L^2}
\]
defines a norm on \( H^1_0(\Omega) \) that is equivalent to the usual norm of \( H^1(\Omega) \).

### 2.2.4 Trace of a function

When a function \( u \in L^2(\Omega) \), where \( \Omega \subset \mathbb{R}^d \), it is not possible to consider its restriction to a zero-measure set since the functions in \( L^2(\Omega) \) are defined except to a zero-measure set. However, the functions in Sobolev spaces are more regular than \( L^2 \) functions. We will now consider the restriction of the function to \( \partial \Omega \), called the trace of the function on the boundary of the domain.

Let consider first the half-space
\[
\Omega = \mathbb{R}^d_+ = \{(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}, \ x_d > 0\}
\]
for which we have
\[
\partial \Omega = \{(x', 0), \ x' \in \mathbb{R}^{d-1}\},
\]
that will be identified to \( \mathbb{R}^{d-1} \).

**Theorem 2.20 (Trace)** There exists a continuous linear map, called the trace operator and denoted:
\[
\gamma_0 : H^1(\mathbb{R}^d_+) \to H^{1/2}(\mathbb{R}^{d-1}),
\]
that extends the usual restriction mapping of the continuous functions. This mapping is surjective and its kernel is \( \text{Ker}(\gamma_0) = H^1_0(\mathbb{R}^d_+) \).

Hence, functions in \( H^1_0(\mathbb{R}^d_+) \) vanishes on the boundary of the domain. then, we extend this notion to Lipschitz domains.

**Theorem 2.21** Consider \( \Omega \) an open set of class \( C^1 \). Then, there exists a continuous linear operator, called trace operator and denoted \( \gamma_0 \) of \( H^1(\Omega) \) in \( L^2(\partial \Omega) \) that coincide with the usual restriction operator for continuous functions. Its kernel is \( \text{Ker}(\gamma_0) = H^1_0(\Omega) \).
Definition 2.16 Consider an open subset of class $C^m$, $m \in \mathbb{N}^*$ and $\gamma_0$ the trace operator introduced in the previous theorem. We define the space:

$$H^{m-1/2}(\partial \Omega) = \gamma_0(H^m(\Omega)),$$

endowed with the norm:

$$\|u\|_{H^{m-1/2}(\partial \Omega)} = \inf_{v \in \gamma_0^{-1}(u)} \|v\|_{H^m(\Omega)}.$$

We denote by $H^{-1}(\Omega)$ the dual space of $H^1_0(\Omega)$ endowed with the norm

$$\|u\|_{H^{-1}} = \sup_{\|v\|_{H^1_0} = 1} \langle u, v \rangle_{H^{-1}, H^1_0}.$$

Hence, a function $u \in L^2(\Omega)$ can be identified with an element of $H^{-1}(\Omega)$ by using the linear form defined by the inner product in $L^2$:

$$\|u\|_{H^{-1}} = \sup_{\|v\|_{H^1_0} = 1} \int_{\Omega} uv.$$

We have then the following inclusion rule

$$H^1_0(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega).$$

Remark 2.7 (i) The trace operator is not defined for $L^2(\Omega)$ functions.

(ii) Thanks to the density of $\mathcal{D}(\bar{\Omega})$ in $H^1(\Omega)$, we can establish the Green formula for the functions of $H^1(\Omega)$:

$$\forall u, v \in H^1(\Omega), \quad \int_{\Omega} u \partial_x v \, dx = - \int_{\Omega} v \partial_x u \, dx + \int_{\partial \Omega} uv n_i \, d\sigma,$$

(2.9)

where $n_i$ denotes the $i$th component of $n$.

We have defined $H^{-1/2}(\partial \Omega)$ as the dual space of $H^{1/2}(\partial \Omega)$. This space plays a role in the definition of the trace of the normal derivative. Let denote $\gamma_1$ the linear operator that associates to $u$ the function $\frac{\partial u}{\partial n}$ defined on $\partial \Omega$:

$$\forall u \in H^2(\Omega), \quad \gamma_1 u = \nabla u \cdot n.$$

We show that $\gamma_1 u \in L^2(\partial \Omega)$ for any function $u \in H^2(\Omega)$ and we have the Green formula for the Laplacian:

$$\forall u \in H^2(\Omega), \forall v \in H^1(\Omega) \quad \int_{\Omega} \Delta uv \, dx = - \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial \Omega} \frac{\partial u}{\partial n} v \, d\sigma,$$

(2.10)

where

$$\frac{\partial u}{\partial n} = \sum_{i=1}^d \gamma_0(\partial_{x_i} u)n_i.$$

And we close this section by enunciating the following result without giving a formal proof.

Theorem 2.22 Suppose $u \in H^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$. Then, $\gamma_1 u$ defines an element of $H^{-1/2}(\Omega)$.