Analytical stability analysis of Coarse-Mesh Finite Difference method

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Abstract

In this paper, an analytical stability analysis is carried out for a Coarse-Mesh Finite Difference method, namely the Modified Average Flux Correction scheme. A simple 2-region heterogeneous slab problem is scanned using local Lyapunov stability theory. The step-characteristics method is used to discretize the transport equation. Two limit cases are analyzed: when transport and diffusion share the same mesh and when the diffusion coarse-mesh is a single region comprising the whole geometry. The first case points out the importance of interface parameters as a key role for stability. The second case helps to comprehend how and when homogenization can stabilize the scheme, in particular when the geometry is extremely heterogeneous and materials have very different optical properties.

1. Introduction

The Coarse-Mesh Finite Difference (CMFD) method for transport calculations is an efficient and effective non-linear technique to accelerate discrete ordinates source iterations (SI). Inspired by the rebalancing methodology, the iterative scheme can be viewed as a two-level angular-spatial multigrid, where the discretized transport plays the role of the fine-mesh operator, while the equivalent diffusion is a coarser operator in angle and space. (Smith, 1980, 1983; Aragones and Ahnert, 1986; Yamamoto, 2005a, 2005b) The equivalence transport $\leftrightarrow$ diffusion is established by dynamically preserving at each iteration neutron balance on the coarse-regions. This entails a flux-weighted homogenization for the cross-sections and an artificial flux-current relation at interfaces of coarse-regions. The later is a typical finite difference relation modified with a transport-computed parameter that guarantees current preservation. Various types of flux-current relation characterize different CMFD methods. (Yamamoto, 2005a, 2005b) The coarse system operator results in a finite difference multidiagonal matrix of reduced dimension. The CMFD has been successfully implemented in transport industrial codes to accelerate lattice calculation (Smith, 1983; Aragones and Ahnert, 1986); it is also an optimum tool to perform quick sensitivity studies. It was recently implemented in the homogeneous/heterogeneous Cartesian cell solver, namely IDT, of the APOLLO2 lattice code. (Masiello and Zmijarevic, 2006; Sanchez et al., 1988)

However, convergence of the accelerated iterations is not assured. This is the case for the CMFD acceleration for which we have observed non-convergence in some particular cases. The accelerat-
tion can default for very thick or very thin homogeneous regions in scattering-dominated regimes. This behavior has been confirmed in the literature, (Cefus and Larsen, 1990; Yamamoto, 2005a, 2005b), where the Fourier analysis of the acceleration operator has been done by linearization near convergence. This technique was applied to homogeneous problem for which the linearization of the method reduces to the linear acceleration equation already obtained by Cefus and Larsen, 1990, and, independently, by Yamamoto, 2005b.

The aim of this work is to analyze the reasons of such non-convergent behaviors for heterogeneous lattice problems by using the local Lyapunov stability theory. More precisely, the iterative scheme can be viewed as a multivariable explicit dynamical system of the type

\[ \overrightarrow{\phi}^{(i+1)} = F(\overrightarrow{\phi}^{(i)}), \]

Here, \( i \) is the iteration index, \( \overrightarrow{\phi}^{(i)} \) and \( \overrightarrow{\phi}^{(i+1)} \) contains the region-wise average scalar flux at two successive iterations, while \( F(\overrightarrow{\phi}) \) is a nonlinear map, which, in our case, is infinitely differentiable except at \( \overrightarrow{\phi} = 0 \). The Lyapunov stability theorem states that the stability of the solution near a fixed point \( \overrightarrow{\phi} \xi \) depends on the magnitude of the largest eigenvalue of the Jacobian matrix \( D F(\overrightarrow{\phi} \xi) = \nabla_{\overrightarrow{\phi}^{(i)} \xi} \overrightarrow{\phi}^{(i+1)} |_{\overrightarrow{\phi}^{(i)} = \overrightarrow{\phi} \xi} \) and, in particular, the scheme is stable if \( ||D F(\overrightarrow{\phi} \xi)|| < 1 \). (Benoist-Guetal, 2005)

The non-linearity of \( F \) is caused by the homogenized cross-sections and by the interface coefficients used to preserve currents. In this work, we analyze separately the non-linear effects due to the homogenization and to the interface currents preservation. In particular, considering a transport solution on a detailed \( N \)-region spatial mesh and a diffusion solution on a coarse \( K \)-region spatial mesh, with \( K \leq N \), we analyze two limit applications of the CMFD method, i.e. for \( K = N \) and for \( K = 1 \). The first application eliminates the homogenization effects and keeps the non-linearity introduced by the interface-currents. On the other hand, the second application eliminates the coupling effects from the interface-currents so that the non-linearity is introduced only by the flux-weighted cross-sections.

