Tensor fields

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Tensor fields: Outline

1. **Tensor fields: definition**
2. **Properties of second-order tensors**
3. **Tensor field topology**
   - Hyperstreamlines, tensor lines
4. **Tensor interpolation**
5. **Examples of applications**
   - Gradient tensor
   - Stress tensor
   - Diffusion tensor
   - Metric/Curvature tensor
   - Structure tensor
6. **Representation of tensors**
**Second-order real tensors**

**Definition 1.** Let $V$ be a vector space of dimension $n$. A second-order tensor is defined as a bilinear function $T : V \times V \rightarrow \mathbb{R}$.

**Remarks.**

- Remember the dual of a vector field: $V^*$ is the set of linear forms $V \rightarrow \mathbb{R}$.

1. Given $A : V \rightarrow V^*$, we define $T_A(v, w) = A(v)(w)$; it is bilinear.
2. Given $T$ second-order tensor, we define $A_T(v)$ a linear form by $A_T(v)(w) = T(v, w)$.

- Matrix notation for the second-order tensor $T$: let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of Euclidean space $V$,

$$\begin{align*}
T(v, w) &= \left( v_1 \ldots v_n \right) \cdot M \cdot \left( \begin{matrix} w_1 \\ \vdots \\ w_n \end{matrix} \right),
\end{align*}$$

where $v = \sum_{i=1}^{n} v_i e_i$, $w = \sum_{i=1}^{n} w_i e_i$, $M$ is the $n \times n$ matrix representing $T$.

**Tensors of any order**

Let $V$ be a vector space of dimension $n$.

**Definition 2.** A general $(k, l)$-tensor is a function

$$T : \underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \times \underbrace{V \times \cdots \times V}_{l \text{ copies}} \rightarrow \mathbb{R},$$

linear in every variable.

In particular:

- A $(0, 0)$-tensor is a scalar.
- A $(1, 0)$-tensor is a vector: to a function $\varphi : V^* \rightarrow \mathbb{R}$ corresponds a vector $v$ since, when $V$ is finite-dimensional, $V$ is isomorphic to $V^{**}$: consider the isomorphism $\psi : V \rightarrow V^{**}$ defined by $\psi(v)(\varphi) = \varphi(v), \varphi \in V^*$.
- A $(0, 2)$-tensor is what we called a second-order tensor. Through the choice of a basis on $V$, we can see it as a $n \times n$-matrix.
Tensor fields

**Definition 3.** A $(k, l)$-tensor field over $U \subset \mathbb{R}^n$ is the giving of a $(k, l)$-tensor in every point of $U$, varying smoothly with the point.

**Definition 4.** Let $S$ be a regular surface. A tensor field $T$ on $S$ is the assignment to each point $p \in S$ of a tensor $T(p)$ on $T_pS$, such that these tensors vary in a smooth manner.

In the following, we will restrict to second-order tensor fields.

- Considering a second-order tensor field in $U \subset \mathbb{R}^2$, we can see it as a field of $2 \times 2$-matrices, $T : U \rightarrow M_2(\mathbb{R})$.
- Considering a second-order tensor field in $U \subset \mathbb{R}^3$, we can see it as a field of $3 \times 3$-matrices, $T : U \rightarrow M_3(\mathbb{R})$.
- Let $S$ be a regular surface patch given by a parametrization $f : U \rightarrow \mathbb{R}^3$. In every point $p$, the second-order tensor field $T$ gives a second-order tensor $T(p)$ on the tangent plane $T_pS$. Since the vectors $\frac{\partial f}{\partial x}(u, v)$ and $\frac{\partial f}{\partial y}(u, v)$ form a basis of $T_{f(u,v)}S$, we can write the second-order tensor as a $2 \times 2$-matrix. Again, we have a map $T : U \rightarrow M_2(\mathbb{R})$.

Change of basis for tensors

Suppose that we have two basis $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ of $V$.

1. For a linear map $L : V \rightarrow V$:

   $$B = P^{-1} A P,$$

   a matrix of $L$ in the first basis of $V$, $B$ in the second, $P$ is the matrix with column vectors $f_i$ expressed in the old basis. $A$ and $B$ are similar.