More explicitly, the accelerated SI, as a results of the action of \( F \) on the iterative flux \( \overrightarrow{\phi}^{(i)}_N = \{ \phi_n^{(i)}, n = 1, N \} \), are characterized by

- a transport iteration
  \[ \overrightarrow{\phi}_N = A_T [\Sigma_s,N\overrightarrow{\phi}_N^{(i)} + \overrightarrow{S}_N], \] (1)
- the construction of non-linear restriction and prolongation operators, together with equivalence parameters
  \[ \Sigma_{r,K} = P_{K,N}(\overrightarrow{\phi}_N)\Sigma_{r,N}, \quad P_{N,K}(\overrightarrow{\phi}_N), \] (2)
- the solution of the non-linear diffusion system
  \[ A_D(\overrightarrow{\phi}_N^{(i)})\overrightarrow{\phi}_K = \overrightarrow{V S}_K, \] (3)
- and the reconstruction of the accelerated flux \( \overrightarrow{\phi}_N^{(i+1)} \),
  \[ \overrightarrow{\phi}_N^{(i+1)} = P_{N,K}(\overrightarrow{\phi}_N)\overrightarrow{\phi}_K. \] (4)

Here we have used the following notation, \( \overrightarrow{\phi}_N = \{ \phi_n, n = 1, N \} \) is the transport solution and \( A_T = \{ A_{n,m}, m,n = 1, N \} \) the discretized transport matrix, while \( \overrightarrow{\phi}_K = \{ \phi_k, k = 1, K \} \) is the coarse diffusion solution and \( A_D(\overrightarrow{\phi}_N^{(i)}) = \{ A_{k,l}, k,l = 1, K \} \) is the non-linear diffusion matrix. \( \overrightarrow{S}_N = \{ S_n, n = 1, N \} \) and \( \overrightarrow{V S}_K = \{ \sum_{n \in k} V_n S_n, k = 1, K \} \) are the external source and the integrated source on the coarse-mesh, respectively, while \( \Sigma_{r,N} = diag\{ \Sigma_{r,n}, n = 1, N \}, \Sigma_{r,K} = diag\{ \Sigma_{r,k}, k = 1, K \} \) and with \( r = a, s \) indicating the absorption and the isotropic scattering, \( a,s = N - \Sigma_{s,n} = N - \Sigma_{n,s} \). The restriction operator and the prolongation operator result the first in an homogenization process,

\[ P_{K,N}(\phi_N)\overrightarrow{S}_{r,N} = \left\{ \sum_{n \in k} V_n \phi_n \sum_{n \in k} V_n \phi_n, k = 1, K \right\}, \] (5)

while the second in a shape factor,

\[ P_{N,K}(\phi_N)\overrightarrow{\phi}_K = \left\{ \frac{\phi_n \phi_k}{\sum_{n \in k} V_n \phi_n / \sum_{n \in k} V_n}, n \in k, k = 1, K \right\}. \] (6)

The nonlinearities introduced by current preservation and homogenization are implicitly contained in \( A_D(\overrightarrow{\phi}_N^{(i)}) \) and could entail analytical complication for analysis. In this work, \( A_T \) and \( A_D(\overrightarrow{\phi}_N^{(i)}) \) will be constructed for a 2-region slab problem to carry out analytical calculation, in particular \( A_D(\overrightarrow{\phi}_N^{(i)}) \) is build following the Modified Average Flux Correction (MAFC) procedure.
The rest of the paper is organized as follows: Sections 2 and 3 describes the transport matrix and the MAFC matrix, while Sections 4 and 5 contain results of analytical calculation on a simple one-dimensional slab problem composed by 2 regions having different set of cross sections, and finally Section 5 is dedicated to conclusions.

2. Step characteristics matrix for the 2-region slab problem

To handle analytical calculation, we simplify further the problem by making the following assumptions: 1) the numerical method for transport equation is the step characteristics scheme accelerated by the Modified Average Flux Correction (MAFC) scheme 2) the boundary conditions are specular reflection on both sides. The first assumption eliminates involved algebraic manipulations needed to deal with the high order spatial moments. Also, MAFC offers a simple manner of writing the acceleration matrix, which is very close to acceleration by rebalancing. The second assumption helps to eliminate boundary conditions from the transport iterations. In particular, by doubling symmetrically the pitch of the geometry, one can use translation boundary conditions to write the iterative transport matrix in function of scalar fluxes only.

The step characteristic equations in region $n = 1, 2$ are

$$
\psi_1^+ = T_n \psi_1^- + E_n q_n,
$$

$$
\psi_2^- = I_n \psi_2^+ + C_n q_n,
$$

where $\psi_1^+ (\mu)$ and $\psi_2^- (\mu)$ are the entering and exiting fluxes, $\psi_n (\mu)$ is the region averaged flux, $q_n = (\Sigma_{sn} \phi_n^{(i)} + S_n)/2$ is the source and

$$
T_n (\mu) = e^{-2\tau_n/|\mu|}, \quad E_n (\mu) = \frac{1 - T_n (\mu)}{\Sigma_n}, \quad I_n (\mu) = |\mu| \frac{E_n (\mu)}{2\tau_n}, \quad C_n (\mu) = \frac{1 - I_n (\mu)}{\Sigma_n}, \quad n = 1, 2,
$$

with $\tau_n = \Delta_n \Sigma_n$, are the transmission, escape, incoming and collision coefficients for the system. Using the translation boundary conditions, $\psi_2^- = \psi_1^- = \psi_{bd}$ for $\mu > 0$ and $\psi_1^- = \psi_2^- = \psi_{bd}$ for $\mu < 0$, we obtain

$$
\psi_{bd} (\mu) = \psi_n (\mu) = T \times \left\{ \begin{array}{ll}
(\Sigma_1 \phi_1^{(i)} + T_1 E_2 q_2^{(i)}), & \mu > 0, \\
(\Sigma_2 \phi_2^{(i)} + T_2 E_1 q_1^{(i)}), & \mu < 0,
\end{array} \right.
$$

where $\psi_n (\mu)$ is the interface flux, $\psi_{bd} (\mu)$ is the boundary flux and $T = (1 - T_1 T_2)^{-1}$. We can now use this result to compute $\psi_1 (\mu)$ and $\psi_2 (\mu)$ and, by integration in $\mu$, derive the expression for the scalar fluxes

$$
\overrightarrow{\phi}_N = A_T (\Sigma_s \overrightarrow{\phi}_N^{(i)} + \overrightarrow{S}_N).
$$

(8)

Also, with $n$ and $m$ denoting the two regions

$$
A_{nn} = \int_0^1 (C_n + I_n T_n E_n) (\mu) d\mu, \quad (9)
$$

$$
A_{nm} = \frac{1}{2} \int_0^1 [I_n E_m (2 - T_1^2 T_2^2)] (\mu) d\mu.
$$

(9)

Moreover, we give the interface net current $J_s = \int_{-1}^1 \mu \psi_s (\mu) d\mu$ and the interface second order angular moment $\theta_s = \int_{-1}^1 \mu^2 \psi_s (\mu) d\mu$.

$$
J_s = \overrightarrow{K} \cdot (\Sigma_s \overrightarrow{\phi}_N^{(i)} + \overrightarrow{S}_N), \quad \theta_s = \overrightarrow{L} \cdot (\Sigma_s \overrightarrow{\phi}_N^{(i)} + \overrightarrow{S}_N),
$$

$$
K_1 = \frac{1}{2} \int_0^1 \mu [(1 - T_2) T E_1] (\mu) d\mu, \quad (10)
$$

$$
K_2 = -\frac{1}{2} \int_0^1 \mu [(1 - T_1) T E_2] (\mu) d\mu,
$$

$$
L_n = \frac{1}{2} \int_0^1 \mu^2 [(1 + T_m) T E_n] (\mu) d\mu.
$$

which are essential to construct the MAFC scheme.