2. For a second-order tensor (or equivalently a bilinear form) $T : V \times V \rightarrow \mathbb{R}$:

   $$(X')^T B Y' = T(x, y) = X^T A Y = (PX')^T A (PY') = (X')^T P^T A (PY'),$$

   where $A$ is the matrix of the tensor in the first basis, $B$ is the matrix of the tensor in second one, $X, Y$ and $X', Y'$ are the coordinates of $x, y$ in the first and the second basis. $A$ and $B$ are congruent:

   $$B = P^T A P.$$

   It means that to a second-order tensor corresponds a congruence class of matrices.
Tensor diagonalization

The matrix representation of a tensor becomes especially simple in a basis made of eigenvectors, when there is one.

Remember that a $3 \times 3$ symmetric matrix always has 3 real eigenvalues, and that the associated eigenvectors $u_1, u_2, u_3$ are orthogonal. The complete transformation of $T$ from an arbitrary basis into the eigenvector basis is then given by

$$U^T T U = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues and $U$ is the orthogonal matrix that is composed of the unit eigenvectors $u_1, u_2, u_3$, i.e. $U = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}$.

Tensor properties

**Definition 5.** • A second-order tensor is said to be symmetric if $S(v, w) = S(w, v)$, for all $v, w \in V$. In matrix notation: $s_{ij} = s_{ji}$ for all $i, j \in \{1, \ldots, n\}$. Number of degrees of freedom: $\frac{1}{2} n(n + 1)$.

• A second-order tensor is said antisymmetric if $A(v, w) = -A(w, v)$, for all $v, w \in V$. In matrix notation: $s_{ij} = -s_{ji}$ for all $i, j \in \{1, \ldots, n\}$. Number of degrees of freedom: $\frac{1}{2} n(n - 1)$.

• A second-order tensor is said to be a traceless tensor if $\tr(T) = 0$, for $T$ a matrix representing the tensor. Since the trace is invariant with respect to congruence, it is well defined to speak of the trace of a second-order tensor.
Properties of second-order tensors

Symmetric tensor properties

Let $T$ be a symmetric second-order tensor.

**Definition 6.** • $T$ is said **positive definite** if $T(v, v) > 0$, for every non-zero vector $v$. It means that all eigenvalues are positive.
• $T$ is said **positive semi-definite** if $T(v, v) \geq 0$, for every non-zero vector $v$. It means that all eigenvalues are non-negative.
• $T$ is said **negative definite** if $T(v, v) < 0$, for every non-zero vector $v$. It means that all eigenvalues are negative.
• $T$ is said **negative semi-definite** if $T(v, v) \leq 0$, for every non-zero vector $v$. It means that all eigenvalues are non-positive.
• $T$ is **indefinite** if it is neither positive definite nor negative definite. The eigenvalues have different signs.

Decomposition in symmetric and anti-symmetric parts

The decomposition of tensors in distinctive parts can help in analyzing them. Each part can reveal information that might not be easily obtained from the original tensor.

Let $T$ be a second-order tensor. If it is not symmetric, it is common to decompose it in a symmetric part $S$ and an antisymmetric part $A$:

$$T = \frac{1}{2}(T + T^T) + \frac{1}{2}(T - T^T) = S + A.$$
Properties of second-order tensors

Decomposition in isotropic and deviatoric parts

Let $T$ be a symmetric second-order tensor on $\mathbb{R}^3$:

$$T = \frac{1}{3} \text{tr}(T) I_3 + (T - T_{iso}) D.$$

- The isotropic part $T_{iso}$ represents a direction independent transformation (uniform scaling, uniform pressure).
- The deviatoric part $D$ represents the distortion.

Decomposition in shape and orientation

Let $T$ be a symmetric second-order tensor on $\mathbb{R}^3$:

- The eigenvalues give information about the shape.
- The eigenvectors give information about the direction.

For a tensor field, the orientation field defined by the eigenvectors is not a vector field, due to the bidirectionality of eigenvectors. It is sometimes of interest to consider shape and orientation separately, for the interpolation, or in order to define features on them.
Hyperstreamlines

Let $T$ be a symmetric second-order tensor field on $\mathbb{R}^3$. In every point $p$, there are three real eigenvalues, say $\lambda_1(p) \geq \lambda_2(p) \geq \lambda_3(p)$ and corresponding eigenvectors $e_1(p), e_2(p), e_3(p)$, characterizing $T(p)$.