3. Modified Average Flux Correction scheme for the 2-region slab problem

We use a CMFD method proposed by Yamamoto, namely the Average Flux Correction (AFC) scheme, (Yamamoto, 2005a), enhanced by the introduction of the interface Eddington factor, that attenuates errors of the low-order angular representation of the diffusion approximation, (Goldin, 1964; Hiruta and Anistratov, 2006; Anistratov, 2006) For the sake of simplicity, we suppose to deal with a coarse-mesh composed by 2 regions, i.e. $K = 2$, regardless of the number of fine-meshes. Using specular reflection on both sides, the balance equations for the two coarse-meshes are

$$
J_s + \tau_{a1} \phi_1 = \Delta_1 S_1, \quad (11)
$$

$$
- J_s + \tau_{a2} \phi_2 = \Delta_2 S_2,
$$

where $(\tau_{r,k} = \frac{\Sigma_{sn} \bar{T}_n \phi_{bd, k}}{2 \bar{\Sigma}_n}, k = 1, 2, r = a, s)$ are the optical thicknesses of coarse-meshes. The trans-
port net current $J_s$ between the two regions is preserved by the finite difference relation,

$$
J_s = -d_{s1}(\phi_s - \phi_1) - a_{s1}\phi_1,
$$

$$
J_s = -d_{s2}(\phi_2 - \phi_s) - a_{s2}\phi_2,
$$

(12)

here, $(a_{sk}, k = 1, 2)$ are the interface equivalence parameters computed by inverting Eqs. (12) with transport-computed fluxes and currents. The MAFC differs from the original AFC by coefficients $(d_{sk}, k = 1, 2)$, which contain the Quasidiffusion closure. More specifically, we introduce the transport-computed Eddington factor $\frac{\theta_s}{\phi_s}$ in $d_{sk},$

$$
d_{sk} = \frac{2}{\tau_k}\frac{\theta_s}{\phi_s},
$$

The matrix-form of balance Eqs. (11) gives the expression of $A_D(\overrightarrow{\phi}_N^{(i)})$; by using the continuity of $\phi_s$ and $J_s$ in Eqs. (12) together with the expression of coefficients $d_{sk}$ and $a_{sk}$ we get

$$
A_D(\overrightarrow{\phi}_N^{(i)}) = \begin{pmatrix}
\frac{\tau_1 J_s + 2\theta_s}{\tau_1 + 2\phi_1} & \frac{\tau_2 J_s - 2\theta_s}{\tau_2 + 2\phi_2} \\
\frac{\tau_1 J_s + 2\theta_s}{\tau_1 + 2\phi_1} & \frac{\tau_2 J_s - 2\theta_s}{\tau_2 + 2\phi_2} + \tau_{a2}
\end{pmatrix}
$$

(13)

The dependence of $A_D$ by $\overrightarrow{\phi}_N^{(i)}$is implicitly contained in the transport quantities $J_s, \theta_s$ and $(\phi_k = \sum_{n \in k} \Delta_n \phi_n / \sum_{n \in k} \Delta_n, k = 1, 2).$

4. Case 1: interface parameter effect (K=N=2)

Thanks to the simplifying assumptions, when the coarse and fine operator deals with the same mesh, the restriction and prolongation operator are identity matrices and the non-linear map can be written in the simple form:

$$
\overrightarrow{\phi}_N = A_T[\Sigma_s \overrightarrow{\phi}_N^{(i)} + \overrightarrow{S}_N],
$$

$$
A_D(\overrightarrow{\phi}_N^{(i)}) \overrightarrow{\phi}_N^{(i+1)} = \overrightarrow{V_S} S_N.
$$

(14)

In our particular 2-region one-dimensional slab problem $K = N = 2$, the matrices $A_T$ and $A_D$ are given by expressions (9) and (13), respectively.

In order to compute the Jacobian matrix $D F(\overrightarrow{\phi}_N^{(i)}) = \nabla_{\overrightarrow{\phi}_N^{(i)}} \overrightarrow{\phi}_N^{(i)}$ we write $A_D$ as the sum of a singular matrix, which contains boundary informations, plus a diagonal matrix $B_d$, which contains the absorption thickness of the regions,

$$
A_D(\overrightarrow{\phi}_N^{(i)}) = B_d(\overrightarrow{\phi}_N^{(i)}) D(\overrightarrow{\phi}) + B_d,
$$

(15)

where $B_d = \text{diag} [\tau_{an}, n = 1, 2], D(\overrightarrow{\phi}) = \text{diag} [1/\phi_n, n = 1, 2]$ and

$$
B_d(\overrightarrow{\phi}_N^{(i)}) = \frac{1}{\tau_1 + \tau_2} \begin{pmatrix}
\tau_1 J_s + 2\theta_s & \tau_2 J_s - 2\theta_s \\
-\tau_1 J_s + 2\theta_s & -(\tau_2 J_s - 2\theta_s)
\end{pmatrix}.
$$

Eq. (15) shows that $A_D(\overrightarrow{\phi}_N^{(i)})$ is singular when $B_d = 0$, moreover, for $B_d \sim 0$ the diffusion matrix is ill-conditioned.

To obtain an explicit expression for the Jacobian matrix we differentiate the second Eq. of system (14) with respect to $\phi_n^{(i)}$:

$$
\nabla_{\overrightarrow{\phi}_N^{(i)}} \overrightarrow{\phi}_N^{(i+1)} = -A_D^{-1}(\nabla_{\overrightarrow{\phi}_N^{(i)}} A_D) \overrightarrow{\phi}_N^{(i+1)},
$$

(16)

with

$$
\frac{\partial A_D}{\partial \phi_n^{(i)}} = \frac{\partial B_d}{\partial \phi_n^{(i)}} D_d(\overrightarrow{\phi}_N) + \Sigma_{an} B_d(\overrightarrow{\phi}_N^{(i)}) \sum_{m=1,2} \frac{\partial D}{\partial \phi_m}(\overrightarrow{\phi}_N) A_{mn},
$$

where use has been made of (15) together with (8) and

$$
\frac{\partial B_d}{\partial \phi_n^{(i)}} = \Sigma_{an} \left( \frac{\tau_1 K_n + 2L_n}{\tau_1 + 2\phi_1} \frac{\tau_2 K_n + 2L_n}{\tau_2 + 2\phi_2} \right),
$$

(17)

$$
\frac{\partial D}{\partial \phi_1} = \begin{pmatrix}
-\frac{1}{\phi_1} & 0 \\
0 & -\frac{1}{\phi_2}
\end{pmatrix},
$$

$$
\frac{\partial D}{\partial \phi_2} = \begin{pmatrix}
0 & 0 \\
0 & -\frac{1}{\phi_2}
\end{pmatrix}.
$$

The explicit behavior of the Jacobian during iterations depends on transport coefficients $K$ and $L$, the transport matrix $A_T$, Eqs. (9) and (10), and the transport flux $\overrightarrow{\phi}_N$. The fixed point $\overrightarrow{\xi}$ is the converged solution of transport iterations:

$$
\overrightarrow{\xi} = [1 - A_T(\overrightarrow{\xi}) \Sigma_s]^{-1} A_T(\overrightarrow{\xi}) S.
$$

This expression shows that the stability of the iterative scheme depends also on the source distribution $S$.

To simplify further the problem and to eliminate the dependence on $\overrightarrow{\xi}$, one can chose a flat fixed point such that $\xi_1 = \xi_2 = \xi$, which entails $S_n / \Sigma_{an} = \xi$, $J = 0$ and $\theta = \xi/3$, to find

$$
A_D(\overrightarrow{\xi}) = \frac{2}{\tau_1 + \tau_2} \begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix} + \begin{pmatrix}
\tau_{a1} & 0 \\
0 & \tau_{a2}
\end{pmatrix}.
$$

(18)
Using (17) for the contraction $\nabla_{\phi_N} A_D \phi_N^{(i+1)}|_{\phi_N^{(i+1)}} = \xi$, we get

$$\nabla_{\phi_N} A_D (\xi) = \left( \begin{array}{c} \Sigma s_1[K_1 - \frac{2}{3} (A_{11} - A_{21})] \\
\Sigma s_2[K_2 + \frac{2}{3} (A_{12} - A_{22})] \\
\Sigma s_3[K_1 - \frac{2}{3} (A_{11} - A_{21})] \\
\Sigma s_2[K_2 + \frac{2}{3} (A_{12} - A_{22})] \end{array} \right)$$

Finally, by collecting Eqs. (18) and Eq. (19) into Eq. (16), one can compute the analytical expression for the non zero eigenvalue $\lambda$ of $DF(\xi)$,

$$\lambda = -\frac{1}{\det(A_D)} \left[ \{\tau_2 \Sigma s_1 K_1 - \tau_1 \Sigma s_2 K_2\} - 2 \frac{\tau_2 \Sigma s_2 (A_{11} - A_{21}) + \tau_1 \Sigma s_2 (A_{12} - A_{22})}{(\tau_1 + \tau_2)} \right]$$

where $\det(A_D) = \tau_1 \tau_2 + (2/3)(\tau_1 + \tau_2)/\tau_1 + \tau_2$. (Note that the other eigenvalue is zero because $\nabla_{\phi_N} A_D$ is a singular matrix.)