- The eigenvector fields $e_i(p)$ are no vector fields: they are line or orientation fields.
- An hyperstreamline (or tensor line) is a curve that is tangent to an eigenvector field $e_i(p)$ everywhere along its path.
- Two hyperstreamlines for different eigenvalues can only intersect at right angles, since eigenvectors belonging to different eigenvalues must be mutually perpendicular.
- $T$ is said to be degenerate in one point $p$ if the eigenvalues in this point are not all distinct: for example: $\lambda_1(p) = \lambda_2(p) \geq \lambda_3(p)$. The point $p$ is said to be a degenerate point of $T$.

2D case: the degenerate points.

Consider a second-order tensor field over $\mathbb{R}^2$.

- The eigenvalues $\lambda_1 \geq \lambda_2$ define major and minor tensor lines.
- The stable degenerate points are isolated.
- The tensor index can be used to classify degenerate points: It is computed along a closed non self-intersecting curve as the number of rotations of the eigenvectors when traveling once along the curve in counterclockwise direction. Because of the lack of orientation of eigenvectors, the tensor index is a multiple of $\frac{1}{2}$.

Computation of the index for a trisector [Tricoche]. Travelling along this curve counterclockwise the angle $\theta$ varies from $\pi$ to 0, giving the index value of $\frac{\pi}{2\pi} = \frac{1}{2}$.
2D case: the first order degenerate points.

- The first order degenerate points are wedges and trisectors. Note that these degeneracies cannot appear as singularities of a vector field (they are possible for tensors because no direction is given for tensor lines). They are called first-order degenerate points because they correspond to a linear approximation of the tensor around the degenerate point.

- In the neighborhood of a degenerate point, the regions where tensor lines pass the singularity by in both directions are called hyperbolic. The regions where they reach the singularity, on the contrary, are called parabolic.

\[ \text{First order degenerate points [Tricoche].} \]

2D case: higher order degenerate points.

They correspond to unstable configurations, slight perturbations can break them into first order degenerate points:

\[ \text{Decomposition of higher order degeneracies. [Delmarcelle]} \]
2D case: the tensor field topology.

The topological skeleton consists of:

- degenerate points with their type
- separatrices: they segment the tensor field such that they bound regions of qualitative homogeneous eigenvector behaviour. They are specific hyperstreamlines that are either limit cycles, or trajectories emanating from degenerate points and lying on the border of hyperbolic sectors.

First order degenerate points, with major and minor separatrices and tensor lines [Kratz].

Example of the tensor field topology.

Shows the tensor topology, with major and minor tensor lines, showing the trisectors and wedges [Chen].
Example of the tensor field topology (2)

Shows the tensor topology, the LIC shows the major tensor lines, and the color shows the magnitude of the corresponding eigenvalue. W for Wedge, T for trisector. [Delmarcelle].

3D case: the degenerate curves.

- There are **major**, **medium** and **minor** tensor lines corresponding to the three eigenvalues.

- Triple degeneracies are possible. They are numerically instable, can disappear under arbitrarily small perturbations.

- The stable degeneracies consist of curves of double degenerate points. We call a degenerate point **linear** if the minor eigenvalue has multiplicity two and **planar** if the major eigenvalue has multiplicity two. Either all points of the curve are planar, or they are linear.

- A **separating surface** is the union of all hyperstreamlines emanating from a degenerate curve following one of the eigenvector fields. There are three sheets of such surfaces around the trisector segments along a degenerate curve and one or three sheets around the wedge segments.
Example of a 3D case.

Degenerate lines in deformation tensors of flow past a cylinder with a hemispherical cap. Feature lines are colored by the eigen-difference $K = 2\lambda_2(\lambda_1 + \lambda_3)$, with eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$.

Interpolation of sampled second-order tensor fields

For a sampled tensor field, there are many ways to interpolate between the tensors given at the vertices. Among them, you can imagine interpolating the components of the tensor, or interpolating the eigenvectors and eigenvalues. The second interpolation is much more shape-preserving, the change of directions is more uniform.

When designing new interpolation schemes: you want to preserve the characteristics of the original data (positive definiteness, determinant...).

Two examples comparing the results of linear component-wise tensor interpolation (top row) and linear interpolation of eigenvectors and eigenvalues (bottom row) between two positions [Hotz].
Interpolation of sampled second-order tensor fields (2)

Component-wise and eigenvectors-based interpolations inside a triangle without degenerate point.
Upper row: ellipses, lower row: tensor lines [Hotz].