We have conducted a parametric study around a flat-flux solution for the case when the two regions have the same width $\Delta = 1$. We write

$$\tau_1 = \tau = \Sigma s_1 \Delta, \quad c_1 = c, \quad \tau_2 = (1-c+pc)\tau, \quad c_2 = pc/(1-c+pc),$$

where $p$ is a scaling parameter. This is equivalent to preserving the 'absorption' component $(1-c)\tau$ of $\tau$ for both regions while multiplying the 'scattering' component $c\tau$ times $p$ for region-2 and has the advantage of having a flat-flux $\xi = \Delta S/(1-c)\tau$ equilibrium solution (fixed point of $F$.)

With these assumptions the nonlinear map $F$ is a function $F(\phi_N,p,c,\tau)$ and the largest eigenvalue $\lambda$ of the map $DF(\xi)$ is given by the simple expression

$$\lambda(p,c,\tau) = -\frac{1}{(1-c)\tau + \frac{1}{3} \tau^2 - c + pc} \times$$

$$\left[ c\tau(K_1 - \tau_2 p K_2) - 2c(A_{11} - A_{21}) + p(A_{12} - A_{22}) \right].$$

In Fig. 1 we have plotted the eigenvalue $\lambda(p,c,\tau)$ versus $p$ for $c \in \{0.1, 0.5, 0.9, 0.99, 1\}$ and $\tau \in \{0.1, 1, 10\}$. The figure shows that the eigenvalue becomes rapidly greater than unity when the number of secondaries $c$ of the left region gets close to unity. This result is confirmed by directly replacing $c = 1$ in Eq. (22) to obtain

$$\lambda(p,c) = 1, \tau = -\frac{3\tau^2}{4} (\tau + p) (K_1 - p K_2) -$$

$$\frac{2}{3} (A_{11} - A_{21}) + p (A_{12} - A_{22})] =$$

Because the transport coefficients $A_{12}, A_{22}$ and $K_2$ are bounded smooth functions of $p$, the eigenvalue $\lambda$ varies almost parabolically with respect to $p$ ($\sim -\frac{3\tau^2}{4} p^2 |K_2|$). As the optical thickness of the first region increases, the spectral radius for $c = 0.9$ ensures convergence, whereas, for $c = 0.99$ and $c = 1$, the eigenvalue runs rapidly out of the stability range. The behavior of the eigenvalue for $p \to \infty$ can be obtained from (7), $\Delta_2 = \Delta = 1$,

$$T_2(\mu) \sim 0, \quad I_2(\mu) \sim \frac{|\mu|}{pc\tau},$$

$$E_2(\mu), C_2(\mu) \sim \frac{1}{pc\tau}, \quad p \to \infty.$$
This analysis demonstrates that the accelerated solution can be an unstable fixed point of the nonlinear iterator. In particular, when the difference between the optical thicknesses of two neighbor cells is large, the iterative scheme is affected by cyclic trajectories and by oscillatory behavior. This is illustrated in the bifurcation diagram of Fig. 2 where we have plotted the numerical iterative behavior in the neighborhood of the fixed point for the case $\tau = 0.1$ and $c = 0.99$. The iterations becomes instable for $p \sim 10^2$, following a classical nonlinear bifurcation. This is confirmed by the analytical results in Fig. (1). The top graphic in this figure plainly shows that, for $\tau = 0.1$ and $c = 0.99$, $|\lambda|$ exceeds $1$ for $p \sim 10^2$. A doubling period interval, a chaotic interval and a cyclic interval are observed in succession in the numerical evaluation. Although we do not provide an analytical proof for the chaotic behavior, our numerical results substantiate the existence of such behavior.

Plotted for 5 different values of $c$ (0.1, 0.5, 0.9, 0.99, 1) An S16 level symmetric quadrature formula is used.

Fig. 2. Bifurcation of the flux $\phi_2^{(i)}(p)$ for a flat fixed point $\phi_2^{(\infty)}(p) = \phi_2^{(\infty)}(p) = 1$. The graph was obtained numerically for $\tau = 0.1$ and $c = 0.99$ by running the code with 200 inner accelerated iterations. For each point $p$ are plotted the twenty latest iterations.

5. Case 2: homogenization effect (K=1)

We consider the case when SI are accelerated with one single coarse-mesh comprising the whole domain and, as before, with reflective boundary conditions. The iterative scheme (1)-(4) reduces itself to a practical vector-form that can be easily studied,

$$\overrightarrow{\phi}_N = A_T [\Sigma_{s,N} \overrightarrow{\phi}_N + \overrightarrow{S}_N], \quad (25)$$

$$\overrightarrow{\phi}_N^{(i+1)} = \alpha(\overrightarrow{\phi}_N) \overrightarrow{\phi}_N,$$

with the coefficient

$$\alpha(\overrightarrow{\phi}_N) = \frac{(V_N \cdot \overrightarrow{S}_N)}{(V_{\Sigma_{s,N}} \cdot \overrightarrow{\phi}_N)}. \quad (26)$$

For this case, the $A_D$ matrix collapses in a scalar, i.e. the total flux-weighted absorption $A_D(\overrightarrow{\phi}_N) = (V_N \cdot T_N)(V_{\Sigma_{s,N}} \cdot \overrightarrow{\phi}_N)/(V_N \cdot \overrightarrow{\phi}_N)$, while the restriction and prolongation operator are respectively

$P_{1,N}(\overrightarrow{\phi}_N) \Sigma_{1,N} = (V_{\Sigma_{s,N}} \cdot \overrightarrow{\phi}_N)/(V_N \cdot \overrightarrow{\phi}_N)$ and $P_{N1}(\overrightarrow{\phi}_N) = (V_N \cdot \overrightarrow{T}_N)/(V_N \cdot \overrightarrow{\phi}_N)$.

By deriving the system (25) we obtain a general form for the Jacobian matrix

$$\nabla_{\overrightarrow{\phi}_N} \overrightarrow{\phi}_N^{(i+1)} = \alpha(\overrightarrow{\phi}_N) P(\overrightarrow{\phi}_N) A_T \Sigma_{s,N}, \quad (27)$$

here $P(\overrightarrow{\phi}_N)$ is the projector

$$P(\overrightarrow{\phi}_N) = I = \overrightarrow{\phi}_N \otimes \frac{V_{\Sigma_{s,N}}}{(V_{\Sigma_{s,N}} \cdot \overrightarrow{\phi}_N)}, \quad (28)$$
which has as null-space the space defined by the transport flux, i.e., \( P(\phi_N) \phi_N = 0 \). Here, \( I \) is the identity matrix and \( V\Sigma_a N = \{ V_N \Sigma_a, n = 1, N \} \). Moreover, since \( \| PA_T \Sigma_{s,n} \| \leq \| A_T \Sigma_{s,n} \| \), it easy to show that

\[
\left\| \nabla \frac{\phi_{N}^{(i)}}{\phi_{N}^{(i+1)}} \right\| \leq \alpha(\phi_N) \| A_T \Sigma_{s,n} \| \leq \alpha(\phi_N) c_{\text{max}},
\]

where \( c_{\text{max}} = \max\{ \Sigma_{s,n}/\Sigma_{n}, n = 1, N \} \). The scheme is locally stable if

\[
\alpha(\phi_N) < c_{\text{max}}^{-1}.
\]

Such a general stability condition depends upon the ratio (26) between the total source and the total absorption rate. (For realistic multigroup eigenvalue problems, condition (29) is generally fulfilled because of the normalization of the fission source by the \( k_{\text{eff}} \).)

We introduce the previous one-dimensional two-region problem and carry out the analytical calculation of the largest eigenvalue imposing flat-flux solution, i.e. \( \xi_1 = \xi_2 = \xi \) and consequently \( (S_n/\Sigma_{an} = \xi, n = 1, 2) \). For this case, because \( \alpha(\phi_N) = 1 \), the stability is matched. Moreover, getting use of Eq. (28), (27) becomes

\[
\nabla \frac{\phi_{N}^{(i)}}{\phi_{N}^{(i+1)}} = \\
\left\{ (1 - a) \Sigma_{s,1}(A_{11} - A_{21}) - (1 - a) \Sigma_{s,2}(A_{22} - A_{12}) \\
- a \Sigma_{s,1}(A_{11} - A_{12}) + a \Sigma_{s,2}(A_{22} - A_{12}) \right\}
\]

where \( a = \tau_0/\tau_2 + \tau_0 \). The largest eigenvalue is easily computed,

\[
\lambda = (1 - a) \Sigma_{s,1}(A_{11} - A_{21}) + a \Sigma_{s,2}(A_{22} - A_{12})
\]

\[
\lambda(p,c,\tau) = \frac{1}{1 + p} c\tau(A_{11} - A_{21}) + \frac{p^2}{1 + p} c\tau(A_{22} - A_{12})
\]

obtained by replacing (31) in (30) for \( \Delta = 1 \), \( c \in \{0.1,0.5,0.9,0.99,1\} \) and \( \tau \in \{0.1,1,10\} \). The

Fig. 3. Case \( N = 2, K = 1 \). Eigenvalue portraits for \( \tau = 0.1, \tau = 1, \) and \( \tau = 10 \), and for 5 different values of \( c \) \((0.1, 0.5, 0.9, 0.99, 1) \). An S16 level symmetric quadrature formula is used.