Interpolation of sampled second-order tensor fields (3)

Middle and right: Component-wise and eigenvectors-based interpolations inside a triangle with degenerate point.
Upper row: trisector point; bottom row: wedge point.
On the left triangle, you can see at the same time the separatrices for both interpolations.
For the wedge point case, there exist two more radial lines with two additional parabolic sectors for the eigenvector interpolation [Hotz].
Gradient tensor

Let us consider a smooth vector field $v$. Its gradient $\nabla v$ is a second-order tensor field given by the Jacobian matrix. In general, it is indefinite and not symmetric. The trace of $\nabla v$ is the divergence of $v$.

For such tensor fields, it is common to make a decomposition in symmetric and anti-symmetric parts. The existence and strength of vortices is related to the antisymmetric part. The gradient tensor can be a tool to study vector fields.

Stress tensor

In mechanical engineering, when material bodies experience small deformations, stress and strain tensors are central concepts.

- Stress tensor field: describes internal forces or stresses acting within deformable bodies as reaction to external forces.
- Strain tensor field: deformation of the body due to stress.

External forces $f$ are applied to a deformable body. Re-acting forces are described by the stress tensor, symmetric, composed of three normal stresses $\sigma$, and three shear stresses $\tau$.

Given a surface normal $n$ of some cutting plane, the stress tensor maps $n$ into a traction vector $t = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} \cdot n$. This vector describes the internal forces that act on this plane (normal and shear stresses).
Stress tensor (2)

We can define a scalar anisotropy measure for the stress tensor: $\tau = \frac{\lambda_1 - \lambda_3}{2}$. Instead of having the whole tensor to look at, we can summarize the tensor in one scalar value and analyze the yielding evolution much easier.

If there is no rotation (which is, in general fulfilled for infinitesimally small volume elements), the stress tensor field is symmetric.

It is used in mechanical engineering but also in medicine (for the elastic properties of soft tissues for example).

Diffusion tensor

The diffusion tensor $D$ is a material property, containing the strength information of the diffusion, according to the direction. It is a second-order positive semi-definite and symmetric tensor field (in the case where the material is inhomogeneous.

For inhomogeneous anisotropic media, Fick’s law relates the flux $J$ (flow per unit area) to the concentration gradient $\nabla c$ through this tensor field:

$$J = -D \cdot \nabla c.$$
Tensors in differential geometry

Let us take a regular surface patch $S$.

- The first fundamental form is a symmetric positive definite tensor field on it. It can be seen as the tensor giving the infinitesimal distance on the manifold.
- The second fundamental form is a symmetric tensor field on it, it describes the change of the surface normal in any direction.

The metric and curvature tensors [Kratz].

Structure tensor

The structure tensor in image processing provides information about the local structure of an image, makes it possible to detect corners or boundaries. It is a symmetric positive definite second-order tensor.

The structure tensor of a 3D image $f$ can be represented in every point (voxel) $(x, y, z)$ by a $3 \times 3$-matrix

$$M(x, y, z) = \begin{pmatrix}
    \langle f_x, f_x \rangle \omega_{(x,y,z)} & \langle f_x, f_y \rangle \omega_{(x,y,z)} & \langle f_x, f_z \rangle \omega_{(x,y,z)} \\
    \langle f_y, f_x \rangle \omega_{(x,y,z)} & \langle f_y, f_y \rangle \omega_{(x,y,z)} & \langle f_y, f_z \rangle \omega_{(x,y,z)} \\
    \langle f_z, f_x \rangle \omega_{(x,y,z)} & \langle f_z, f_y \rangle \omega_{(x,y,z)} & \langle f_z, f_z \rangle \omega_{(x,y,z)}
\end{pmatrix}$$

where the inner product is defined by

$$\langle f, g \rangle \omega_{(x,y,z)} = \sum_{i,j,k} \omega(x,y,z)(i,j,k) \cdot f(i,j,k) \cdot g(i,j,k)$$

$\omega(x,y,z)$ being a fixed "window function", a distribution of three variables centered in the voxel $(x, y, z)$. 

Representation methods

- 3D tensor fields are sometimes represented by glyphs: different types of glyphs can be designed.
- LIC and textures can also be applied to represent tensor fields.
- For glyphs and for tensor lines, placement is an important issue.
- Tensor topology concerns only symmetric tensors!
- Scalar descriptors (determinant, eigenvalues, trace, application specific functions) can be applied to analyze and represent the tensor field.

Glyphs [Kindlmann].

This lecture is largely inspired from the paper