We introduce a parametrization as done in (21), with the difference that we make varying also the absorption cross section with \( p \), i.e. \( \tau_0 = \tau_1/\tau_2 \), resulting in the following positions

\[
\tau_1 = \tau = \Sigma_1 \Delta, \quad \tau_2 = (\frac{1 - c}{p} + pc)\tau, \quad c_1 = c, \quad c_2 = p^2 c/(1 - c + p^2 c).
\]

\[
\lambda(p,c,\tau) = \frac{1}{1 + p} c\tau(A_{11} - A_{21}) + \frac{p^2}{1 + p} c\tau(A_{22} - A_{12})
\]

Parametrization (31) is such that region-2 becomes an infinite absorber material as \( p \to 0 \) or an infinite diffusive material as \( p \to \infty \). We give the spectral portraits in Fig. 3, of
analysis shows that the scheme is stable over all the set of parameters confirming results obtained by numerical simulation. When region-2 becomes an infinite diffusive material, i.e. $p \to \infty$, using (23) in (32) and remembering (24), one get\[
\lim_{p \to \infty} \lambda = (1 - \frac{2G(\tau)}{\tau}),\]
showing that such a limit case does not depend on $c$. Exploring the other limit, when region-2 becomes an infinite absorber material, i.e. $p \to 0$, and using the expansion $\tau A_{11} \sim (1 - \frac{2G(\tau)}{\tau})$ and $\tau A_{21} \sim 0$ for $p \to 0$, one get\[
\lim_{p \to 0} \lambda = c(1 - \frac{2G(\tau)}{\tau}),\]
thus, $\lambda$ is bounded by $c$ for $p \to 0$. In conclusion, the homogenization scheme give a spectral radius bounded by the value $(1 - \frac{2G(\tau)}{\tau}) < 1$. As confirmed by Fig. 3, the scheme is effective when region-1 is not optically thick, in fact the function $(1 - \frac{2G(\tau)}{\tau}) \to 1$ as $\tau \to \infty$, thus, when $\tau >> 1$ the acceleration scheme could converge slower then transport iterations if also region-2 is optically thick ($p \gtrsim 10^{1\div2}$).

6. Conclusions and further remarks

In this paper we have analyzed one of the CMFD methods, namely MAFC, by making use of the local stability theory. We have conducted two investigations to spread out separately the effects of homogenization and of the interface parameters. To have a stable scheme, the coarse-regions have to be weakly connected, thus, the interface net current has to be as little as possible during iterations. This condition may fail when we deal with extremely optically thick coarse meshes. The stability analysis of a simple 2-region problem has demonstrated accelerated transport iterations could lead to an unstable fixed point. How to stabilize CMFD methods is still an open problem which requires further investigation. However, a classical manner to eliminate instabilities is to use a suitable coarse mesh that smoothes optical differences between regions. As a consequence, a possible path to have a robust CMFD scheme is to dynamically choose the coarse-mesh boundaries by a minimal net current search. In fact, the second problems here analyzed shows that the homogenization process can smooth the instable effects observed when the optical difference between cells is large and the media are collision-dominated. Hence, homogenization can be viewed as means to stabilize CMFD iterations. However, such a technique is problem-dependent and does not offer a general strategy for a systematic application. We think that an alternative consistent way may be to introduce a stability parameter that acts on the non-linear map. The difficulty is then transferred to the estimation of the local Lyapunov coefficients and the way such a parameter would have to act to constraint the spectral radius of the accelerated iterator to the stability range. Another possible way to improve the method is to construct a coarse-mesh finite element operator instead of the coarse-mesh finite difference operator. Indeed, the coarse operator can be enriched introducing high order spatial moments for both current and flux to have a better representation of the gradient throughout the domain. Such a technique could be a solution to enhance the stability of the method. We intend to explore different stabilizing techniques in future works.

References


